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A High–Order Perturbation of Envelopes (HOPE) method for vector electromagnetic scattering by periodic inhomogeneous media: Analytic continuation ☆

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Abstract

Electromagnetic waves interacting with three–dimensional periodic structures occur in many applications of great scientific and engineering interest. These three dimensional interactions are extremely complicated and subtle, so it is unsurprising that practitioners find their rapid, robust, and accurate numerical simulation to be of paramount interest. Among the wide array of possible numerical approaches, the High–Order Spectral algorithms are often preferred due to their surpassing fidelity with a moderate number of unknowns, and here we describe an algorithm that fits into this class. In addition, we take a perturbative approach to the problem which views the deviation of the permittivity from a reference value as the deformation and we conduct a regular perturbation theory. This work concludes a line of research on these methods which began with two-dimensional problems governed by the Helmholtz equation and moved to small perturbations in the fully three-dimensional vector Maxwell equations. We now extend these latter results to large (real) perturbations constituting a rigorous analytic continuation.

MSC: 65N35; 78M22; 78A45; 35J25; 35Q60; 35Q86

Keywords: Maxwell equations; Inhomogeneous media; Layered media; High–Order Spectral methods; High–Order Perturbation of Envelopes methods; Analytic continuation

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1. Introduction

Electromagnetic waves interacting with three–dimensional periodic structures occur in many applications of great scientific and engineering interest. Many examples can be given from fields as different as surface-enhanced spectroscopy [25], extraordinary optical transmission [9], cancer therapy [10], and surface plasmon resonance (SPR) biosensing [17,20,23,31].

Because of their crucial role in these linear scattering applications, all of the classical numerical algorithms for the simulation of solutions to the governing partial differential equations have been brought to bear upon this problem. Among these are the Finite Difference [39,22], Finite Element [19,18], Discontinuous Galerkin [16], Spectral Element [8], and Spectral [15,37,38] methods. While these are compelling choices, due to their *volumetric* character they require a large number of unknowns ($N = N_x N_y N_z$ for a three dimensional simulation) and require the inversion of large, non–symmetric positive definite matrices (of dimension $N \times N$). We point the interested reader to [11,24] for recent developments.

Focusing on the particular example of SPR sensors [17,20,23,31] which is the focus of this work, their utility and ubiquity follows from two key properties of an SPR, namely its extremely strong and sensitive response. Quantitatively, over the range of tens of nanometers in incident wavelength, the reflected energy can reduce from almost 100% by a factor of 10 or even 100 before ascending back to almost 100%. Clearly, to approximate such a structure with the required fidelity, the numerical algorithm should produce surpassingly accurate results in a fast and robust manner. For this reason, we will focus upon High–Order Spectral (HOS) methods [15,37,38] which have exactly these features.

In regard to the classical approaches listed above, one standard method for generating SPRs is via homogeneous layers of material, and it is clearly wasteful to discretize the bulk of each layer. As a result, most prominent solvers feature interfacial unknowns with the understanding that information *inside* a layer can be computed from appropriate integral formulas. Boundary element (BEM) [36] and boundary integral (BIM) [7,21] methods are two popular approaches and can produce highly accurate solutions in a fraction of the time of their volumetric competitors.

In previous work [26,32] the authors investigated a new algorithm which has much in common with these HOS algorithms, but was inspired by the "High–Order Perturbation of Surfaces" (HOPS) methods [29,30] which have proven to be so useful for layered media. A HOPS scheme views the layer interfaces as perturbations of flat ones and then makes recursive corrections to the scattering returns from this exactly solvable configuration [40]. However, our new "High–Order Perturbation of Envelopes" (HOPE) schemes consider a more general permittivity function, $\epsilon(x, y, z)$, which does *not* necessarily have layered structure. Our approach follows the lead of Feng, Lin, and Lorton [13,14] who adopted a perturbative philosophy by studying the permittivity as a perturbation of a trivial one, e.g.,

$$\epsilon(x, y, z) = \bar{\epsilon}(1 - \delta\mathcal{E}(x, y, z)), \quad \bar{\epsilon} \in \mathbf{R}, \quad \mathcal{E}(x + d_x, y + d_y, z) = \mathcal{E}(x, y, z),$$

where $\mathcal{E}$ is a permittivity "envelope." In [26] we focused upon the two–dimensional scalar problems of electromagnetic radiation in Transverse Electric (TE) or Transverse Magnetic (TM) polarization. Building upon this work, in [32] we extended our results to the three-dimensional vector electromagnetic case governed by the full Maxwell equations. These new methods have computational advantages over volumetric solvers in some configurations (e.g., where the support of $\mathcal{E}$ is small or where the set on which $\mathcal{E}$ significantly changes is small). In particular, we considered an approximate indicator function which modeled the absence/presence of a material.

There were several contributions of [26] including a new, and far–reaching, rigorous analysis. In more detail, we proved not only that the domain of analyticity of the scattered field in δ can be extended to a neighborhood of the *entire* real axis (up to topological obstruction), see Fig. 1, but also that this field is *jointly* analytic in parametric and spatial variables provided that $\mathcal{E}(x, y, z)$ is spatially analytic. In our subsequent paper [32] we extended a subset of these results to the three dimensional vector time–harmonic Maxwell equations, in particular, that the scattered field is analytic as a function of δ and *jointly* analytic in both parametric and spatial variables if $\mathcal{E}(x, y, z)$ is spatially analytic. In the current contribution we take up not only the issue of analytic continuation to perturbations δ of arbitrary (real) size, see Fig. 1, but also their joint analyticity with respect to spatial variables, which completes the analysis of these HOPE methods as applied to the Maxwell equations. As we shall see, this requires a significant enhancement of the existing technology. In particular, in contrast to the constant–coefficient analysis of [32] which enabled us to use rather explicit *exact* solution formulas, the theorem presented here required a novel analysis of the *weak formulation* of the relevant variable coefficient Helmholtz problem presented in Bao & Li [2]. It is noteworthy that this new analysis required an accounting of inhomogeneous terms which are not considered in [2] and presented several unforeseen challenges. While this theorem is not technically required to justify the numerical simulations presented in our previous work [32] on scattering returns by periodic structures, it does definitively answer the question: Does δ need to be small? The answer is an emphatic “no,” provided that δ is in the domain of analyticity which includes a neighborhood of the *entire* real axis, see Fig. 1.

The rest of the paper is organized as follows. In § 2 we recall the governing equations and discuss transparent boundary conditions in § 3. We describe the HOPE algorithm in § 4 and begin our theoretical developments with a statement of the relevant function spaces in § 5. We state and prove our results on parametric analytic continuation in § 6 and extend these to joint analyticity in § 7. In § 8 we describe how the numerical algorithm from our companion work [32] can be greatly enhanced by numerical analytic continuation which is justified by the theorems proven here. We provide concluding remarks in § 9. The crucial elliptic estimates upon which these results rely are established in Appendix A and Appendix B.

2. Governing equations

We consider materials whose electromagnetic response is modeled by the time–harmonic Maxwell equations in three dimensions with a constant permeability $\mu = \mu_0$ and no currents or sources,

$$\begin{aligned} \operatorname{curl}[E] - i\omega\mu_0 H = 0, \quad \operatorname{curl}[H] + i\omega\epsilon E = 0, \\ \operatorname{div}[\epsilon E] = 0, \quad \operatorname{div}[H] = 0, \end{aligned} \tag{2.1}$$

where (E, H) are the electric and magnetic vector fields, and we have factored out time dependence of the form $\exp(-i\omega t)$ [2]. The permittivity $\epsilon(x, y, z)$ is biperiodic with periods d_x and d_y, and is specified by

$$\epsilon(x, y, x) = \begin{cases} \epsilon^{(u)}\epsilon_0, & z > h, \\ \epsilon^{(v)}(x, y, z)\epsilon_0, & -h < z < h, \\ \epsilon^{(w)}\epsilon_0, & z < -h, \end{cases}$$

where ϵ_0 is the permittivity of vacuum, $\epsilon^{(u)}, \epsilon^{(w)} \in \mathbf{R}^+$, and $\epsilon^{(v)}(x + d_x, y + d_y, z) = \epsilon^{(v)}(x, y, z)$, and

$$\lim_{z \to h-} \epsilon^{(v)}(x, y, z) = \epsilon^{(u)}, \qquad \lim_{z \to (-h)+} \epsilon^{(v)}(x, y, z) = \epsilon^{(w)}.$$

For future use we define

$$k_0^2 = \omega^2 \epsilon_0 \mu_0 = \frac{\omega^2}{c_0^2}, \quad (k^m)^2 = \epsilon^{(m)} k_0^2, \quad m \in \{u, w\},$$

and $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$ is the speed of light in vacuum.

This structure is illuminated from above by plane–wave incident radiation of the form

$$E^{\text{inc}}(x, y, z) = A \exp(i\alpha x + i\beta y - i\gamma^{(u)} z),$$
$$H^{\text{inc}}(x, y, z) = B \exp(i\alpha x + i\beta y - i\gamma^{(u)} z),$$

where

$$A \cdot \kappa = 0, \quad B = \frac{1}{\omega \mu_0} \kappa \times A, \quad |A| = |B| = 1,$$

and

$$\kappa = \begin{pmatrix} \alpha \\ \beta \\ -\gamma^{(u)} \end{pmatrix} = k^{(u)} \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ -\cos(\theta) \end{pmatrix},$$

where (θ, ϕ) are the angles of incidence.

3. Transparent boundary conditions

Following the lead of Bao & Li [2] we use Transparent Boundary Conditions at $z = \pm h$ to both rigorously specify the appropriate far–field boundary conditions, and reduce the infinite domain to one of finite size. These are specified with Dirichlet–Neumann Operators (DNOs) which map the *tangential* traces of the scattered electric fields at $z = \pm h$ to the traces of the scattered magnetic fields at $z = \pm h$. Such operators are commonly called Capacity Operators [2].

To summarize the developments of [2] (§ 3.2.2) we use the fact that $H = \frac{1}{i\omega\mu_0} \operatorname{curl}[E]$ and define

$$T_u : U \to \tilde{U}, \quad T_w : W \to \tilde{W},$$

where, for $N_u = (0, 0, 1)^T$ and $N_w = (0, 0, -1)^T$,

$$U := N_u \times (E^{\text{scat}}\big|_{z=h} \times N_u), \quad \tilde{U} := \frac{1}{i\omega\mu_0}(\text{curl}\left[E^{\text{scat}}\right]\big|_{z=h} \times N_u),$$
$$W := N_w \times (E^{\text{scat}}\big|_{z=-h} \times N_w), \quad \tilde{W} := \frac{1}{i\omega\mu_0}(\text{curl}\left[E^{\text{scat}}\right]\big|_{z=-h} \times N_w).$$

Using the facts that

$$E = E^{\text{scat}} + E^{\text{inc}}, \quad H = H^{\text{scat}} + H^{\text{inc}}, \quad z > h,$$
$$E = E^{\text{scat}}, \quad H = H^{\text{scat}}, \quad z < -h,$$

and multiplying the definitions of $\{T_u, T_w\}$ by $-(i\omega\mu_0)$, we specify the Transparent Boundary Conditions

$$\text{curl}\left[(E - E^{\text{inc}})\right] \times N_u - (i\omega\mu_0)T_u[N_u \times ((E - E^{\text{inc}}) \times N_u)] = 0, \qquad z = h,$$
$$\text{curl}\,[E] \times N_w - (i\omega\mu_0)T_w[N_w \times (E \times N_w)] = 0, \qquad z = -h,$$

or

$$\text{curl}\,[E] \times N_u - (i\omega\mu_0)T_u[N_u \times (E \times N_u)] = \phi, \qquad z = h,$$
$$\text{curl}\,[E] \times N_w - (i\omega\mu_0)T_w[N_w \times (E \times N_w)] = 0, \qquad z = -h,$$

where

$$\phi = \text{curl}\left[E^{\text{inc}}\right] \times N_u - (i\omega\mu_0)T_u[N_u \times (E^{\text{inc}} \times N_u)], \quad z = h. \tag{3.1}$$

To find a formula for T_u we note that, in the upper domain $\{z > h\}$, separation of variables demands that upward propagating (α, β)–quasiperiodic solutions of the Maxwell equations are

$$E^{\text{scat}} = \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty} \hat{u}_{p,q} \exp(i\alpha_p x + i\beta_q y + i\gamma_{p,q}^{(u)}(z - h)), \quad \hat{u}_{p,q} = \begin{pmatrix} \hat{u}_{p,q}^x \\ \hat{u}_{p,q}^y \\ \hat{u}_{p,q}^z \end{pmatrix},$$

[33,40] where

$$\alpha_p = \alpha + (2\pi/d_x)p, \quad \beta_q = \beta + (2\pi/d_y)q,$$
$$(\gamma_{p,q}^{(m)})^2 = \epsilon^{(m)}k_0^2 - \alpha_p^2 - \beta_q^2, \quad \text{Im}\{\gamma_{p,q}^{(m)}\} \geq 0, \quad m \in \{u, w\}.$$

In particular, for a dielectric ($\epsilon^{(m)} \in \mathbf{R}^+$) we have

$$\gamma_{p,q}^{(m)} := \begin{cases} \sqrt{\epsilon^{(m)}k_0^2 - \alpha_p^2 - \beta_q^2}, & \alpha_p^2 + \beta_q^2 \leq \epsilon^{(m)}k_0^2, \\ i\sqrt{\alpha_p^2 + \beta_q^2 - \epsilon^{(m)}k_0^2}, & \alpha_p^2 + \beta_q^2 > \epsilon^{(m)}k_0^2. \end{cases}$$

Using the relation

$$\begin{pmatrix} U^x(x,y) \\ U^y(x,y) \\ 0 \end{pmatrix} = U(x,y) = N_u \times (E^{\text{scat}}(x,y,h) \times N_u) = \begin{pmatrix} E^{\text{scat},x}(x,y,h) \\ E^{\text{scat},y}(x,y,h) \\ 0 \end{pmatrix},$$

we find that

$$\hat{u}^x_{p,q} = \hat{U}^x_{p,q}, \quad \hat{u}^y_{p,q} = \hat{U}^y_{p,q}.$$

To resolve $\hat{u}^z_{p,q}$ we use the divergence–free condition in the upper layer to deduce that

$$(i\alpha_p)\hat{u}^x_{p,q} + (i\beta_q)\hat{u}^y_{p,q} + (i\gamma^{(u)}_{p,q})\hat{u}^z_{p,q} = 0.$$

We now make the assumption that we are away from Rayleigh Singularities (commonly referred to as Wood's Anomalies)

$$\gamma^{(u)}_{p,q} \neq 0, \quad \forall\ p,q \in \mathbf{Z},$$

which gives

$$\hat{u}^z_{p,q} = \frac{-\alpha_p \hat{u}^x_{p,q} - \beta_q \hat{u}^y_{p,q}}{\gamma^{(u)}_{p,q}} = \frac{-\alpha_p \hat{U}^x_{p,q} - \beta_q \hat{U}^y_{p,q}}{\gamma^{(u)}_{p,q}}.$$

Therefore we can express

$$E^{\text{scat}} = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \begin{pmatrix} \hat{U}^x_{p,q} \\ \hat{U}^y_{p,q} \\ \frac{-\alpha_p \hat{U}^x_{p,q} - \beta_q \hat{U}^y_{p,q}}{\gamma^{(u)}_{p,q}} \end{pmatrix} \exp(i\alpha_p x + i\beta_q y + i\gamma^{(u)}_{p,q}(z-h)).$$

Now, it is a simple matter to compute

$$\begin{aligned} \tilde{U} = H^{\text{scat}} \times N_u &= \frac{1}{i\omega\mu_0} (\text{curl}\left[E^{\text{scat}}\right]\big|_{z=h} \times N_u) \\ &= \frac{1}{i\omega\mu_0} \begin{pmatrix} \partial_z E^{\text{scat},x}(x,y,h) - \partial_x E^{\text{scat},z}(x,y,h) \\ \partial_z E^{\text{scat},y}(x,y,h) - \partial_y E^{\text{scat},z}(x,y,h) \\ 0 \end{pmatrix}. \end{aligned}$$

For this we observe that

$$\partial_x E^{\text{scat}}(x,y,h) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (i\alpha_p) \begin{pmatrix} \hat{U}^x_{p,q} \\ \hat{U}^y_{p,q} \\ \frac{-\alpha_p \hat{U}^x_{p,q} - \beta_q \hat{U}^y_{p,q}}{\gamma^{(u)}_{p,q}} \end{pmatrix} \exp(i\alpha_p x + i\beta_q y),$$

and

$$\partial_y E^{\mathrm{scat}}(x,y,h) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (i\beta_q) \begin{pmatrix} \hat{U}^x_{p,q} \\ \hat{U}^y_{p,q} \\ \frac{-\alpha_p \hat{U}^x_{p,q} - \beta_q \hat{U}^y_{p,q}}{\gamma^{(u)}_{p,q}} \end{pmatrix} \exp(i\alpha_p x + i\beta_q y),$$

and

$$\partial_z E^{\mathrm{scat}}(x,y,h) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} (i\gamma^{(u)}_{p,q}) \begin{pmatrix} \hat{U}^x_{p,q} \\ \hat{U}^y_{p,q} \\ \frac{-\alpha_p \hat{U}^x_{p,q} - \beta_q \hat{U}^y_{p,q}}{\gamma^{(u)}_{p,q}} \end{pmatrix} \exp(i\alpha_p x + i\beta_q y).$$

Therefore,

$$\begin{aligned} \tilde{U} &= T_u[U] \\ &= \frac{1}{i\omega\mu_0} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \begin{pmatrix} (i\gamma^{(u)}_{p,q})\hat{U}^x_{p,q} + \frac{(i\alpha_p)}{\gamma^{(u)}_{p,q}} \left\{ \alpha_p \hat{U}^x_{p,q} + \beta_q \hat{U}^y_{p,q} \right\} \\ (i\gamma^{(u)}_{p,q})\hat{U}^y_{p,q} + \frac{(i\beta_p)}{\gamma^{(u)}_{p,q}} \left\{ \alpha_p \hat{U}^x_{p,q} + \beta_q \hat{U}^y_{p,q} \right\} \\ 0 \end{pmatrix} \exp(i\alpha_p x + i\beta_q y). \end{aligned}$$

In a similar fashion

$$\begin{aligned} \tilde{W} &= T_w[W] \\ &= \frac{1}{i\omega\mu_0} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \begin{pmatrix} (i\gamma^{(w)}_{p,q})\hat{W}^x_{p,q} + \frac{(i\alpha_p)}{\gamma^{(w)}_{p,q}} \left\{ \alpha_p \hat{W}^x_{p,q} + \beta_q \hat{W}^y_{p,q} \right\} \\ (i\gamma^{(w)}_{p,q})\hat{W}^y_{p,q} + \frac{(i\beta_p)}{\gamma^{(w)}_{p,q}} \left\{ \alpha_p \hat{W}^x_{p,q} + \beta_q \hat{W}^y_{p,q} \right\} \\ 0 \end{pmatrix} \exp(i\alpha_p x + i\beta_q y). \end{aligned}$$

At this point we can state our governing equations with full rigor. Eliminating the magnetic field from (2.1) and gathering our full set of governing equations we find the following problem to solve.

$$\operatorname{curl}[\operatorname{curl}[E]] - \epsilon^{(v)} k_0^2 E = 0, \quad \text{in } \Omega, \tag{3.2a}$$

$$-\operatorname{div}\left[\epsilon^{(v)} k_0^2 E\right] = 0, \quad \text{in } \Omega, \tag{3.2b}$$

$$\operatorname{curl}[E] \times N_u - (i\omega\mu_0) T_u[N_u \times (E \times N_u)] = \phi, \quad \text{at } \Gamma_u, \tag{3.2c}$$

$$\operatorname{curl}[E] \times N_w - (i\omega\mu_0) T_w[N_w \times (E \times N_w)] = 0, \quad \text{at } \Gamma_w, \tag{3.2d}$$

$$E(x+d_x, y+d_y, z) = \exp(i\alpha d_x + i\beta d_y) E(x,y,z), \tag{3.2e}$$

where

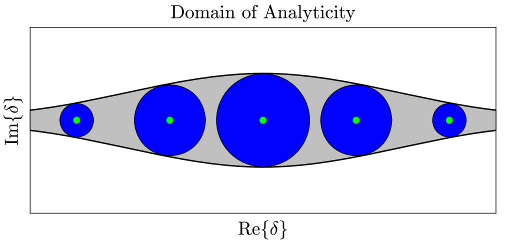

Fig. 1. The domain of analyticity (gray) in the complex plane of the field, (4.1), as a function of the perturbation parameter, δ. We demonstrate existence of *disks* (blue) of analyticity around arbitrary values $\rho_0 \in \mathbf{R}$ (green) which fills out the entire gray region. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

$$\Omega := (0, d_x) \times (0, d_y) \times (-h, h),$$
$$\Gamma_u := (0, d_x) \times (0, d_y) \times \{z = h\}, \quad \Gamma_w := (0, d_x) \times (0, d_y) \times \{z = -h\}.$$

Remark 3.1. Equation (3.2b) is, of course, a simple consequence of the divergence operator applied to (3.2a). However, we include it *explicitly* in order to highlight its importance in our subsequent elliptic estimates and analyticity theory.

Remark 3.2. We point out that sufficiently regular solutions of (2.1) are solutions of (3.2a)–(3.2b), and vice versa.

4. A High–Order Perturbation of Envelopes method

In our previous work [26,32] we pursued the solution of (3.2) not by a classical volumetric approach, but rather by a perturbative one where we thought of our configuration as a *small* deviation from a simpler, constant, structure,

$$\epsilon^{(v)}(x, y, z) = \bar{\epsilon}(1 - \delta \mathcal{E}(x, y, z)) = \bar{\epsilon} - \delta(\bar{\epsilon}\mathcal{E}(x, y, z)),$$

where $\delta \ll 1$. We showed that, provided that $\mathcal{E}$ is sufficiently smooth, the solution depends analytically on δ and can be expressed as a convergent Taylor series. Now, it is known that the coefficients of this series determine the solution throughout the *entire* domain of analyticity of E which, we now show, is much larger than the disk of convergence of this series. In fact, it contains the *whole* real line; see Fig. 1.

To investigate this claim we study perturbations of the form

$$\epsilon^{(v)}(x, y, z) = \bar{\epsilon}(1 - \rho \mathcal{E}(x, y, z)), \quad \rho \in \mathbf{R},$$

by setting $\rho = \rho_0 + \delta$ where ρ_0 is arbitrary, but real, while $\delta \ll 1$. We then write

$$\begin{aligned} \epsilon^{(v)}(x, y, z) &= \bar{\epsilon}(1 - \rho \mathcal{E}(x, y, z)) \\ &= \bar{\epsilon}(1 - \rho_0 \mathcal{E}(x, y, z)) - \delta(\bar{\epsilon}\mathcal{E}(x, y, z)) \\ &= \bar{\epsilon}_0(x, y, z) - \delta(\bar{\epsilon}\mathcal{E}(x, y, z)), \end{aligned}$$

which defines the base permittivity envelope

$$\bar{\epsilon}_0(x,y,z) := \bar{\epsilon}(1-\rho_0\mathcal{E}(x,y,z)).$$

In contrast to our previous work [32], this base value is *not* constant, but rather dependent upon all of the spatial variables, (x,y,z).

We posit that the field $E = E(x,y,z;\delta)$ depends analytically upon δ so that

$$E = E(x,y,z;\delta) = \sum_{\ell=0}^{\infty} E_\ell(x,y,z)\delta^\ell, \tag{4.1}$$

converges strongly in a function space. In this way we establish that the field, E, is analytic in a disk about an *arbitrary* real value of $\rho = \rho_0 \in \mathbf{R}$. It is not difficult to see that these E_ℓ must satisfy

$$\operatorname{curl}[\operatorname{curl}[E_\ell]] - \bar{\epsilon}_0 k_0^2 E_\ell = \bar{\epsilon}_0 k_0^2 F_\ell, \quad \text{in } \Omega, \tag{4.2a}$$

$$-\operatorname{div}\left[\bar{\epsilon}_0 k_0^2 E_\ell\right] = \operatorname{div}\left[\bar{\epsilon}_0 k_0^2 F_\ell\right], \quad \text{in } \Omega, \tag{4.2b}$$

$$\operatorname{curl}[E_\ell] \times N_u - (i\omega\mu_0)T_u[N_u \times (E_\ell \times N_u)] = \delta_{\ell,0}\phi, \quad \text{at } \Gamma_u, \tag{4.2c}$$

$$\operatorname{curl}[E_\ell] \times N_w - (i\omega\mu_0)T_w[N_w \times (E_\ell \times N_w)] = 0, \quad \text{at } \Gamma_w, \tag{4.2d}$$

$$E_\ell(x+d_x, y+d_y, z) = \exp(i\alpha d_x + i\beta d_y)E_\ell(x,y,z), \tag{4.2e}$$

where

$$F_\ell = F_\ell(x,y,z) = -\frac{\bar{\epsilon}\mathcal{E}(x,y,z)}{\bar{\epsilon}_0(x,y,z)} E_{\ell-1}(x,y,z), \tag{4.2f}$$

and $\delta_{\ell,0}$ is the Kronecker delta function.

There are many possibilities for the envelope function $\mathcal{E}(x,y,z)$ and each leads to a slightly different perturbation approach. For instance, consider the function

$$\Phi_{a,b}(z) := \frac{\tanh(w(z-a)) - \tanh(w(z-b))}{2},$$

with sharpness parameter w, which is effectively zero outside the interval (a,b) while being essentially one inside (a,b), cf. [26]. We can approximate a slab of material (of permittivity ϵ') with thickness $2d$ and a gap of width $2g$ in vacuum by selecting [32]

$$\bar{\epsilon} = 1, \quad \mathcal{E}(x,y,z) = \left(\frac{\bar{\epsilon}-\epsilon'}{\bar{\epsilon}}\right)\Phi_{-d,d}(z)\left\{1 - \Phi_{-g,g}(x)\right\}, \quad \rho_0 = 0, \quad \delta = 1.$$

See Fig. 2 with the choices $d = 1/4$, $g = 1/10$, and $w = 50$ on the cell $[-1,1] \times [-1,1]$.

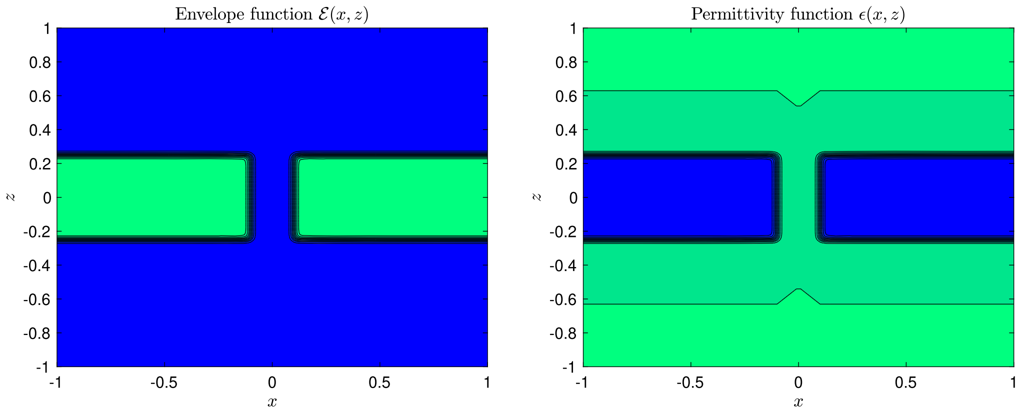

Fig. 2. Contour plots of $\mathcal{E}(x,z)$ (left) and $\epsilon^{(v)}(x,z)$ (right).

5. Function spaces

In this section, we present function spaces and theoretical notions that are necessary for our analysis. Due to the very weak formulation of the Maxwell equations we employ, espoused by Bao & Li [2], our function spaces are quite different from those used in our previous work [26,32]. In fact we move to the functional framework outlined in Section 3.3.1 of [2] of (α,β)–quasiperiodic $H(\text{curl})$ and $H(\text{div})$ functions. More specifically

$$\begin{aligned} H(\text{curl}) = H(\text{curl},\Omega) = \Big\{ & u \in L^2(\Omega)^3 \mid \text{curl}\,[u] \in L^2(\Omega)^3, \\ & e^{i\alpha d_x} u(0,y,z)\times\hat{n}_x = u(d_x,y,z)\times\hat{n}_x, \\ & e^{i\beta d_y} u(x,0,z)\times\hat{n}_y = u(x,d_y,z)\times\hat{n}_y \Big\}, \end{aligned}$$

where $\hat{n}_x = (1,0,0)$ and $\hat{n}_y = (0,1,0)$, and

$$\|u\|^2_{H(\text{curl})} := \|u\|^2_{L^2} + \|\text{curl}\,[u]\|^2_{L^2}.$$

Additionally,

$$\begin{aligned} H(\text{div}) = H(\text{div},\Omega) = \Big\{ & u \in L^2(\Omega)^3 \mid \text{div}\,[u] \in L^2(\Omega), \\ & e^{i\alpha d_x} u(0,y,z)\times\hat{n}_x = u(d_x,y,z)\times\hat{n}_x, \\ & e^{i\beta d_y} u(x,0,z)\times\hat{n}_y = u(x,d_y,z)\times\hat{n}_y \Big\}, \end{aligned}$$

and

$$\|u\|^2_{H(\text{div})} := \|u\|^2_{L^2} + \|\text{div}\,[u]\|^2_{L^2}.$$

Due to the particular structure of the inhomogeneous Maxwell equations, we can simplify the statement and proof of our theorems by introducing the $(\bar{\epsilon}_0(x,y,z)k_0^2)$–dependent space

$$X = X(\bar{\epsilon}_0, \Omega) := \left\{ u \in H(\text{curl}, \Omega) \mid (\bar{\epsilon}_0 k_0^2 u) \in H(\text{div}, \Omega) \right\},$$

with norm

$$\|u\|_X^2 := \|u\|_{H(\text{curl})}^2 + \left\| \bar{\epsilon}_0 k_0^2 u \right\|_{H(\text{div})}^2 .$$

In addition, we require interfacial versions of the spaces $H(\text{curl})$ and $H(\text{div})$ at the artificial boundaries $\Gamma_u = \{z = h\}$ and $\Gamma_w = \{z = -h\}$, namely

$$\begin{aligned} H^{-1/2}(\text{curl}) &= H^{-1/2}(\text{curl}, \Gamma_m) \\ &= \left\{ u \in H^{-1/2}(\Gamma_m)^3, \text{curl}_{\Gamma_m} u \in H^{-1/2}(\Gamma_m)^3, u^z = 0 \right\}, \\ H^{-1/2}(\text{div}) &= H^{-1/2}(\text{div}, \Gamma_m) \\ &= \left\{ u \in H^{-1/2}(\Gamma_m)^3, \text{div}_{\Gamma_m} u \in H^{-1/2}(\Gamma_m), u^z = 0 \right\}. \end{aligned}$$

We point out that the final condition in the definition of these spaces, $u^z = 0$, is due to the fact that the boundaries, Γ_m, are *flat* (see [2], page 64). For these, the norms can be computed [2] from

$$\|u\|_{H^{-1/2}(\text{curl})}^2 := d_x d_y \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{\left|\hat{u}_{p,q}^x\right|^2 + \left|\hat{u}_{p,q}^y\right|^2 + \left|\alpha_p \hat{u}_{p,q}^y - \beta_q \hat{u}_{p,q}^x\right|^2}{\sqrt{1 + \alpha_p^2 + \beta_q^2}},$$

$$\|u\|_{H^{-1/2}(\text{div})}^2 := d_x d_y \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \frac{\left|\hat{u}_{p,q}^x\right|^2 + \left|\hat{u}_{p,q}^y\right|^2 + \left|\alpha_p \hat{u}_{p,q}^x + \beta_q \hat{u}_{p,q}^y\right|^2}{\sqrt{1 + \alpha_p^2 + \beta_q^2}}.$$

We also recall the space of s-times continuously differentiable functions with Hölder norm

$$|v|_{C^s} := \max_{0 \le \ell + r \le s} \max_{m \in \{x,y,z\}} \left| \partial_x^\ell \partial_y^r v^m \right|_{L^\infty} .$$

We close with an essential result [12,27] required for our later proofs.

Lemma 5.1. *Let $g \in C^1(\Omega)$, $u \in H(\text{curl}, \Omega)$, $v \in H(\text{div}, \Omega)$, where Ω is a subset of $\mathbf{R}^3$, then $gu \in H(\text{curl}, \Omega)$ and $gv \in H(\text{div}, \Omega)$. Furthermore,*

$$\begin{aligned} \|gu\|_{H(\text{curl})} &\le M(\Omega) \, |g|_{C^1} \, \|u\|_{H(\text{curl})} , \\ \|gv\|_{H(\text{div})} &\le M(\Omega) \, |g|_{C^1} \, \|v\|_{H(\text{div})} , \end{aligned}$$

where M is some positive constant.

Finally, we recall the following elementary result [28,26].

Lemma 5.2. *Let $s \geq 0$ be an integer, then there exists a constant $S > 0$ such that*

$$\sum_{j=0}^{s} \frac{(s+1)^2}{(s-j+1)^2(j+1)^2} < S, \quad \sum_{j=0}^{s}\sum_{r=0}^{j} \frac{(s+1)^2}{(s-j+1)^2(j-r+1)^2(r+1)^2} < S^2.$$

6. Analytic continuation

At this point we are in a position to establish analyticity of the full electric field in a neighborhood of *any* real value ρ_0 by demonstrating the analytic dependence of $E = E(x, y, z; \delta)$ upon δ sufficiently small. More specifically, we show that the expansion (4.1) converges strongly in an appropriate function space.

For this we require an elliptic estimate for our inductive proof which is established in Appendix A. For future convenience, we define the following differential operators associated to the Maxwell system

$$\begin{aligned} \mathcal{L}_0 E &:= \text{curl}\,[\text{curl}\,[E]] - \bar{\epsilon}_0(x,y,z)k_0^2 E, && \text{in } \Omega, \\ \mathcal{B}_m E &:= \text{curl}\,[E] \times N_m - (i\omega\mu_0)T_m[N_m \times (E \times N_m)], && \text{at } \Gamma_m, \end{aligned}$$

for $m \in \{u, w\}$. As is well known [2], the issue of *uniqueness* of solutions to the Maxwell problem

$$\mathcal{L}_0 V = 0, \quad \text{in } \Omega, \tag{6.1a}$$

$$-\text{div}\left[\bar{\epsilon}_0 k_0^2 V\right] = 0, \quad \text{in } \Omega, \tag{6.1b}$$

$$\mathcal{B}_u V = 0, \quad \text{at } \Gamma_u, \tag{6.1c}$$

$$\mathcal{B}_w V = 0, \quad \text{at } \Gamma_w, \tag{6.1d}$$

$$V(x+d_x, y+d_y, z) = \exp(i\alpha d_x + i\beta d_y)V(x,y,z), \tag{6.1e}$$

cf. (4.2), which should have only the *trivial* solution $V \equiv 0$, is a subtle one and certain illuminating frequencies ω (alternatively wavenumbers k_0) will induce non–uniqueness in some configurations. Unfortunately a precise characterization of the set of forbidden frequencies is elusive and all that is known is that it is countable and accumulates at infinity [2]. To accommodate this state of affairs we define the set of permissible configurations

$$\mathcal{P} := \{(\omega, \bar{\epsilon}_0) \mid V \equiv 0 \text{ is the unique solution of } (6.1)\}. \tag{6.2}$$

With this we can now state the following fundamental elliptic regularity result.

Theorem 6.1. *If $(\omega, \bar{\epsilon}_0) \in \mathcal{P}$, $\bar{\epsilon}_0 \in L^\infty(\Omega)$, $(\bar{\epsilon}_0 k_0^2 F) \in H(\text{div}, \Omega)$, $Q \in H^{-1/2}(\text{div}, \Gamma_u)$, and $R \in H^{-1/2}(\text{div}, \Gamma_w)$, then there exists a unique solution $E \in X(\bar{\epsilon}_0, \Omega)$ of*

$$\mathcal{L}_0 E = \bar{\epsilon}_0 k_0^2 F, \quad \textit{in } \Omega, \tag{6.3a}$$

$$-\text{div}\left[\bar{\epsilon}_0 k_0^2 E\right] = \text{div}\left[\bar{\epsilon}_0 k_0^2 F\right], \quad \textit{in } \Omega, \tag{6.3b}$$

$$\mathcal{B}_u E = Q, \qquad \text{at } \Gamma_u, \tag{6.3c}$$

$$\mathcal{B}_w E = R, \qquad \text{at } \Gamma_w, \tag{6.3d}$$

$$E(x + d_x, y + d_y, z) = \exp(i\alpha d_x + i\beta d_y)E(x, y, z), \tag{6.3e}$$

satisfying

$$\|E\|_X \le C_e \left(\left\| \bar{\epsilon}_0 k_0^2 F \right\|_{H(\text{div})} + \|Q\|_{H^{-1/2}(\text{div})} + \|R\|_{H^{-1/2}(\text{div})} \right), \tag{6.4}$$

where $C_e > 0$ *is a positive constant.*

We can now prove the following analytic continuation result.

Theorem 6.2. *If* $(\omega, \bar{\epsilon}_0) \in \mathcal{P}$, $\bar{\epsilon}_0 \in L^\infty(\Omega)$, *and* $(\mathcal{E}/\bar{\epsilon}_0) \in C^1(\Omega)$ *then the series* (4.1) *converges strongly. More precisely,*

$$\|E_\ell\|_X \le K B^\ell, \quad \forall\, \ell \ge 0, \tag{6.5}$$

for some constants $K, B > 0$.

Proof. We prove the estimate (6.5) by induction. For $\ell = 0$ the system (4.2) can be written as

$$\begin{aligned}
&\mathcal{L}_0 E_0 = 0, && \text{in } \Omega,\\
&-\operatorname{div}\left[\bar{\epsilon}_0 k_0^2 E_0\right] = 0, && \text{in } \Omega,\\
&\mathcal{B}_u E_0 = \phi, && \text{at } \Gamma_u,\\
&\mathcal{B}_w E_0 = 0, && \text{at } \Gamma_w,\\
&E_0(x + d_x, y + d_y, z) = \exp(i\alpha d_x + i\beta d_y)E_0(x, y, z).
\end{aligned}$$

We can apply Theorem 6.1 with $F \equiv 0$, $Q = \phi$, and $R \equiv 0$ to obtain (6.5)

$$\|E_\ell\|_X \le C_e \|\phi\|_{H^{-1/2}(\text{div})} =: K.$$

Next, we assume that (6.5) is true for all $\ell < L$ and apply Theorem 6.1 to the system (4.2) for E_L with $F = -\bar{\epsilon}(\mathcal{E}/\bar{\epsilon}_0)E_{L-1}$ and $Q \equiv R \equiv 0$. This gives

$$\begin{aligned}
\|E_L\|_X &\le C_e \left\| \bar{\epsilon}_0 k_0^2 F_L \right\|_{H(\text{div})}\\
&\le C_e \left\| (\bar{\epsilon}_0 k_0^2)(-\bar{\epsilon}(\mathcal{E}/\bar{\epsilon}_0)E_{L-1}) \right\|_{H(\text{div})}\\
&\le C_e \bar{\epsilon} M \, |\mathcal{E}/\bar{\epsilon}_0|_{C^1} \left\| \bar{\epsilon}_0 k_0^2 E_{L-1} \right\|_{H(\text{div})}\\
&\le C_e \bar{\epsilon} M \, |\mathcal{E}/\bar{\epsilon}_0|_{C^1} \, K B^{L-1}\\
&\le K B^L,
\end{aligned}$$

provided that

$$B > C_e \bar{\epsilon} M \left|\mathcal{E}/\bar{\epsilon}_0\right|_{C^1},$$

and we are done. □

Remark 6.1. At this point we comment on the smoothness requirements we make on $\mathcal{E}(x, y, z)$ and $\bar{\epsilon}_0(x, y, z) = \bar{\epsilon}(1 - \rho_0 \mathcal{E}(x, y, z))$. First, we ask that $\bar{\epsilon}_0 \in L^\infty$ which imposes a (completely appropriate) boundedness requirement on $\mathcal{E}(x, y, z)$. However, we also ask that $\mathcal{E}(x, y, z)/\bar{\epsilon}_0(x, y, z) \in C^1$ which puts more complicated size requirements on $\rho_0 \mathcal{E}(x, y, z)$ measured in the C^1 norm. However, it is clear that $|\rho_0| \left|\mathcal{E}(x, y, z)\right|_{C^1}$ cannot be large and must not be too small.

From this we can derive the exponential order of convergence of this HOPE method. More precisely, defining the L–th partial sum of (4.1)

$$E^L(x, y, z; \delta) := \sum_{\ell=0}^{L} E_\ell(x, y, z)\delta^\ell,$$

we obtain the following error estimate.

Theorem 6.3. *If $(\omega, \bar{\epsilon}_0) \in \mathcal{P}$, $\bar{\epsilon}_0 \in L^\infty(\Omega)$, and E is the unique solution of (3.2), under the assumptions of Theorem 6.2 we have the estimate*

$$\left\| E - E^L \right\|_X \leq 2K(B\delta)^{L+1},$$

for some constants $K, B > 0$, provided that $|\delta| < 1/(2B)$.

Proof. Since

$$E(x, y, z) - E^L(x, y, z) = \sum_{\ell=L+1}^{\infty} E_\ell(x, y, z)\delta^\ell,$$

we have, by Theorem 6.2,

$$\left\| E - E^L \right\|_X \leq \sum_{\ell=L+1}^{\infty} \|E_\ell\|_X \, \delta^\ell \leq \sum_{\ell=L+1}^{\infty} K B^\ell \delta^\ell.$$

By gathering terms and re–indexing we have

$$\left\| E - E^L \right\|_X \leq K(B\delta)^{L+1} \sum_{\ell=0}^{\infty} (B\delta)^\ell = \frac{K(B\delta)^{L+1}}{1 - B\delta},$$

for $|\delta B| < 1$, where we have used the elementary fact that

$$\sum_{\ell=0}^{\infty} \alpha^{\ell} = \frac{1}{1-\alpha},$$

provided that $|\alpha| < 1$. If we choose, say $|\delta B| \leq 1/2$ then

$$\frac{1}{1-(B\delta)} \leq 2,$$

and

$$\left\| E - E^{L} \right\|_{X} \leq 2K(B\delta)^{L+1}. \quad \square$$

Remark 6.2. Returning to the consideration of smoothness properties of the permittivity envelope, $\mathcal{E}(x, y, z)$, we note that this information is reflected in the *size* of B as we have just demonstrated in the previous two results: Larger $|\mathcal{E}|_{C^1}$ demands smaller B and a more restricted set of δ. In addition, for the *same* value of δ the convergence will be slower if B is larger. These issues in the two dimensional setting are investigated in our previous work [26] and we refer the interested reader to this.

7. Joint analyticity

In this section we show that the solution $E(x, y, z; \delta)$ of (3.2) is jointly analytic with respect to both parameter, δ, and spatial variables, (x, y, z). For this we need an appropriate notion of analyticity which we give in the following definition of C_q^{ω}.

Definition 7.1. Given an integer $q \geq 0$, if the functions $f = f(x, y)$ and $\mathcal{E} = \mathcal{E}(x, y, z)$ are *real analytic* and satisfy the following estimates

$$\left| \frac{\partial_x^r \partial_y^t}{(r+t)!} f \right|_{C^q} \leq C_f \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

$$\left| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \mathcal{E} \right|_{C^q} \leq C_{\mathcal{E}} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2},$$

for all $r, t, s \geq 0$ and constants $C_f, C_{\mathcal{E}}, \eta, \theta, \zeta > 0$, then $f \in C_q^{\omega}(\Gamma_m)$, $m \in \{u, w\}$, and $\mathcal{E} \in C_q^{\omega}(\Omega)$.

Here C_q^{ω} is the space of real analytic functions with radius of convergence (specified by η, θ, and ζ) measured in the C^q norm. It is clear that the incident radiation function ϕ, (3.1), is jointly analytic in x and y as we now explicitly state.

Lemma 7.1. *The function $\phi(x, y)$ defined in (3.1) is real analytic and satisfies*

$$\left\| \frac{\partial_x^r \partial_y^t}{(r+t)!} \phi \right\|_{H^{-1/2}(\mathrm{div})} \leq C_{\phi} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

for all $r, t \geq 0$ and some constants $C_{\phi}, \eta, \theta > 0$.

Now we present the fundamental elliptic estimate which is required in our estimates. (It is proven in Appendix B.)

Theorem 7.1. *Given any integer $q \geq 0$, if $(\omega, \bar{\epsilon}_0) \in \mathcal{P}$, $\bar{\epsilon}_0 \in C_q^{\omega}(\Omega)$, such that*

$$\left| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \bar{\epsilon}_0 \right|_{C^q} \leq C_{\bar{\epsilon}_0} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2},$$

for all $r, t, s \geq 0$ and some constants $C_{\bar{\epsilon}_0}, \eta, \theta, \zeta > 0$, and $(\bar{\epsilon}_0 k_0^2 F) \in C^{\omega}(\Omega)$ such that

$$\left\| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \left[\bar{\epsilon}_0 k_0^2 F \right] \right\|_{H(\text{div})} \leq C_F \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2},$$

for all $r, t, s \geq 0$ and some constant $C_F > 0$, and $Q \in C^{\omega}(\Gamma_u)$ and $R \in C^{\omega}(\Gamma_w)$ satisfying

$$\left\| \frac{\partial_x^r \partial_y^t}{(r+t)!} Q \right\|_{H^{-1/2}(\text{div})} \leq C_Q \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

$$\left\| \frac{\partial_x^r \partial_y^t}{(r+t)!} R \right\|_{H^{-1/2}(\text{div})} \leq C_R \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

for all $r, t \geq 0$ and some constants $C_R, C_Q > 0$. Then, there exists a unique solution $E \in C^{\omega}(\Omega)$ of

$$\mathcal{L}_0 E = \bar{\epsilon}_0 k_0^2 F, \quad \text{in } \Omega, \tag{7.1a}$$

$$-\operatorname{div}\left[\bar{\epsilon}_0 k_0^2 E\right] = \operatorname{div}\left[\bar{\epsilon}_0 k_0^2 F\right], \quad \text{in } \Omega, \tag{7.1b}$$

$$\mathcal{B}_u E = Q, \quad \text{at } \Gamma_u, \tag{7.1c}$$

$$\mathcal{B}_w E = R, \quad \text{at } \Gamma_w, \tag{7.1d}$$

$$E(x + d_x, y + d_y, z) = e^{i\alpha d_x + i\beta d_y} E(x, y, z), \tag{7.1e}$$

satisfying

$$\left\| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} E \right\|_X \leq \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2}, \tag{7.2}$$

for all $r, t, s \geq 0$ where

$$\underline{C}_e = \overline{C}(C_F + C_Q + C_R) > 0,$$

and $\overline{C} > 0$ is a constant.

We now give the recursive estimate which is essential for our joint analyticity result.

Lemma 7.2. *Given any integer $q \geq 0$, if $(\omega, \bar{\epsilon}_0) \in \mathcal{P}$; $\bar{\epsilon}_0, (\mathcal{E}/\bar{\epsilon}_0) \in C_q^{\omega}(\Omega)$, such that*

$$\left| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \bar{\epsilon}_0 \right|_{C^q} \leq C_{\bar{\epsilon}_0} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2},$$

$$\left| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} [\mathcal{E}/\bar{\epsilon}_0] \right|_{C^q} \leq C_{\mathcal{E}/\bar{\epsilon}_0} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2},$$

for all $r, t, s \geq 0$ and some constants $C_{\bar{\epsilon}_0}, C_{\mathcal{E}/\bar{\epsilon}_0}, \eta, \theta, \zeta > 0$, and

$$\left\| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} E_\ell \right\|_X \leq K B^\ell \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2}, \quad \forall\, \ell < L,$$

for all $r, t, s \geq 0$ and for some constants $K, B > 0$. Then,

$$\left\| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \left[(\bar{\epsilon}_0 k_0^2) F_L \right] \right\|_{H(\mathrm{div})} \leq \tilde{C} K B^{L-1} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2}, \tag{7.3}$$

for all $r, t, s \geq 0$ and some constant $\tilde{C} > 0$.

Proof. Using Leibniz's rule, we have that

$$\begin{aligned}
\frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \left[(\bar{\epsilon}_0 k_0^2) F_L \right] &= \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \left[(\bar{\epsilon}_0 k_0^2)(-\bar{\epsilon}(\mathcal{E}/\bar{\epsilon}_0) E_{L-1}) \right] \\
&= -\bar{\epsilon} \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \left[(\mathcal{E}/\bar{\epsilon}_0) \bar{\epsilon}_0 k_0^2 E_{L-1} \right] \\
&= -\bar{\epsilon} \frac{r!t!s!}{(r+t+s)!} \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{s} \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{s-\ell}}{(s-\ell)!} [\mathcal{E}/\bar{\epsilon}_0] \right) \\
&\quad \times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} \left[\bar{\epsilon}_0 k_0^2 E_{L-1} \right] \right).
\end{aligned}$$

Using the inequality $r!t!s! \leq (r+t+s)!$, we obtain

$$\begin{aligned}
&\left\| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \left[(\bar{\epsilon}_0 k_0^2) F_L \right] \right\|_{H(\mathrm{div})} \\
&\quad \leq \bar{\epsilon} \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{s} \left\| \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{s-\ell}}{(s-\ell)^2} [\mathcal{E}/\bar{\epsilon}_0] \right) \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} \left[\bar{\epsilon}_0 k_0^2 E_{L-1} \right] \right) \right\|_{H(\mathrm{div})}
\end{aligned}$$

Continuing

$$
\begin{aligned}
\left\| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \left[(\bar{\epsilon}_0 k_0^2) F_L \right] \right\|_{H(\mathrm{div})} &\le \bar{\epsilon} M \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{s} \left| \frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{s-\ell}}{(s-\ell)^2} [\mathcal{E}/\bar{\epsilon}_0] \right|_{C^1} \\
&\quad \times \left\| \frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} \left[\bar{\epsilon}_0 k_0^2 E_{L-1} \right] \right\|_{H(\mathrm{div})} \\
&\le \bar{\epsilon} M \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{s} C_{\mathcal{E}/\bar{\epsilon}_0} \frac{\eta^{r-j}}{(r-j+1)^2} \\
&\quad \times \frac{\theta^{t-k}}{(t-k+1)^2} \frac{\zeta^{s-\ell}}{(s-\ell+1)^2} \\
&\quad \times K B^{L-1} \frac{\eta^j}{(j+1)^2} \frac{\theta^k}{(k+1)^2} \frac{\zeta^\ell}{(\ell+1)^2} \\
&\le \bar{\epsilon} M C_{\mathcal{E}/\bar{\epsilon}_0} K B^{L-1} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2} \\
&\quad \times \sum_{j=0}^{r} \frac{(r+1)^2}{(r-j+1)^2 (j+1)^2} \sum_{k=0}^{t} \frac{(t+1)^2}{(t-k+1)^2 (k+1)^2} \\
&\quad \times \sum_{\ell=0}^{s} \frac{(s+1)^2}{(s-\ell+1)^2 (\ell+1)^2} \\
&\le \bar{\epsilon} M C_{\mathcal{E}/\bar{\epsilon}_0} S^3 K B^{L-1} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2},
\end{aligned}
$$

where S comes from Lemma 5.2. If we choose

$$
\tilde{C} \ge \bar{\epsilon} M C_{\mathcal{E}/\bar{\epsilon}_0} S^3,
$$

the proof is complete. □

We conclude with our joint analyticity theorem.

Theorem 7.2. *Given any integer $q \ge 0$, if $(\omega, \bar{\epsilon}_0) \in \mathcal{P}$; $\bar{\epsilon}_0, (\mathcal{E}/\bar{\epsilon}_0) \in C_q^\omega(\Omega)$, such that*

$$
\begin{aligned}
\left| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} \bar{\epsilon}_0 \right|_{C^q} &\le C_{\bar{\epsilon}_0} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2}, \\
\left| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} [\mathcal{E}/\bar{\epsilon}_0] \right|_{C^q} &\le C_{\mathcal{E}/\bar{\epsilon}_0} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2},
\end{aligned}
$$

for all $r, t, s \ge 0$ and some constants $C_{\bar{\epsilon}_0}, C_{\mathcal{E}/\bar{\epsilon}_0}, \eta, \theta, \zeta > 0$. Then the series (4.1) converges strongly. Moreover the $E_\ell(x, y, z)$ satisfy the joint analyticity estimate

$$\left\| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} E_\ell \right\|_X \le K B^\ell \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2}, \tag{7.4}$$

for all $\ell, r, t, s \ge 0$ and constants $K, B > 0$.

Proof. We prove (7.4) by induction, beginning with $\ell = 0$. Applying Theorem 7.1 with $F \equiv 0$, $Q = \phi$, and $R \equiv 0$ we obtain

$$\left\| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} E_0 \right\|_X \le C_\phi \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

for all $r, t, s \ge 0$ which establishes (7.4) with $K := C_\phi$.

Next we assume that (7.4) is valid for all $\ell < L$. With $\ell = L$ we invoke Lemma 7.2 and apply Theorem 7.1 with $F \equiv F_L$, $C_F = \tilde{C} K B^{L-1}$, $Q \equiv 0$, and $R \equiv 0$, to arrive at

$$\left\| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} E_L \right\|_X \le \overline{C} \tilde{C} K B^{L-1} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2},$$

for all $r, t, s \ge 0$. The proof is complete by choosing $B > \overline{C}\tilde{C}$. □

8. Numerical analytic continuation

In our companion paper [32] we described an algorithm which implements a numerical approximation of the recursions (4.2). We demonstrated the rapid, stable, and highly accurate behavior an implementation of this method is able to achieve [32]. However, the code was severely challenged by the large size of the perturbation parameter, $\delta = 1$, which is required to realize the goal geometry (identically zero/one to indicate the absence/presence of a material). The reason for this behavior is that this value of δ is near the boundary of the disk of convergence of the expansion (4.1) and we simply summed a truncation of this series in a straightforward manner, a procedure we term "Taylor summation." There are, of course, alternative approaches to approximating this sum, many of which attempt a numerical analytic continuation. Using the most popular of these, Padé approximation [1], we now investigate how the convergence of the most challenging of the computations found in [32] can be significantly enhanced using this technique, a result which the theory of this manuscript justifies. We now provide a brief discussion of the numerical method and demonstrate our results.

8.1. Implementation

In [32] we described a numerical HOPE algorithm which begins with a truncation of the expansion (4.1) at order $\ell = L$,

$$E(x, y, z; \delta) \approx E^L(x, y, z; \delta) := \sum_{\ell=0}^{L} E_\ell(x, y, z) \delta^\ell. \tag{8.1}$$

Each of the functions E_ℓ must approximately satisfy (4.2) which we enforce with a High–Order Spectral (HOS) method [15,6,4,38]. Due to the quasiperiodic lateral boundary conditions, we used a Fourier–Chebyshev method which approximates

$$E_\ell \approx E_\ell^{N_x,N_y,N_z} := \sum_{p=-N_x/2}^{N_x/2-1} \sum_{q=-N_y/2}^{N_y/2-1} \sum_{r=0}^{N_z} \hat{E}_{\ell,p,q,r} T_r(z/h) e^{i\alpha_p x + i\beta_q y}, \quad \hat{E}_{\ell,p,q,r} \in \mathbf{C}^3,$$

and T_r is the r–th Cheybshev polynomial. To find the $\hat{E}_{\ell,p,q,r}$ we adopt the collocation approach: The equations (4.2) must be true at the gridpoints

$$\{x_j = j(d_x/N_x) \mid 0 \le j \le N_x - 1\}, \quad \{y_k = k(d_y/N_y) \mid 0 \le k \le N_y - 1\},$$
$$\{z_m = h\cos(\pi m/N_z) \mid 0 \le m \le N_z\}.$$

This demands the solution of a system of linear equations which can be achieved in an efficient and stable manner with the repeated use of fast Fourier and Chebyshev transforms [15,6,4,38].

In the theorems above we have demonstrated that the region of analyticity of the series (4.1) contains the entire real axis. One way to not only access this extended region of analyticity, but also realize improved rates of convergence *inside* the disk of analyticity centered at the origin, is the classical technique of Padé approximation [1]. Padé approximation seeks to estimate the truncated Taylor series $f(\delta) = \sum_{\ell=0}^{L} f_\ell \delta^\ell$ by the rational function

$$\left[\frac{M}{N}\right](\delta) := \frac{a^M(\delta)}{b^N(\delta)} = \frac{\sum_{m=0}^{M} a_m \delta^m}{\sum_{n=0}^{N} b_n \delta^n}, \quad M + N = L,$$

and

$$\left[\frac{M}{N}\right](\delta) = f(\delta) + \mathcal{O}(\delta^{M+N+1});$$

well–known formulas for the coefficients $\{a_m, b_n\}$ can be found in [1]. These Padé approximants have stunning properties of enhanced convergence, and we point the interested reader to § 2.2 of [1] and the calculations in § 8.3 of [3] for a complete discussion.

8.2. The Method of Manufactured Solutions

To test our implementation, we utilized the Method of Manufactured Solutions (MMS) [5, 34,35]. For this consider the general system of partial differential equations subject to generic boundary conditions

$$\mathcal{P} v = 0, \qquad \text{in } \Omega,$$
$$\mathcal{B} v = 0, \qquad \text{at } \partial\Omega.$$

It is usually just as easy to implement a numerical algorithm to solve the nonhomogeneous version of this set of equations

$$\mathcal{P}v = \mathcal{F}, \qquad \text{in } \Omega,$$
$$\mathcal{B}v = \mathcal{J}, \qquad \text{at } \partial\Omega.$$

To validate our code we began with the “manufactured solution,” $\tilde{v}$, and set

$$\mathcal{F}_v := \mathcal{P}\tilde{v}, \quad \mathcal{J}_v := \mathcal{B}\tilde{v}.$$

Therefore, given $\{\mathcal{F}_v, \mathcal{J}_v\}$ we had an *exact* solution of the nonhomogeneous problem, namely $\tilde{v}$. While this does not prove an implementation to be correct, if the function $\tilde{v}$ is chosen to imitate the behavior of desired solutions (e.g., satisfying the boundary conditions exactly) then this gives us confidence in our algorithm.

In the present setting we considered the representative Maxwell problem, cf. (3.2),

$$\begin{aligned}
&\operatorname{curl}\left[\operatorname{curl}\left[\tilde{v}\right]\right] - \epsilon^{(v)} k_0^2 \tilde{v} = F_v, && \text{in } \Omega,\\
&-\operatorname{div}\left[\epsilon^{(v)} k_0^2 \tilde{v}\right] = \operatorname{div}\left[\epsilon^{(v)} k_0^2 F_v\right], && \text{in } \Omega,\\
&\operatorname{curl}\left[\tilde{v}\right] \times N_u - (i\omega\mu_0) T_u [N_u \times (\tilde{v} \times N_u)] = Q_v, && \text{at } \Gamma_u,\\
&\operatorname{curl}\left[\tilde{v}\right] \times N_w - (i\omega\mu_0) T_w [N_w \times (\tilde{v} \times N_w)] = R_v, && \text{at } \Gamma_w,\\
&\tilde{v}(x + d_x, y + d_y, z) = \exp(i\alpha d_x + i\beta d_y)\tilde{v}(x, y, z),
\end{aligned}$$

with

$$d_x = d_y = 2\pi, \quad h = 5/2, \quad k_0 = 1.3, \quad \theta = \phi = 0,$$

and the (biperiodic) manufactured solution,

$$\tilde{v}(x, y, z) = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} e^{i\alpha_s x + i\beta_t y + i\gamma_{s,t}^{(v)} z},$$
$$A_1 = 1, \quad A_2 = \frac{\pi}{4}, \quad A_3 = -\frac{A_1(i\alpha_s) + A_2(i\beta_t)}{i\gamma_{s,t}^{(v)}}, \quad s = t = 1.$$

We coupled this with (essentially) the choice of the envelope function mentioned above in § 4,

$$\bar{\epsilon} = 0.9, \quad \mathcal{E}(x, y, z) = \left(\frac{\bar{\epsilon} - \epsilon'}{\bar{\epsilon}}\right) \Phi_{-d,d}(z) \left\{1 - \Phi_{a,b}(x)\right\},$$

where we selected

$$a = \pi/2, \quad b = 3\pi/2, \quad d = 1/4, \quad \delta = 1.$$

For our test we supplied the “exact” input data, $\{F_v, Q_v, R_v\}$ to our HOPE algorithm and compared the output of this, $\tilde{v}^{\text{approx}}$, with $\tilde{v}$ by computing the error

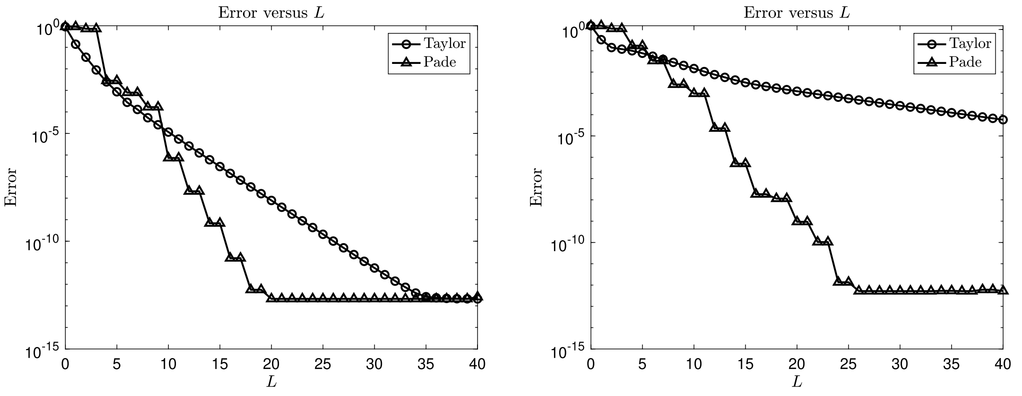

Fig. 3. Errors, (8.2), in HOPE simulation, with both Taylor and Padé summation, of a large deviation configuration ($\epsilon' = 1.6$) with transition parameter $w = 2$ (left: smooth transition) and $w = 200$ (right: sharp transition) versus perturbation order L. ($N_x = N_y = N_z = 24$, $\delta = 1$.)

$$\text{Error} := \left|\tilde{v} - \tilde{v}^{\text{approx}}\right|_{L^\infty} . \tag{8.2}$$

To exhibit the behavior of our scheme we describe results in the “large deviation” ($\epsilon' = 1.6$) regime [32], which was further challenged by studying the effect of the sharpness of the transition from $\bar{\epsilon}$ to ϵ' in $\mathcal{E}(x, y, z; w)$ by choosing $w = 2$ (smooth transition) and $w = 200$ (sharp transition).

We display our findings in Fig. 3 with transition parameter choices $w = 2$ (left: smooth transition) and $w = 200$ (right: sharp transition), for both Taylor and Padé summation. We chose $\delta = 1$ and $N_x = N_y = N_z = 24$, and note the steady and stable convergence of our method, both with Taylor and Padé approximation. However, it is clear that Padé summation enables *greatly* enhanced accuracy in each case. In particular, for $w = 2$ full precision (error on the order 10^{-13}) is realized after only 20 terms with Padé while Taylor requires 35 orders for similar results. Additionally, when $w = 200$ we achieved a relative error of 10^{-13} by perturbation order $L = 25$ with Padé, while Taylor summation only ever realizes an error of 10^{-4} by $L = 40$.

9. Conclusions

In this paper we have demonstrated how the solutions of the vector Maxwell equations in three dimensions, which govern the scattering of incident radiation by a periodic structure, can be analytically continued in perturbation parameter, δ, beyond the disk of convergence centered at the origin to a neighborhood of the *entire* real axis. Furthermore, we have demonstrated the joint analyticity of these solutions with respect to all spatial variables provided that the permittivity envelope itself is jointly analytic. These results complete the analysis of HOPE methods for the study of analyticity properties of solutions to the relevant vector Maxwell equations. The proof of these results demanded significant enhancements of the current analytical machinery, in particular we required a new analysis of the weak formulation of the problem which takes into account inhomogeneous terms which was not considered in the previous work of Bao & Li [2].

CRediT authorship contribution statement

All authors contributed to the study conception and design. Material preparation, data collection and analysis were performed by all authors. The first draft of the manuscript was written by D.P. Nicholls and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

Appendix A. The Proof of Theorem 6.1

Following Bao & Li [2] we dot the Maxwell equation (6.3a) with $\bar{w} \in H(\text{curl})$ and integrate over Ω

$$\int_\Omega (\text{curl}\,[\text{curl}\,[E]]) \cdot \bar{w}\, dV - k_0^2 \int_\Omega \bar{\epsilon}_0 E \cdot \bar{w}\, dV = \int_\Omega (\bar{\epsilon}_0 k_0^2 F) \cdot \bar{w}\, dV.$$

We now use the first vector Green theorem [2]

$$\int_\Omega u \cdot (\text{curl}\,[\sigma\, \text{curl}\,[v]])\, dV = \int_\Omega \sigma\, \text{curl}\,[u] \cdot \text{curl}\,[v]\; dV - \oint_{\partial\Omega} \sigma (u \times \text{curl}\,[v]) \cdot \nu\, dS,$$

to obtain (with $u = \bar{w}$, $v = E$, and $\sigma = 1$)

$$\int_\Omega \text{curl}\,[E] \cdot \text{curl}\,[\bar{w}]\; dV - \oint_{\partial\Omega} (\bar{w} \times \text{curl}\,[E]) \cdot N\, dS - k_0^2 \int_\Omega \epsilon^{(v)} E \cdot \bar{w}\, dV = \int_\Omega (\bar{\epsilon}_0 k_0^2 F) \cdot \bar{w}\, dV.$$

Using the triple product identity $a \cdot (b \times c) = b \cdot (c \times a)$ we find

$$\int_\Omega \text{curl}\,[E] \cdot \text{curl}\,[\bar{w}]\; dV - \oint_{\partial\Omega} \bar{w} \cdot (\text{curl}\,[E] \times N)\, dS - k_0^2 \int_\Omega \epsilon^{(v)} E \cdot \bar{w}\, dV = \int_\Omega (\bar{\epsilon}_0 k_0^2 F) \cdot \bar{w}\, dV.$$

As E and w are (α, β)–quasiperiodic, the contributions from the boundary of Ω reduce to Γ_u and Γ_w,

$$\int_\Omega \text{curl}\,[E] \cdot \text{curl}\,[\bar{w}]\; dV - \int_{\Gamma_u} \bar{w} \cdot (\text{curl}\,[E] \times N_u)\, dS - \int_{\Gamma_w} \bar{w} \cdot (\text{curl}\,[E] \times N_w)\, dS$$
$$- k_0^2 \int_\Omega \epsilon^{(v)} E \cdot \bar{w}\, dV = \int_\Omega (\bar{\epsilon}_0 k_0^2 F) \cdot \bar{w}\, dV.$$

Using the boundary conditions at $z = \pm h$ we find

$$\int_\Omega \text{curl}\,[E] \cdot \text{curl}\,[\bar{w}]\; dV - k_0^2 \int_\Omega \epsilon^{(v)} E \cdot \bar{w}\, dV$$

$$- \int_{\Gamma_u} \bar{w} \cdot \{(i\omega\mu_0) T_u [N_u \times (E \times N_u)] + Q\} \, dS$$
$$- \int_{\Gamma_w} \bar{w} \cdot \{(i\omega\mu_0) T_w [N_w \times (E \times N_w)] + R\} \, dS = \int_{\Omega} (\bar{\epsilon}_0 k_0^2 F) \cdot \bar{w} \, dV.$$

We write this as

$$a(E, w) = L[w], \tag{A.1}$$

where

$$a(E, w) = (\text{curl}\,[E], \text{curl}\,[w])_{\Omega} - k_0^2 \left(\epsilon^{(v)} E, w \right)_{\Omega}$$
$$- \langle (i\omega\mu_0) T_u [N_u \times (E \times N_u)], w \rangle_{\Gamma_u} - \langle (i\omega\mu_0) T_w [N_w \times (E \times N_w)], w \rangle_{\Gamma_w},$$

and

$$L[w] = \left((\bar{\epsilon}_0 k_0^2 F), w \right)_{\Omega} + \langle Q, w \rangle_{\Gamma_u} + \langle R, w \rangle_{\Gamma_w}.$$

In these we use the duality pairings

$$(u, v)_{\Omega} := \int_{\Omega} u \cdot \bar{v} \, dV, \quad \langle u, v \rangle_{\Gamma_m} := \int_{\Gamma_m} u \cdot \bar{v} \, dS.$$

We now seek a solution, $E \in H(\text{curl})$, of this weak formulation by writing the $E, w \in H(\text{curl})$ in the form

$$E = \vec{u} + \nabla u, \quad w = \vec{v} + \nabla v,$$
$$\vec{u}, \vec{v} \in \mathbb{H}, \quad u, v \in H_0^1, \quad \nabla u, \nabla v \in \mathbb{H}^{\perp}.$$

In these we use the spaces $\mathbb{H}$ and $\mathbb{H}^{\perp}$ defined in Bao & Li [2]

$$\mathbb{H} = \left\{ \vec{u} \in H(\text{curl}) \mid \text{div}\left[\bar{\epsilon}_0 \vec{u}\right] = 0 \text{ in } \Omega, \right.$$
$$\left. -k_0^2 \bar{\epsilon}_0 \vec{u} \cdot N_m + \text{div}_{\Gamma_m} \left[(i\omega\mu_0) T_m [\vec{u}_{\Gamma_m}] \right] = 0 \text{ at } \Gamma_m \right\},$$
$$\mathbb{H}^{\perp} = \left\{ \vec{u} \mid \vec{u} = \nabla u, u \in H_0^1 \right\},$$
$$H_0^1 = \left\{ u \in H^1 \;\middle|\; \int_{\Omega} u \, dV = 0 \right\}.$$

Inserting these into our weak form we find

$$a(\vec{u} + \nabla u, \vec{v} + \nabla v) = L[\vec{v} + \nabla v].$$

Using the fact that $a(\vec{u}, \nabla v) = 0$, [2], we have

$$a(\vec{u}, \vec{v}) + a(\nabla u, \vec{v}) + a(\nabla u, \nabla v) = L[\vec{v}] + L[\nabla v],$$

which we solve in two phases: First for the terms involving ∇v, and then for the remaining terms.

A.1. Finding a solution I: $\mathbb{H}^{\perp}$

We begin by determining u from

$$a(\nabla u, \nabla v) = L[\nabla v].$$

To proceed we note the following result.

Lemma A.1. *We have*

$$\begin{aligned}
\langle Q, \nabla v\rangle_{\Gamma_u} &= \langle -\operatorname{div}_{\Gamma_u}[Q], v\rangle_{\Gamma_u}, \\
\langle R, \nabla v\rangle_{\Gamma_w} &= \langle -\operatorname{div}_{\Gamma_w}[R], v\rangle_{\Gamma_w}, \\
\left((\bar{\epsilon}_0 k_0^2 F), \nabla v\right)_{\Omega} &= \left(-\operatorname{div}\left[(\bar{\epsilon}_0 k_0^2 F)\right], v\right)_{\Omega} + \langle(\bar{\epsilon}_0 k_0^2 F)\cdot N, v\rangle_{\Gamma_u} + \langle(\bar{\epsilon}_0 k_0^2 F)\cdot N, v\rangle_{\Gamma_w}.
\end{aligned}$$

Proof. From integration–by–parts,

$$\begin{aligned}
\langle Q, \nabla v\rangle_{\Gamma_u} &= \int_{\Gamma_u} Q \cdot (\nabla_{\Gamma_u}\bar{v})\, dS \\
&= \int_{\Gamma_u} \operatorname{div}_{\Gamma_u}[Q\cdot\bar{v}]\, dS - \int_{\Gamma_u} \operatorname{div}_{\Gamma_u}[Q]\,\bar{v}\, dS \\
&= [Q\cdot\bar{v}]_{x=0}^{x=d} - \int_{\Gamma_u} \operatorname{div}_{\Gamma_u}[Q]\,\bar{v}\, dS \\
&= \langle -\operatorname{div}_{\Gamma_u}[Q], v\rangle_{\Gamma_u},
\end{aligned}$$

by the periodicity of the product $Q\cdot\bar{v}$. In a similar fashion we have

$$\begin{aligned}
(F, \nabla v)_{\Omega} &= \int_{\Omega} (\bar{\epsilon}_0 k_0^2 F)\cdot\nabla\bar{v}\, dV \\
&= \int_{\Omega} \operatorname{div}\left[(\bar{\epsilon}_0 k_0^2 F)\bar{v}\right] dV - \int_{\Omega} \operatorname{div}\left[(\bar{\epsilon}_0 k_0^2 F)\right]\bar{v}\, dV \\
&= \oint_{\partial\Omega} ((\bar{\epsilon}_0 k_0^2 F)\cdot N)\bar{v}\, dS - \int_{\Omega} \operatorname{div}\left[(\bar{\epsilon}_0 k_0^2 F)\right]\bar{v}\, dV
\end{aligned}$$

$$= -\int_{\Omega} \operatorname{div}\left[(\bar{\epsilon}_0 k_0^2 F)\right] \bar{v}\, dV + \int_{\Gamma_u} ((\bar{\epsilon}_0 k_0^2 F)\cdot N)\bar{v}\, dS + \int_{\Gamma_w} ((\bar{\epsilon}_0 k_0^2 F)\cdot N)\bar{v}\, dS$$

$$= \left(-(\bar{\epsilon}_0 k_0^2 F), v\right)_{\Omega} + \langle(\bar{\epsilon}_0 k_0^2 F)\cdot N, v\rangle_{\Gamma_u} + \langle(\bar{\epsilon}_0 k_0^2 F)\cdot N, v\rangle_{\Gamma_w}. \quad \square$$

Using the surface gradient notation

$$\nabla_{\Gamma_m} u = N_m \times (\nabla u \times N_m),$$

and the fact that $\operatorname{curl}[\nabla u] = \operatorname{curl}[\nabla v] = 0$, we find that

$$a(\nabla u, \nabla v) = -k_0^2 (\bar{\epsilon}_0 \nabla u, \nabla v)_{\Omega} \\ - \langle (i\omega\mu_0) T_u[\nabla_{\Gamma_u} u], \nabla_{\Gamma_u} v\rangle_{\Gamma_u} - \langle (i\omega\mu_0) T_w[\nabla_{\Gamma_w} u], \nabla_{\Gamma_w} v\rangle_{\Gamma_w},$$

and

$$L[\nabla v] = \left((\bar{\epsilon}_0 k_0^2 F), \nabla v\right)_{\Omega} + \langle Q, \nabla_{\Gamma_u} v\rangle_{\Gamma_u} + \langle R, \nabla_{\Gamma_w} v\rangle_{\Gamma_w}.$$

Using Lemma A.1 we discover

$$a(\nabla u, \nabla v) = k_0^2 (\operatorname{div}[\bar{\epsilon}_0 \nabla u], v)_{\Omega} \\ - k_0^2 \langle \bar{\epsilon}_0 \nabla u \cdot N_u, v\rangle_{\Gamma_u} - k_0^2 \langle \bar{\epsilon}_0 \nabla u \cdot N_w, v\rangle_{\Gamma_w} \\ + \langle \operatorname{div}_{\Gamma_u}\left[(i\omega\mu_0) T_u[\nabla_{\Gamma_u} u]\right], v\rangle_{\Gamma_u} + \langle \operatorname{div}_{\Gamma_w}\left[(i\omega\mu_0) T_w[\nabla_{\Gamma_w} u]\right], v\rangle_{\Gamma_w},$$

and

$$L[\nabla v] = -\left(\operatorname{div}\left[(\bar{\epsilon}_0 k_0^2 F)\right], v\right)_{\Omega} + \langle(\bar{\epsilon}_0 k_0^2 F)\cdot N_u, v\rangle_{\Gamma_u} + \langle(\bar{\epsilon}_0 k_0^2 F)\cdot N_w, v\rangle_{\Gamma_w} \\ - \langle \operatorname{div}_{\Gamma_u}[Q], v\rangle_{\Gamma_u} - \langle \operatorname{div}_{\Gamma_w}[R], v\rangle_{\Gamma_w}.$$

In this way (cf. Lemma 3.24 of [2]) we see that $a(\nabla u, \nabla v) = L(\nabla v)$ is the weak formulation of the elliptic problem

$$\begin{aligned} k_0^2 \operatorname{div}[\bar{\epsilon}_0 \nabla u] &= -\operatorname{div}\left[(\bar{\epsilon}_0 k_0^2 F)\right], && \Omega, \\ -k_0^2 \bar{\epsilon}_0 \nabla u \cdot N_u + \operatorname{div}_{\Gamma_u}\left[(i\omega\mu_0) T_u[\nabla_{\Gamma_u} u]\right] &= (\bar{\epsilon}_0 k_0^2 F)\cdot N - \operatorname{div}_{\Gamma_u}[Q], && \Gamma_u, \\ -k_0^2 \bar{\epsilon}_0 \nabla u \cdot N_w + \operatorname{div}_{\Gamma_w}\left[(i\omega\mu_0) T_w[\nabla_{\Gamma_w} u]\right] &= (\bar{\epsilon}_0 k_0^2 F)\cdot N - \operatorname{div}_{\Gamma_w}[R], && \Gamma_w, \end{aligned}$$

which has a unique solution $u \in H_0^1(\Omega)$ [2] that satisfies

$$\|u\|_{H_0^1} \le C_e \left\{ \left\|\operatorname{div}\left[(\bar{\epsilon}_0 k_0^2 F)\right]\right\|_{L^2} + \left\|(\bar{\epsilon}_0 k_0^2 F)\cdot N_u\right\|_{H^{-1/2}} + \left\|\operatorname{div}_{\Gamma_u}[Q]\right\|_{H^{-1/2}} \\ + \left\|(\bar{\epsilon}_0 k_0^2 F)\cdot N_w\right\|_{H^{-1/2}} + \left\|\operatorname{div}_{\Gamma_w}[R]\right\|_{H^{-1/2}} \right\},$$

or

$$\|u\|_{H_0^1} \leq C_e \left\{ \left\| (\bar{\epsilon}_0 k_0^2 F) \right\|_{H(\mathrm{div})} + \|Q\|_{H^{-1/2}(\mathrm{div})} + \|R\|_{H^{-1/2}(\mathrm{div})} \right\}. \tag{A.2}$$

A.2. Finding a solution II: $\mathbb{H}$

We continue by considering

$$a(\vec{u}, \vec{v}) + a(\nabla u, \vec{v}) = L[\vec{v}],$$

which, as we have recovered u, we rewrite as

$$a(\vec{u}, \vec{v}) = L[\vec{v}] - a(\nabla u, \vec{v}).$$

Bao & Li [2] (Theorem 3.28) demonstrate that, save for a countable number of frequencies ω (alternatively wavenumbers k_0), there exists a unique solution $\vec{u} \in H(\mathrm{curl})$. Furthermore, from the inf–sup estimate of Bao & Li [2] (Equation (4.83))

$$\sup_{0 \neq \vec{v} \in H(\mathrm{curl})} \frac{a(\vec{u}, \vec{v})}{\|\vec{v}\|_{H(\mathrm{curl})}} \geq \gamma_1 \|\vec{u}\|_{H(\mathrm{curl})}, \quad \forall\, \vec{u} \in H(\mathrm{curl}),$$

for some $\gamma_1 > 0$, we have the inequality

$$a(\vec{u}, \vec{v}) \geq \gamma_1 \|\vec{u}\|_{H(\mathrm{curl})} \|\vec{v}\|_{H(\mathrm{curl})}.$$

Therefore, from the continuity estimate established in the proof of Theorem 3.28 from Bao & Li [2],

$$|a(\vec{u}, \vec{v})| \leq C \|\vec{u}\|_{H(\mathrm{curl})} \|\vec{v}\|_{H(\mathrm{curl})},$$

we can show that

$$\begin{aligned} \gamma_1 \|\vec{u}\|_{H(\mathrm{curl})} \|\vec{v}\|_{H(\mathrm{curl})} &\leq |a(\vec{u}, \vec{v})| \leq |L[\vec{v}]| + |a(\nabla u, \vec{v})| \\ &\leq \left| \left((\bar{\epsilon}_0 k_0^2 F), \vec{v} \right)_\Omega \right| + \left| \langle Q, \vec{v} \rangle_{\Gamma_u} \right| + \left| \langle R, \vec{v} \rangle_{\Gamma_w} \right| \\ &\quad + C \|\nabla u\|_{H(\mathrm{curl})} \|\vec{v}\|_{H(\mathrm{curl})}. \end{aligned}$$

From Lemma 3.15 of [2] we have

$$\left| \langle \vec{u}, \vec{v} \rangle_{\Gamma_m} \right| \leq C \|\vec{u}\|_{H^{-1/2}(\mathrm{div})} \|\vec{v}\|_{H^{-1/2}(\mathrm{div})},$$

so that

$$\gamma_1 \|\vec{u}\|_{H(\mathrm{curl})} \|\vec{v}\|_{H(\mathrm{curl})} \le \left\| (\bar{\epsilon}_0 k_0^2 F) \right\|_{L^2} \|\vec{v}\|_{L^2} + \left(\|Q\|_{H^{-1/2}(\mathrm{div})} + \|R\|_{H^{-1/2}(\mathrm{div})} \right) \|\vec{v}\|_{H^{-1/2}(\mathrm{curl})} + C \|u\|_{H_0^1} \|\vec{v}\|_{H(\mathrm{curl})} .$$

Lemma 3.16 of [2] establishes that, for a given $\gamma_0 > 0$,

$$\|\vec{u}\|_{H^{-1/2}(\mathrm{curl})} \le \gamma_0 \|\vec{u}\|_{H(\mathrm{curl})} ,$$

so that

$$\gamma_1 \|\vec{u}\|_{H(\mathrm{curl})} \|\vec{v}\|_{H(\mathrm{curl})} \le \left\{ \left\| (\bar{\epsilon}_0 k_0^2 F) \right\|_{L^2} + \gamma_0 \left(\|Q\|_{H^{-1/2}(\mathrm{div})} + \|R\|_{H^{-1/2}(\mathrm{div})} \right) + C \|u\|_{H_0^1} \right\} \|\vec{v}\|_{H(\mathrm{curl})} .$$

Clearly, by canceling $\|\vec{v}\|_{H(\mathrm{curl})}$, there exists a constant $C_e > 0$ such that

$$\|\vec{u}\|_{H(\mathrm{curl})} \le C_e \left\{ \left\| (\bar{\epsilon}_0 k_0^2 F) \right\|_{H(\mathrm{div})} + \|Q\|_{H^{-1/2}(\mathrm{div})} + \|R\|_{H^{-1/2}(\mathrm{div})} \right\} .$$

By combining this with (A.2) we find

$$\|E\|_{H(\mathrm{curl})} \le C_e \left\{ \left\| (\bar{\epsilon}_0 k_0^2 F) \right\|_{H(\mathrm{div})} + \|Q\|_{H^{-1/2}(\mathrm{div})} + \|R\|_{H^{-1/2}(\mathrm{div})} \right\} . \tag{A.3}$$

Next, simply applying the L^2 norm to both sides of (6.3b) we obtain

$$\left\| \mathrm{div} \left[\bar{\epsilon}_0 k_0^2 E \right] \right\|_{L^2} = \left\| \mathrm{div} \left[\bar{\epsilon} k_0^2 F \right] \right\|_{L^2} .$$

Since we now know that $E \in H(\mathrm{curl})$ we have $E \in L^2$, and, since $\bar{\epsilon}_0 \in L^\infty$,

$$\left\| \bar{\epsilon}_0 k_0^2 E \right\|_{L^2} \le |\bar{\epsilon}_0|_{L^\infty} k_0^2 \|E\|_{L^2} \le |\bar{\epsilon}_0|_{L^\infty} k_0^2 \|E\|_{H(\mathrm{curl})} .$$

Now, recalling the definition of the $H(\mathrm{div})$ norm,

$$\left\| \bar{\epsilon}_0 k_0^2 E \right\|_{H(\mathrm{div})}^2 = \left\| \bar{\epsilon}_0 k_0^2 E \right\|_{L^2}^2 + \left\| \mathrm{div} \left[\bar{\epsilon}_0 k_0^2 E \right] \right\|_{L^2}^2 \le |\bar{\epsilon}_0|_{L^\infty}^2 k_0^4 \|E\|_{H(\mathrm{curl})}^2 + \left\| \bar{\epsilon}_0 k_0^2 F \right\|_{H(\mathrm{div})}^2 .$$

Estimate (A.3) delivers (6.4).

Appendix B. The Proof of Theorem 7.1

To establish Theorem 7.1 we work by induction on the orders of spatial derivatives, beginning with the x–derivative, moving to the y–derivative, and concluding with the z–derivative. To begin this project we start with the following result on analyticity in x.

Theorem B.1. *Given any integer $q \geq 0$, if $(\omega, \bar{\epsilon}_0) \in \mathcal{P}$, $\bar{\epsilon}_0 \in C_q^\omega(\Omega)$, such that*

$$\left| \frac{\partial_x^r}{r!} \bar{\epsilon}_0 \right|_{C^q} \leq C_{\bar{\epsilon}_0} \frac{\eta^r}{(r+1)^2},$$

for all $r \geq 0$ and some constants $C_{\bar{\epsilon}_0}, \eta > 0$, and $(\bar{\epsilon}_0 k_0^2 F) \in C^\omega(\Omega)$ such that

$$\left\| \frac{\partial_x^r}{r!} \left[\bar{\epsilon}_0 k_0^2 F \right] \right\|_{H(\mathrm{div})} \leq C_F \frac{\eta^r}{(r+1)^2},$$

for all $r \geq 0$ and some constant $C_F > 0$, and $Q \in C^\omega(\Gamma_u)$ and $R \in C^\omega(\Gamma_w)$ satisfying

$$\left\| \frac{\partial_x^r}{r!} Q \right\|_{H^{-1/2}(\mathrm{div})} \leq C_Q \frac{\eta^r}{(r+1)^2},$$
$$\left\| \frac{\partial_x^r}{r!} R \right\|_{H^{-1/2}(\mathrm{div})} \leq C_R \frac{\eta^r}{(r+1)^2},$$

for all $r \geq 0$ and some constants $C_R, C_Q > 0$. Then, there exists a unique solution $E \in C^\omega(\Omega)$ of

$$\begin{aligned}
&\mathcal{L}_0 E = \bar{\epsilon}_0 k_0^2 F, && \text{in } \Omega,\\
&-\mathrm{div}\left[\bar{\epsilon}_0 k_0^2 E\right] = \mathrm{div}\left[\bar{\epsilon}_0 k_0^2 F\right], && \text{in } \Omega,\\
&\mathcal{B}_u E = Q, && \text{at } \Gamma_u,\\
&\mathcal{B}_w E = R, && \text{at } \Gamma_w,\\
&E(x + d_x, y + d_y, z) = e^{i\alpha d_x + i\beta d_y} E(x, y, z), &&
\end{aligned}$$

satisfying

$$\left\| \frac{\partial_x^r}{r!} E \right\|_X \leq \underline{C}_e \frac{\eta^r}{(r+1)^2}, \tag{B.1}$$

for all $r \geq 0$ where

$$\underline{C}_e = \overline{C}(C_F + C_Q + C_R) > 0,$$

and $\overline{C} > 0$ is a constant.

Proof. We prove this theorem by induction on $r \geq 0$. When $r = 0$ we apply Theorem 6.1 to conclude that

$$\|E\|_X \leq C_e \left(\left\| \bar{\epsilon}_0 k_0^2 F \right\|_{H(\mathrm{div})} + \|Q\|_{H^{-1/2}(\mathrm{div})} + \|R\|_{H^{-1/2}(\mathrm{div})} \right) \leq \underline{C}_e.$$

We now assume (B.1) for all $r < \bar{r}$ and seek to establish this estimate at $r = \bar{r}$. Applying the differential operator $\partial_x^r / r!$ to (7.1) delivers

$$\begin{aligned}
&\mathcal{L}_0 \left[\frac{\partial_x^r}{r!} E \right] = \bar{\epsilon}_0 k_0^2 X_r, && \text{in } \Omega, \\
&-\operatorname{div} \left[\bar{\epsilon}_0 k_0^2 \frac{\partial_x^r}{r!} E \right] = \operatorname{div} \left[\bar{\epsilon}_0 k_0^2 X_r \right], && \text{in } \Omega, \\
&\mathcal{B}_u \left[\frac{\partial_x^r}{r!} E \right] = \frac{\partial_x^r}{r!} Q, && \text{at } \Gamma_u, \\
&\mathcal{B}_w \left[\frac{\partial_x^r}{r!} E \right] = \frac{\partial_x^r}{r!} R, && \text{at } \Gamma_w, \\
&\frac{\partial_x^r}{r!} E(x + d_x, y + d_y, z) = e^{i\alpha d_x + i\beta d_y} \frac{\partial_x^r}{r!} E(x, y, z),
\end{aligned}$$

where

$$X_r := \frac{1}{\bar{\epsilon}_0 k_0^2} \left\{ \frac{\partial_x^r}{r!} \left[\bar{\epsilon}_0 k_0^2 F \right] + \left[\mathcal{L}_0, \frac{\partial_x^r}{r!} \right] E \right\},$$

and $[A, B] := AB - BA$ is the commutator. With this we can express

$$AB = BA + [A, B] = BA - [B, A].$$

From Theorem 6.1 we have

$$\left\| \frac{\partial_x^{\bar{r}}}{\bar{r}!} E \right\|_X \leq C_e \left\{ \left\| \bar{\epsilon}_0 k_0^2 X_{\bar{r}} \right\|_{H(\mathrm{div})} + \left\| \frac{\partial_x^{\bar{r}}}{\bar{r}!} Q \right\|_{H^{-1/2}(\mathrm{div})} + \left\| \frac{\partial_x^{\bar{r}}}{\bar{r}!} R \right\|_{H^{-1/2}(\mathrm{div})} \right\},$$

while Lemma B.1 gives

$$\left\| \frac{\partial_x^{\bar{r}}}{\bar{r}!} E \right\|_X \leq C_e \left\{ C_F \frac{\eta^{\bar{r}}}{(\bar{r}+1)^2} + \underline{C}_e \tilde{C} \frac{\eta^{\bar{r}-1}}{(\bar{r}+1)^2} + C_Q \frac{\eta^{\bar{r}}}{(\bar{r}+1)^2} + C_R \frac{\eta^{\bar{r}}}{(\bar{r}+1)^2} \right\}.$$

We are done provided that

$$\underline{C}_e \geq 2C_e(C_F + C_Q + C_R), \quad \eta \geq 2C_e \tilde{C}. \quad \square$$

Lemma B.1. *Given any integer $q \geq 0$, if $(\omega, \bar{\epsilon}_0) \in \mathcal{P}$, $\bar{\epsilon}_0 \in C_q^{\omega}(\Omega)$, such that*

$$\left| \frac{\partial_x^r}{r!} \bar{\epsilon}_0 \right|_{C^q} \leq C_{\bar{\epsilon}_0} \frac{\eta^r}{(r+1)^2},$$

for all $r \geq 0$ and some constants $C_{\bar{\epsilon}_0}, \eta > 0$, and $(\bar{\epsilon}_0 k_0^2 F) \in C^{\omega}(\Omega)$ such that

$$\left\| \frac{\partial_x^r}{r!} \left[\bar{\epsilon}_0 k_0^2 F \right] \right\|_{H(\mathrm{div})} \leq C_F \frac{\eta^r}{(r+1)^2},$$

for all $r \geq 0$ and some constant $C_F > 0$. Assuming

$$\left\| \frac{\partial_x^r}{r!} E \right\|_X \leq \underline{C}_e \frac{\eta^r}{(r+1)^2},$$

for all $r < \bar{r}$ then

$$\left\| \bar{\epsilon}_0 k_0^2 X_{\bar{r}} \right\|_{H(\mathrm{div})} \leq C_F \frac{\eta^{\bar{r}}}{(\bar{r}+1)^2} + \underline{C}_e \tilde{C} \frac{\eta^{\bar{r}-1}}{(\bar{r}+1)^2}.$$

Proof. We note that

$$X_{\bar{r}} := \frac{1}{\bar{\epsilon}_0 k_0^2} \left\{ \frac{\partial_x^{\bar{r}}}{\bar{r}!} \left[\bar{\epsilon}_0 k_0^2 F \right] + \left[\mathcal{L}_0, \frac{\partial_x^{\bar{r}}}{\bar{r}!} \right] E \right\},$$

so that,

$$\left\| \bar{\epsilon}_0 k_0^2 X_{\bar{r}} \right\|_{H(\mathrm{div})} \leq \left\| \frac{\partial_x^{\bar{r}}}{\bar{r}!} \left[\bar{\epsilon}_0 k_0^2 F \right] \right\|_{H(\mathrm{div})} + \left\| \left[\mathcal{L}_0, \frac{\partial_x^{\bar{r}}}{\bar{r}!} \right] E \right\|_{H(\mathrm{div})}.$$

The first term can be bounded above by

$$C_F \frac{\eta^{\bar{r}}}{(\bar{r}+1)^2},$$

by assumption. Regarding the second term, we have, for any $r \geq 0$,

$$\begin{aligned}
\left[\mathcal{L}_0, \frac{\partial_x^r}{r!} \right] E &= \mathcal{L}_0 \left[\frac{\partial_x^r}{r!} E \right] - \frac{\partial_x^r}{r!} \left[\mathcal{L}_0[E] \right] \\
&= \mathrm{curl} \left[\mathrm{curl} \left[\frac{\partial_x^r}{r!} E \right] \right] - \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r}{r!} E - \frac{\partial_x^r}{r!} \left[\mathrm{curl}\left[\mathrm{curl}\left[E\right]\right] - \bar{\epsilon}_0 k_0^2 E \right] \\
&= -\bar{\epsilon}_0 k_0^2 \frac{\partial_x^r}{r!} E + \frac{\partial_x^r}{r!} \left[\bar{\epsilon}_0 k_0^2 E \right].
\end{aligned}$$

The Leibniz rule tells us that

$$\left[\mathcal{L}_0, \frac{\partial_x^r}{r!}\right] E = -\bar{\epsilon}_0 k_0^2 \frac{\partial_x^r}{r!} E + k_0^2 \sum_{j=0}^{r} \left(\frac{\partial_x^{r-j}}{(r-j)!} \bar{\epsilon}_0 \right) \frac{\partial_x^j}{j!} E$$
$$= k_0^2 \sum_{j=0}^{r-1} \left(\frac{\partial_x^{r-j}}{(r-j)!} \bar{\epsilon}_0 \right) \frac{\partial_x^j}{j!} E.$$

Setting $r = \bar{r}$ we can estimate

$$\begin{aligned}
\left\| \left[\mathcal{L}_0, \frac{\partial_x^{\bar{r}}}{\bar{r}!}\right] E \right\|_{H(\mathrm{div})} &\le k_0^2 \sum_{j=0}^{\bar{r}-1} \left\| \left(\frac{\partial_x^{\bar{r}-j}}{(\bar{r}-j)!} \bar{\epsilon}_0 \right) \frac{\partial_x^j}{j!} E \right\|_{H(\mathrm{div})} \\
&\le k_0^2 M \sum_{j=0}^{\bar{r}-1} \left| \frac{\partial_x^{\bar{r}-j}}{(\bar{r}-j)!} \bar{\epsilon}_0 \right|_{C^1} \left\| \frac{\partial_x^j}{j!} E \right\|_{H(\mathrm{div})} \\
&\le k_0^2 M \sum_{j=0}^{\bar{r}-1} \frac{1}{\bar{r}-j} \left| \frac{\partial_x^{\bar{r}-j-1}}{(\bar{r}-j-1)!} \bar{\epsilon}_0 \right|_{C^2} \left\| \frac{\partial_x^j}{j!} [E] \right\|_{H(\mathrm{div})} \\
&\le k_0^2 M \sum_{j=0}^{\bar{r}-1} C_{\bar{\epsilon}_0} \frac{\eta^{\bar{r}-j-1}}{(\bar{r}-j-1+1)^2} \underline{C}_e \frac{\eta^j}{(j+1)^2} \\
&\le k_0^2 M C_{\bar{\epsilon}_0} \underline{C}_e S \frac{\eta^{\bar{r}-1}}{(\bar{r}+1)^2},
\end{aligned}$$

and we choose

$$\tilde{C} \ge k_0^2 M C_{\bar{\epsilon}_0} S. \quad \square$$

The next step is to prove analyticity jointly in x and y.

Theorem B.2. *Given any integer $q \ge 0$, if $(\omega, \bar{\epsilon}_0) \in \mathcal{P}$, $\bar{\epsilon}_0 \in C_q^{\omega}(\Omega)$, such that*

$$\left| \frac{\partial_x^r \partial_y^t}{(r+t)!} \bar{\epsilon}_0 \right|_{C^q} \le C_{\bar{\epsilon}_0} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

for all $r, t \ge 0$ and some constants $C_{\bar{\epsilon}_0}, \eta, \theta > 0$, and $(\bar{\epsilon}_0 k_0^2 F) \in C^{\omega}(\Omega)$ such that

$$\left\| \frac{\partial_x^r \partial_y^t}{(r+t)!} \left[\bar{\epsilon}_0 k_0^2 F \right] \right\|_{H(\mathrm{div})} \le C_F \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

for all $r, t \ge 0$ and some constant $C_F > 0$, and $Q \in C^{\omega}(\Gamma_u)$ and $R \in C^{\omega}(\Gamma_w)$ satisfying

$$\left\| \frac{\partial_x^r \partial_y^t}{(r+t)!} Q \right\|_{H^{-1/2}(\mathrm{div})} \leq C_Q \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

$$\left\| \frac{\partial_x^r \partial_y^t}{(r+t)!} R \right\|_{H^{-1/2}(\mathrm{div})} \leq C_R \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

for all $r, t \geq 0$ and some constants $C_R, C_Q > 0$. Then, there exists a unique solution $E \in C^\omega(\Omega)$ of

$$\begin{aligned}
&\mathcal{L}_0 E = \bar{\epsilon}_0 k_0^2 F, && \textit{in } \Omega, \\
&-\operatorname{div}\left[\bar{\epsilon}_0 k_0^2 E\right] = \operatorname{div}\left[\bar{\epsilon}_0 k_0^2 F\right], && \textit{in } \Omega, \\
&\mathcal{B}_u E = Q, && \textit{at } \Gamma_u, \\
&\mathcal{B}_w E = R, && \textit{at } \Gamma_w, \\
&E(x+d_x, y+d_y, z) = e^{i\alpha d_x + i\beta d_y} E(x,y,z),
\end{aligned}$$

satisfying

$$\left\| \frac{\partial_x^r \partial_y^t}{(r+t)!} E \right\|_X \leq \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2}, \tag{B.2}$$

for all $r, t \geq 0$ where

$$\underline{C}_e = \overline{C}(C_F + C_Q + C_R) > 0,$$

and $\overline{C} > 0$ is a constant.

Proof. We prove this theorem by induction on $t \geq 0$. When $t = 0$ we apply Theorem B.1 to establish (B.2). We now assume (B.2) for all $r \geq 0$ and $t < \bar{t}$ and seek to establish this estimate at $t = \bar{t}$. Applying the differential operator $(\partial_x^r \partial_y^t)/(r+t)!$ to (7.1) delivers

$$\begin{aligned}
&\mathcal{L}_0 \left[\frac{\partial_x^r \partial_y^t}{(r+t)!} E \right] = \bar{\epsilon}_0 k_0^2 Y_{r,t}, && \text{in } \Omega, \\
&-\operatorname{div}\left[\bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t}{(r+t)!} E\right] = \operatorname{div}\left[\bar{\epsilon}_0 k_0^2 Y_{r,t}\right], && \text{in } \Omega, \\
&\mathcal{B}_u \left[\frac{\partial_x^r \partial_y^t}{(r+t)!} E \right] = \frac{\partial_x^r \partial_y^t}{(r+t)!} Q, && \text{at } \Gamma_u, \\
&\mathcal{B}_w \left[\frac{\partial_x^r \partial_y^t}{(r+t)!} E \right] = \frac{\partial_x^r \partial_y^t}{(r+t)!} R, && \text{at } \Gamma_w, \\
&\frac{\partial_x^r \partial_y^t}{(r+t)!} E(x+d_x, y+d_y, z) = e^{i\alpha d_x + i\beta d_y} \frac{\partial_x^r \partial_y^t}{(r+t)!} E(x,y,z),
\end{aligned}$$

where

$$Y_{r,t} := \frac{1}{\bar{\epsilon}_0 k_0^2} \left\{ \frac{\partial_x^r \partial_y^t}{(r+t)!} \left[\bar{\epsilon}_0 k_0^2 F \right] + \left[\mathcal{L}_0, \frac{\partial_x^r \partial_y^t}{(r+t)!} \right] E \right\}.$$

From Theorem 6.1 we have

$$\left\| \frac{\partial_x^r \partial_y^{\bar{t}}}{(r+\bar{t})!} E \right\|_X \leq C_e \left\{ \left\| \bar{\epsilon}_0 k_0^2 Y_{r,\bar{t}} \right\|_{H(\mathrm{div})} + \left\| \frac{\partial_x^r \partial_y^{\bar{t}}}{(r+\bar{t})!} Q \right\|_{H^{-1/2}(\mathrm{div})} + \left\| \frac{\partial_x^r \partial_y^{\bar{t}}}{(r+\bar{t})!} R \right\|_{H^{-1/2}(\mathrm{div})} \right\},$$

while Lemma B.2 gives

$$\left\| \frac{\partial_x^r \partial_x^{\bar{t}}}{\bar{r}!} E \right\|_X \leq C_e \left\{ C_F \frac{\eta^r}{(r+1)^2} \frac{\theta^{\bar{t}}}{(\bar{t}+1)^2} + \underline{C}_e \tilde{C} \frac{\eta^r}{(r+1)^2} \frac{\theta^{\bar{t}-1}}{(\bar{t}+1)^2} \right.$$
$$\left. + C_Q \frac{\eta^r}{(r+1)^2} \frac{\theta^{\bar{t}}}{(\bar{t}+1)^2} + C_R \frac{\eta^r}{(r+1)^2} \frac{\theta^{\bar{t}}}{(\bar{t}+1)^2} \right\}.$$

We are done provided that

$$\underline{C}_e \geq 2C_e(C_F + C_Q + C_R), \quad \theta \geq 2C_e \tilde{C}. \quad \square$$

Lemma B.2. *Given any integer $q \geq 0$, if $(\omega, \bar{\epsilon}_0) \in \mathcal{P}$, $\bar{\epsilon}_0 \in C_q^\omega(\Omega)$, such that*

$$\left| \frac{\partial_x^r \partial_y^t}{(r+t)!} \bar{\epsilon}_0 \right|_{C^q} \leq C_{\bar{\epsilon}_0} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

for all $r, t \geq 0$ and some constants $C_{\bar{\epsilon}_0}, \eta, \theta > 0$, and $(\bar{\epsilon}_0 k_0^2 F) \in C^\omega(\Omega)$ such that

$$\left\| \frac{\partial_x^r \partial_y^t}{(r+t)!} \left[\bar{\epsilon}_0 k_0^2 F \right] \right\|_{H(\mathrm{div})} \leq C_F \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

for all $r, t \geq 0$ and some constant $C_F > 0$. Assuming

$$\left\| \frac{\partial_x^r \partial_y^t}{(r+t)!} E \right\|_X \leq \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2},$$

for all $r \geq 0$ and $t < \bar{t}$ then

$$\left\| \bar{\epsilon}_0 k_0^2 Y_{r,\bar{t}} \right\|_{H(\mathrm{div})} \leq C_F \frac{\eta^r}{(r+1)^2} \frac{\theta^{\bar{t}}}{(\bar{t}+1)^2} + \underline{C}_e \tilde{C} \frac{\eta^r}{(r+1)^2} \frac{\theta^{\bar{t}-1}}{(\bar{t}+1)^2}.$$

Proof. Very similar to that of Lemma B.1 and therefore omitted. $\square$

We are now in a position to prove Theorem 7.1. Once again, we work by induction, this time on s, the order of the z derivative acting on E. As before, at $s = 0$ we use Theorem B.2 to establish (7.2). We now assume (7.2),

$$\left\| \frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!} E \right\|_X \leq \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^s}{(s+1)^2}, \tag{B.3}$$

for all $r, t \geq 0$ and all $s < \bar{s}$. We now examine this estimate at order $s = \bar{s}$. From the definition of the X–norm we have

$$\begin{aligned}
\left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right\|_X^2 &= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right\|_{H(\text{curl})}^2 + \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right\|_{H(\text{div})}^2 \\
&= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right\|_{L^2}^2 + \left\| \text{curl} \left[\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right] \right\|_{L^2}^2 \\
&\quad + \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right\|_{L^2}^2 + \left\| \text{div} \left[\bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right] \right\|_{L^2}^2 \\
&= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right\|_{L^2}^2 + \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \text{curl}\,[E] \right\|_{L^2}^2 \\
&\quad + \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right\|_{L^2}^2 \\
&\quad + \left\| \nabla \bar{\epsilon}_0 \cdot \left(k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right) \right\|_{L^2}^2 + \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \text{div}\,[E] \right\|_{L^2}^2 \\
&\leq \left\{ 1 + |\bar{\epsilon}_0|_{C^1}^2 k_0^4 \right\} \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E \right\|_{L^2}^2 \\
&\quad + \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \text{curl}\,[E] \right\|_{L^2}^2 + \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \text{div}\,[E] \right\|_{L^2}^2 \\
&=: \left\{ 1 + |\bar{\epsilon}_0|_{C^1}^2 k_0^4 \right\} I_1 + I_2 + I_3.
\end{aligned}$$

The estimate of I_1: Using the notation $E = (E^x, E^y, E^z)^T$ we obtain

$$\begin{aligned}
I_1 &= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E^x \right\|_{L^2}^2 + \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E^y \right\|_{L^2}^2 + \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} E^z \right\|_{L^2}^2 \\
&= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z E^x \right\|_{L^2}^2 + \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z E^y \right\|_{L^2}^2 + \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z E^z \right\|_{L^2}^2
\end{aligned}$$

$$:= Z_1 + Z_2 + Z_3.$$

To estimate Z_1 and Z_2, we notice that

$$\begin{aligned}\operatorname{curl}[E] &= \left(\partial_y E^z - \partial_z E^y, \partial_z E^x - \partial_x E^z, \partial_x E^y - \partial_y E^x\right) \\ &= \left(\partial_y E^z, -\partial_x E^z, \partial_x E^y - \partial_y E^x\right) + \left(-\partial_z E^y, \partial_z E^x, 0\right),\end{aligned}$$

we obtain

$$\left(-\partial_z E^y, \partial_z E^x, 0\right) = \operatorname{curl}[E] - \left(\partial_y E^z, -\partial_x E^z, \partial_x E^y - \partial_y E^x\right).$$

Therefore,

$$\begin{aligned}\{Z_1 + Z_2\}^{1/2} &= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} (-\partial_z E^y, \partial_z E^x, 0)^T \right\|_{L^2} \\ &\le \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \operatorname{curl}[E] \right\|_{L^2} \\ &\quad + \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} (\partial_y E^z, -\partial_x E^z, \partial_x E^y - \partial_y E^x) \right\|_{L^2} \\ &\le \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \operatorname{curl}[E] \right\|_{L^2} \\ &\quad + \left\| \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} E^z \right\|_{L^2} + \left\| \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} E^z \right\|_{L^2} \\ &\quad + \left\| \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} E^y \right\|_{L^2} + \left\| \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} E^x \right\|_{L^2}.\end{aligned}$$

Using the inequality $(r+t+\bar{s}-1)! \le (r+t+\bar{s})!$ and the inductive hypothesis, (B.3), we have

$$\begin{aligned}\{Z_1 + Z_2\}^{1/2} &\le \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}-1}}{(\bar{s}-1+1)^2} \\ &\quad + 2\underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^{t+1}}{(t+1+1)^2} \frac{\zeta^{\bar{s}-1}}{(\bar{s}-1+1)^2} \\ &\quad + 2\underline{C}_e \frac{\eta^{r+1}}{(r+1+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}-1}}{(\bar{s}-1+1)^2} \\ &= \underline{C}_e \left\{ \frac{1}{\zeta} \frac{(\bar{s}+1)^2}{\bar{s}^2} + 2\theta \frac{(t+1)^2}{(t+2)^2} \frac{1}{\zeta} \frac{(\bar{s}+1)^2}{\bar{s}^2} + 2\eta \frac{(r+1)^2}{(r+2)^2} \frac{1}{\zeta} \frac{(\bar{s}+1)^2}{\bar{s}^2} \right\} \\ &\quad \times \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2}.\end{aligned}$$

With this and using the inequalities

$$\frac{(\bar{s}+1)^2}{\bar{s}^2} \le 4, \qquad \bar{s} \ge 1,$$
$$\frac{(t+1)^2}{(t+2)^2} < 1, \qquad t \ge 0,$$

we obtain

$$\begin{aligned}\{Z_1 + Z_2\}^{1/2} &\le \underline{C}_e \left\{ \frac{4 + 8\theta + 8\eta}{\zeta} \right\} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2}, \\ &\le \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2},\end{aligned}$$

where $\zeta \ge (4 + 8\theta + 8\eta)$. Next, we estimate Z_3 by using the fact that

$$\partial_z E^z = \operatorname{div}[E] - \partial_x E^x - \partial_y E^y.$$

Therefore,

$$\begin{aligned}Z_3^{1/2} &= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} (0, 0, \partial_z E^z)^T \right\|_{L^2} \\ &\le \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \operatorname{div}[E] \right\|_{L^2} + \left\| \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} E^x \right\|_{L^2} + \left\| \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} E^y \right\|_{L^2} \\ &=: W_1 + W_2 + W_3.\end{aligned}$$

While W_2 and W_3 can be directly controlled by using the inductive hypothesis (B.3), W_1 can be estimated as follows:

$$\begin{aligned}W_1 &= \left\| \frac{\bar{\epsilon}_0 k_0^2}{\bar{\epsilon}_0 k_0^2} \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \operatorname{div}[E] \right\|_{L^2} \\ &\le \frac{1}{\min|\bar{\epsilon}_0|\, k_0^2} \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \operatorname{div}[E] \right\|_{L^2} \\ &\le \frac{1}{\min|\bar{\epsilon}_0|\, k_0^2} \frac{(r+t+\bar{s}-1)!}{(r+t+\bar{s})!} \frac{(\bar{s}+1)^2}{\bar{s}^2 \zeta} \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2} \\ &\le \frac{1}{\min|\bar{\epsilon}_0|\, k_0^2} \frac{4}{\zeta} \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2} \\ &\le \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2},\end{aligned}$$

where $\zeta > 4/(\min|\bar{\epsilon}_0|\, k_0^2)$, which completes our estimation of I_1.

The estimate of I_2: We begin with the fact that

$$\operatorname{curl}\left[\partial_z E\right] = \left(\partial_y \partial_z E^z - \partial_z^2 E^y, \partial_z^2 E^x - \partial_x \partial_z E^z, \partial_x \partial_z E^y - \partial_y \partial_z E^x\right).$$

So,

$$\begin{aligned}
I_2^{1/2} &= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z \operatorname{curl}[E] \right\|_{L^2} \\
&= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \operatorname{curl}\left[\partial_z E\right] \right\|_{L^2} \\
&\leq \left\| \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z E^z \right\|_{L^2} + \left\| \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z E^z \right\|_{L^2} \\
&\quad + \left\| \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z E^y \right\|_{L^2} + \left\| \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z E^x \right\|_{L^2} \\
&\quad + \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z^2 E^y \right\|_{L^2} + \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z^2 E^x \right\|_{L^2} \\
&=: X_1 + X_2 + X_3 + X_4 + X_5 + X_6.
\end{aligned}$$

The terms X_1 and X_2 are similar to Z_3 while the terms X_3 and X_4 can be controlled by using the same techniques as shown in bounding Z_1 and Z_2. Since X_5 and X_6 are quite similar to estimate, we fix on the term X_5.

To begin this estimation, we can write the governing equation, $\mathcal{L}_0 E = \bar{\epsilon}_0 k_0^2 F$, as

$$-\Delta E + \nabla \operatorname{div}[E] - k_0^2 \bar{\epsilon}_0 E = \bar{\epsilon}_0 k_0^2 F,$$

which implies that

$$\partial_z^2 E^y = -\bar{\epsilon}_0 k_0^2 F^y - k_0^2 \bar{\epsilon}_0 E^y - \partial_x^2 E^y + \partial_x \partial_y E^x + \partial_y \partial_z E^z.$$

Therefore,

$$\begin{aligned}
X_5 &= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z^2 E^y \right\|_{L^2} \\
&\leq \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} (\bar{\epsilon}_0 k_0^2 F^y) \right\|_{L^2} + \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} (\bar{\epsilon}_0 k_0^2 E^y) \right\|_{L^2} + \left\| \frac{\partial_x^{r+2} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} E^y \right\|_{L^2} \\
&\quad + \left\| \frac{\partial_x^{r+1} \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} E^x \right\|_{L^2} + \left\| \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z E^z \right\|_{L^2} \\
&:= P_1 + P_2 + P_3 + P_4 + P_5.
\end{aligned}$$

It is clear that P_1 can be estimated by using the analyticity assumption on $\bar{\epsilon}_0 k_0^2 F$, and the terms P_3 and P_4 can be controlled by the inductive hypothesis (B.3). On the other hand, the term P_5 can be estimated by using the same techniques we applied to Z_3. It remains to bound P_2. Using Leibniz's rule, for any $r, t, s \geq 0$, we have

$$\frac{\partial_x^r \partial_y^t \partial_z^s}{(r+t+s)!}(\bar{\epsilon}_0 k_0^2 E) = \frac{r!t!s!}{(r+t+s)!} \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{s} \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{s-\ell}}{(s-\ell)^2} \bar{\epsilon}_0 k_0^2 \right) \times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} E \right).$$

Using the triangle inequality, the fact that $r!t!(\bar{s}-1)! \leq (r+t+\bar{s})!$, the hypotheses on $\bar{\epsilon}_0$, and the inductive hypothesis (B.3) we obtain

$$\begin{aligned}
P_2 &= \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!}(\bar{\epsilon}_0 k_0^2 E) \right\|_{L^2} \\
&\leq \frac{r!t!(\bar{s}-1)!}{(r+t+\bar{s})!} \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{\bar{s}-1} \left\| \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{\bar{s}-1-\ell}}{(\bar{s}-1-\ell)^2} \bar{\epsilon}_0 k_0^2 \right) \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} E \right) \right\|_{L^2} \\
&\leq \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{\bar{s}-1} \left| \frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{\bar{s}-1-\ell}}{(\bar{s}-1-\ell)!} \bar{\epsilon}_0 k_0^2 \right|_{L^\infty} \left\| \frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} E \right\|_{L^2} \\
&\leq \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{\bar{s}-1} C_{\bar{\epsilon}_0} \frac{\eta^{r-j}}{(r-j+1)^2} \frac{\theta^{t-k}}{(t-k+1)^2} \frac{\zeta^{\bar{s}-1-\ell}}{(\bar{s}-1-\ell+1)^2} \\
&\quad \times \underline{C}_e \frac{\eta^j}{(j+1)^2} \frac{\theta^k}{(k+1)^2} \frac{\zeta^\ell}{(\ell+1)^2} \\
&\leq \underline{C}_e C_{\bar{\epsilon}_0} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}-1}}{(\bar{s}-1+1)^2} \\
&\quad \times \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{\bar{s}-1} \frac{(r+1)^2(t+1)^2}{(r-j+1)^2(j+1)^2(t-k+1)^2(k+1)^2} \frac{(\bar{s}-1+1)^2}{(\bar{s}-1-\ell+1)^2(\ell+1)^2} \\
&\leq \underline{C}_e C_{\bar{\epsilon}_0} S^3 \frac{(\bar{s}+1)^2}{\bar{s}^2 \zeta} \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2} \\
&\leq \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2},
\end{aligned}$$

for some $\zeta \geq 4 C_{\bar{\epsilon}_0} S^3$.

The estimate of I_3: We conclude with the computation

$$I_3^{1/2} = \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \operatorname{div}[E] \right\|_{L^2}$$

$$
\begin{aligned}
&\le \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_x E^x \right\|_{L^2} + \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_y E^y \right\|_{L^2} \\
&\quad + \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_z E^z \right\|_{L^2} \\
&\le \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z E^x \right\|_{L^2} + \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z E^y \right\|_{L^2} \\
&\quad + \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_z E^z \right\|_{L^2} \\
&=: L_1 + L_2 + L_3.
\end{aligned}
$$

The terms L_1 and L_2 are similar to Z_1 and Z_2, and we estimate them in the same fashion. We estimate L_3 in the following way. We begin by noting that $\operatorname{div}\left[\bar{\epsilon}_0 k_0^2 E\right] = -\operatorname{div}\left[\bar{\epsilon}_0 k_0^2 F\right]$ implies that

$$
\partial_z(\bar{\epsilon}_0 k_0^2 E^z) = -\operatorname{div}\left[\bar{\epsilon}_0 k_0^2 F\right] - \partial_x(\bar{\epsilon}_0 k_0^2 E^x) - \partial_y(\bar{\epsilon}_0 k_0^2 E^y),
$$

which also gives

$$
\bar{\epsilon}_0 k_0^2 \partial_z E^z = -\operatorname{div}\left[\bar{\epsilon}_0 k_0^2 F\right] - \partial_x(\bar{\epsilon}_0 k_0^2 E^x) - \partial_y(\bar{\epsilon}_0 k_0^2 E^y) - \partial_z(\bar{\epsilon}_0 k_0^2) E^z. \tag{B.4}
$$

In anticipation of our future estimation of higher derivatives of this term we use the commutator notation to express

$$
\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}\left[(\bar{\epsilon}_0 k_0^2)\partial_z E^z\right] = \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}\left[\partial_z E^z\right] + \left[\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}, \bar{\epsilon}_0 k_0^2\right] \partial_z E^z,
$$

where, by Leibniz's Rule,

$$
\begin{aligned}
\left[\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}, \bar{\epsilon}_0 k_0^2\right] \partial_z E^z &= \frac{r!t!\bar{s}!}{(r+t+\bar{s}!)} \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{\bar{s}} \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{\bar{s}-\ell}}{(\bar{s}-\ell)!} \left[\bar{\epsilon}_0 k_0^2\right] \right) \\
&\quad \times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} \left[\partial_z E^z\right] \right) - (\bar{\epsilon}_0 k_0^2) \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_z E^z \\
&= \frac{r!t!\bar{s}!}{(r+t+\bar{s}!)} \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{\bar{s}-1} \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{\bar{s}-\ell}}{(\bar{s}-\ell)!} \left[\bar{\epsilon}_0 k_0^2\right] \right) \\
&\quad \times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} \left[\partial_z E^z\right] \right)
\end{aligned}
$$

$$
+ \frac{r!t!\bar{s}!}{(r+t+\bar{s}!)} \sum_{j=0}^{r} \sum_{k=0}^{t} \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \left[\bar{\epsilon}_0 k_0^2 \right] \right)
\times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^{\bar{s}}}{\bar{s}!} \left[\partial_z E^z \right] \right) - (\bar{\epsilon}_0 k_0^2) \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_z E^z.
$$

Continuing

$$
\left[\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}, \bar{\epsilon}_0 k_0^2 \right] \partial_z E^z = \frac{r!t!\bar{s}!}{(r+t+\bar{s}!)} \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{\bar{s}-1} \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{\bar{s}-\ell}}{(\bar{s}-\ell)!} \left[\bar{\epsilon}_0 k_0^2 \right] \right)
\times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} \left[\partial_z E^z \right] \right)
+ \frac{r!t!\bar{s}!}{(r+t+\bar{s}!)} \sum_{j=0}^{r} \sum_{k=0}^{t-1} \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \left[\bar{\epsilon}_0 k_0^2 \right] \right)
\times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^{\bar{s}}}{\bar{s}!} \left[\partial_z E^z \right] \right)
+ \frac{r!t!\bar{s}!}{(r+t+\bar{s}!)} \sum_{j=0}^{r} \left(\frac{\partial_x^{r-j}}{(r-j)!} \left[\bar{\epsilon}_0 k_0^2 \right] \right)
\times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^t}{t!} \frac{\partial_z^{\bar{s}}}{\bar{s}!} \left[\partial_z E^z \right] \right) - (\bar{\epsilon}_0 k_0^2) \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_z E^z,
$$

and finally,

$$
\left[\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}, \bar{\epsilon}_0 k_0^2 \right] \partial_z E^z = \frac{r!t!\bar{s}!}{(r+t+\bar{s}!)} \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{\bar{s}-1} \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{\bar{s}-\ell}}{(\bar{s}-\ell)!} \left[\bar{\epsilon}_0 k_0^2 \right] \right)
\times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} \left[\partial_z E^z \right] \right)
+ \frac{r!t!\bar{s}!}{(r+t+\bar{s}!)} \sum_{j=0}^{r} \sum_{k=0}^{t-1} \left(\frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \left[\bar{\epsilon}_0 k_0^2 \right] \right)
\times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^{\bar{s}}}{\bar{s}!} \left[\partial_z E^z \right] \right)
+ \frac{r!t!\bar{s}!}{(r+t+\bar{s}!)} \sum_{j=0}^{r-1} \left(\frac{\partial_x^{r-j}}{(r-j)!} \left[\bar{\epsilon}_0 k_0^2 \right] \right)
$$

$$\times \left(\frac{\partial_x^j}{j!} \frac{\partial_y^t}{t!} \frac{\partial_z^{\bar{s}}}{\bar{s}!} \left[\partial_z E^z \right] \right).$$

With this commutator notation, (B.4) implies that

$$\begin{aligned}
\bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_z E^z &= \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \left[(\epsilon_0 k_0^2) \partial_z E^z \right] - \left[\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}, \bar{\epsilon}_0 k_0^2 \right] \partial_z E^z \\
&= -\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \operatorname{div} \left[\bar{\epsilon}_0 k_0^2 F \right] - \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \left[\bar{\epsilon}_0 k_0^2 E^x \right] \\
&\quad - \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \left[\bar{\epsilon}_0 k_0^2 E^y \right] - \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \left[\partial_z (\bar{\epsilon}_0 k_0^2) E^z \right] \\
&\quad - \left[\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}, \bar{\epsilon}_0 k_0^2 \right] \partial_z E^z.
\end{aligned}$$

Proceeding, we find

$$\begin{aligned}
\bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_z E^z &= -\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \operatorname{div} \left[\bar{\epsilon}_0 k_0^2 F \right] - \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z \left[\bar{\epsilon}_0 k_0^2 E^x \right] \\
&\quad - \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z \left[\bar{\epsilon}_0 k_0^2 E^y \right] - \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \partial_z \left[\partial_z (\bar{\epsilon}_0 k_0^2) E^z \right] \\
&\quad - \left[\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}, \bar{\epsilon}_0 k_0^2 \right] \partial_z E^z.
\end{aligned}$$

Using the product rule we find

$$\begin{aligned}
\bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_z E^z &= -\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \operatorname{div} \left[\bar{\epsilon}_0 k_0^2 F \right] - \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\partial_z (\bar{\epsilon}_0 k_0^2) E^x \right] \\
&\quad - \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\bar{\epsilon}_0 k_0^2 \partial_z E^x \right] - \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\partial_z (\bar{\epsilon}_0 k_0^2) E^y \right] \\
&\quad - \frac{\partial_x^r \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\bar{\epsilon}_0 k_0^2 \partial_z E^y \right] - \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\partial_z^2 (\bar{\epsilon}_0 k_0^2) E^z \right] \\
&\quad - \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\partial_z (\bar{\epsilon}_0 k_0^2) \partial_z E^z \right] \\
&\quad - \left[\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}, \bar{\epsilon}_0 k_0^2 \right] \partial_z E^z.
\end{aligned}$$

With this, we can estimate L_3 as follows:

$$
\begin{aligned}
L_3 &= \left\| \bar{\epsilon}_0 k_0^2 \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \partial_z E^z \right\|_{L^2} \\
&\leq \left\| \frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!} \operatorname{div} \left[\bar{\epsilon}_0 k_0^2 F \right] \right\|_{L^2} \\
&\quad + \left\| \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\partial_z (\bar{\epsilon}_0 k_0^2) E^x \right] \right\|_{L^2} + \left\| \frac{\partial_x^{r+1} \partial_y^t \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\bar{\epsilon}_0 k_0^2 \partial_z E^x \right] \right\|_{L^2} \\
&\quad + \left\| \frac{\partial_x^{r} \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\partial_z (\bar{\epsilon}_0 k_0^2) E^y \right] \right\|_{L^2} + \left\| \frac{\partial_x^{r} \partial_y^{t+1} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[(\bar{\epsilon}_0 k_0^2) \partial_z E^y \right] \right\|_{L^2} \\
&\quad + \left\| \frac{\partial_x^{r} \partial_y^{t} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\partial_z^2 (\bar{\epsilon}_0 k_0^2) E^z \right] \right\|_{L^2} + \left\| \frac{\partial_x^{r} \partial_y^{t} \partial_z^{\bar{s}-1}}{(r+t+\bar{s})!} \left[\partial_z (\bar{\epsilon}_0 k_0^2) \partial_z E^z \right] \right\|_{L^2} \\
&\quad + \left\| \left[\frac{\partial_x^r \partial_y^t \partial_z^{\bar{s}}}{(r+t+\bar{s})!}, \bar{\epsilon}_0 k_0^2 \right] \partial_z E^z \right\|_{L^2} \\
&=: U_1 + U_2 + U_3 + U_4 + U_5 + U_6 + U_7 + U_8.
\end{aligned}
$$

It is clear that U_1 can be bounded by using the analyticity assumption on $\bar{\epsilon}_0 k_0^2 F$. Next, the terms U_2, U_4, U_6 can be readily controlled by using the same techniques we used in bounding P_2, with the help of the Leibniz's rule and the inductive hypothesis (B.3). On the other hand, U_3 and U_5 are similar, so we simply present the estimation of U_3. (We turn to U_7 and U_8 in a moment.)

Using the Leibniz's rule, we obtain

$$
U_3 \leq \frac{r!t!\bar{s}!}{(r+t+\bar{s})!} \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{\bar{s}-1} \left| \frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{\bar{s}-1-\ell}}{(\bar{s}-1-\ell)^2} \left[\bar{\epsilon}_0 k_0^2 \right] \right|_{L^\infty} \left\| \frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} \partial_z E^x \right\|_{L^2}.
$$

Following the approach of Z_1, we obtain

$$
\begin{aligned}
\left\| \frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} \partial_z E^x \right\|_{L^2} &\leq \left\| \frac{\partial_x^j}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} \operatorname{curl}[E] \right\|_{L^2} + \left\| \frac{\partial_x^j}{j!} \frac{\partial_y^{k+1}}{k!} \frac{\partial_z^\ell}{\ell!} E^z \right\|_{L^2} \\
&\quad + \left\| \frac{\partial_x^{j+1}}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} E^z \right\|_{L^2} + \left\| \frac{\partial_x^{j+1}}{j!} \frac{\partial_y^k}{k!} \frac{\partial_z^\ell}{\ell!} E^y \right\|_{L^2} \\
&\quad + \left\| \frac{\partial_x^j}{j!} \frac{\partial_y^{k+1}}{k!} \frac{\partial_z^\ell}{\ell!} E^x \right\|_{L^2}.
\end{aligned}
$$

With this, we can bound U_3 as follows

$$
U_3 \leq \sum_{j=0}^{r} \sum_{k=0}^{t} \sum_{\ell=0}^{\bar{s}-1} \left| \frac{\partial_x^{r-j}}{(r-j)!} \frac{\partial_y^{t-k}}{(t-k)!} \frac{\partial_z^{\bar{s}-1-\ell}}{(\bar{s}-1-\ell)^2} \left[\bar{\epsilon}_0 k_0^2 \right] \right|_{L^\infty}
$$

$$\times\left\{\left\|\frac{\partial_x^j}{j!}\frac{\partial_y^k}{k!}\frac{\partial_z^\ell}{\ell!}\operatorname{curl}[E]\right\|_{L^2}+\left\|\frac{\partial_x^j}{j!}\frac{\partial_y^{k+1}}{k!}\frac{\partial_z^\ell}{\ell!}E^z\right\|_{L^2}+\left\|\frac{\partial_x^{j+1}}{j!}\frac{\partial_y^k}{k!}\frac{\partial_z^\ell}{\ell!}E^z\right\|_{L^2}\right.$$
$$\left.+\left\|\frac{\partial_x^{j+1}}{j!}\frac{\partial_y^k}{k!}\frac{\partial_z^\ell}{\ell!}E^y\right\|_{L^2}+\left\|\frac{\partial_x^j}{j!}\frac{\partial_y^{k+1}}{k!}\frac{\partial_z^\ell}{\ell!}E^x\right\|_{L^2}\right\}.$$

Using the analyticity assumption on $\bar{\epsilon}_0$ and the inductive hypothesis (B.3), we get

$$U_3\le\sum_{j=0}^{r}\sum_{k=0}^{t}\sum_{\ell=0}^{\bar{s}-1}C_{\bar{\epsilon}_0}\frac{\eta^{r-j}}{(r-j+1)^2}\frac{\theta^{t-k}}{(t-k+1)^2}\frac{\zeta^{\bar{s}-1-\ell}}{(\bar{s}-1-\ell+1)^2}$$
$$\times\underline{C}_e\left\{\frac{\eta^j}{(j+1)^2}\frac{\theta^k}{(k+1)^2}\frac{\zeta^\ell}{(\ell+1)^2}+2\frac{\eta^j}{(j+1)^2}\frac{\theta^{k+1}}{(k+2)^2}\frac{\zeta^\ell}{(\ell+1)^2}\right.$$
$$\left.+2\frac{\eta^{j+1}}{(j+2)^2}\frac{\theta^k}{(k+1)^2}\frac{\zeta^\ell}{(\ell+1)^2}\right\}$$
$$\le\sum_{j=0}^{r}\sum_{k=0}^{t}\sum_{\ell=0}^{\bar{s}-1}C_{\bar{\epsilon}_0}\frac{\eta^{r-j}}{(r-j+1)^2}\frac{\theta^{t-k}}{(t-k+1)^2}\frac{\zeta^{\bar{s}-1-\ell}}{(\bar{s}-1-\ell+1)^2}$$
$$\times\underline{C}_e(1+2\theta+2\eta)\frac{\eta^j}{(j+1)^2}\frac{\theta^k}{(k+1)^2}\frac{\zeta^\ell}{(\ell+1)^2}$$
$$\le\underline{C}_eC_{\bar{\epsilon}_0}(1+2\theta+2\eta)\frac{\eta^r}{(r+1)^2}\frac{\theta^t}{(t+1)^2}\frac{\zeta^{\bar{s}-1}}{(\bar{s}-1+1)^2}$$
$$\times\sum_{j=0}^{r}\sum_{k=0}^{t}\sum_{\ell=0}^{\bar{s}-1}\frac{(r+1)^2(t+1)^2}{(r-j+1)^2(j+1)^2(t-k+1)^2(k+1)^2}\frac{(\bar{s}-1+1)^2}{(\bar{s}-1-\ell+1)^2(\ell+1)^2}$$
$$\le\underline{C}_eC_{\bar{\epsilon}_0}(1+2\theta+2\eta)\frac{1}{\zeta}S^3\frac{(\bar{s}+1)^2}{\bar{s}^2}\frac{\eta^r}{(r+1)^2}\frac{\theta^t}{(t+1)^2}\frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2}$$
$$\le\underline{C}_e\frac{\eta^r}{(r+1)^2}\frac{\theta^t}{(t+1)^2}\frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2},$$

for some $\zeta\ge 4C_{\bar{\epsilon}_0}(1+2\theta+2\eta)S^3$.

Finally, we observe that, by using Leibniz's rule, U_7 and U_8 can be controlled in a similar fashion. So, it suffices to estimate U_8, and for this we compute

$$U_8=\left\|\left[\frac{\partial_x^r\partial_y^t\partial_z^{\bar{s}}}{(r+t+\bar{s})!},\bar{\epsilon}_0k_0^2\right]\partial_zE^z\right\|_{L^2}$$
$$\le\frac{r!t!\bar{s}!}{(r+t+\bar{s}!)}\sum_{j=0}^{r}\sum_{k=0}^{t}\sum_{\ell=0}^{\bar{s}-1}\left|\frac{\partial_x^{r-j}}{(r-j)!}\frac{\partial_y^{t-k}}{(t-k)!}\frac{\partial_z^{\bar{s}-\ell}}{(\bar{s}-\ell)!}\left[\bar{\epsilon}_0k_0^2\right]\right|_{L^\infty}\left\|\frac{\partial_x^j}{j!}\frac{\partial_y^k}{k!}\frac{\partial_z^\ell}{\ell!}\left[\partial_zE^z\right]\right\|_{L^2}$$
$$+\frac{r!t!\bar{s}!}{(r+t+\bar{s}!)}\sum_{j=0}^{r}\sum_{k=0}^{t-1}\left|\frac{\partial_x^{r-j}}{(r-j)!}\frac{\partial_y^{t-k}}{(t-k)!}\left[\bar{\epsilon}_0k_0^2\right]\right|_{L^\infty}\left\|\frac{\partial_x^j}{j!}\frac{\partial_y^k}{k!}\frac{\partial_z^{\bar{s}}}{\bar{s}!}\left[\partial_zE^z\right]\right\|_{L^2}$$

$$+ \frac{r!t!\bar{s}!}{(r+t+\bar{s}!)} \sum_{j=0}^{r-1} \left| \frac{\partial_x^{r-j}}{(r-j)!} \left[\bar{\epsilon}_0 k_0^2 \right] \right|_{L^\infty} \left\| \frac{\partial_x^j}{j!} \frac{\partial_y^t}{t!} \frac{\partial_z^{\bar{s}}}{\bar{s}!} \left[\partial_z E^z \right] \right\|_{L^2} .$$

Using the inductive hypothesis, (B.3), and following the same approach as used to address U_3, we get

$$U_8 \leq \underline{C}_e \frac{\eta^r}{(r+1)^2} \frac{\theta^t}{(t+1)^2} \frac{\zeta^{\bar{s}}}{(\bar{s}+1)^2},$$

for some $\zeta \geq 4C_{\bar{\epsilon}_0}(1+\theta+\eta)S^3$, and the proof is complete.

Data availability

Data will be made available on request.

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