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Analyticity and Stable Computation of Dirichlet–Neumann Operators for Laplace's Equation Under Quasiperiodic Boundary Conditions in Two and Three Dimensions

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ABSTRACT

Dirichlet–Neumann operators (DNOs) are important to the formulation, analysis, and simulation of many crucial models found in engineering and the sciences. For instance, these operators permit moving-boundary problems, such as the classical water wave problem (free-surface ideal fluid flow under the influence of gravity and capillarity), to be restated in terms of interfacial quantities, which not only eliminates the boundary tracking problem, but also reduces the problem's dimension. While these DNOs have been the object of much recent study regarding their numerical simulation and rigorous analysis, they have yet to be examined in the setting of laterally quasiperiodic boundary conditions. The purpose of this contribution is to begin this investigation with a particular focus on the more realistic simulation of two- and three-dimensional free-surface water waves. Here, we not only carefully define the DNO with respect to these boundary conditions for Laplace's equation, but we also show the rigorous analyticity of these operators with respect to sufficiently smooth boundary perturbations. These theoretical developments suggest a novel algorithm for the stable and high-order simulation of the DNO, which we implement and extensively test.

1 | Introduction

Many models arising in engineering and science couple partial differential equations (PDEs) for volumetric field quantities to their Dirichlet and Neumann traces at domain boundaries and interfaces. Often, the governing PDEs in the bulk are rather trivial (e.g., linear and constant coefficient) and the simulation challenges (numerical and analytical) arise from the complicated nature of the problem geometry (e.g., nonseparable or unbounded) or from nonlinear conditions at the interfaces. For these problems, not only can these technical challenges be effectively addressed, but also the dimension of the problem can be reduced by restating them entirely in terms of interfacial quantities (the Dirichlet data). However, this requires that the corresponding surface normal derivatives (the Neumann data) can be recovered by solving the underlying PDE. Dirichlet–Neumann operators (DNOs) accomplish this procedure. These DNOs arise in a wide array of applications ranging from linear acoustic [1], electromagnetic [2, 3], and elastic [4] scattering to solid [5] and fluid [6, 7] mechanics. Of particular interest in this contribution are the DNOs which arise in the study of the water waves problem [6, 8, 9] which models the free-surface evolution of an ideal fluid under the influence of gravity and capillarity. In light of these considerations, it is clearly desirable to have a thorough understanding of not only the analytical properties of these DNOs, but also accurate and stable methods for their numerical simulation.

All classical methods for the numerical approximation of solutions of PDEs have been brought to bear upon this problem: Finite Difference Methods [10, 11], Finite Element Methods [12, 13], Spectral Methods [14–16], and their various generalizations and refinements. However, for the problems that we have in mind, which feature *homogeneous* PDEs in the bulk, these *volumetric* solvers are needlessly expensive as they discretize the full domain rather than simply the boundaries. Consequently, interfacial methods such as Boundary Integral/Element Methods (BIM/BEM) have very successfully been applied to the problem of simulating DNOs [17–20]. Of particular note in the two-dimensional setting, are the conformal mapping techniques [21–25], which map the fluid domain to the lower half plane or a horizontal strip. This simplifies the computation of the DNOs to Hilbert transforms which can be easily computed in Fourier space. Another major advantage of the conformal mapping techniques is that they can be used to compute water waves that are not graphs of functions, such as overhanging waves. However, these techniques cannot be easily extended to three dimensions.

In this work, we advocate for a different class of efficient and accurate interfacial methods for simulating DNOs that view the problem geometry as a deviation from a simpler configuration (e.g., planar, circular). In summary, the DNO can be shown to depend *analytically* upon the deviation magnitude, ε, [26] implying that it can be expressed as a convergent Taylor series in ε. The numerical algorithm then consists of approximating, for instance, the first N terms in this series and performing the (finite) summation [27]. Initially, one assumes that the perturbation parameter ε is sufficiently small, however, it can be demonstrated that, under certain reasonable geometrical constraints, the domain of analyticity includes the *entire* real axis [28] so that (real) perturbations of any size can be simulated provided one has an appropriate analytic continuation strategy (e.g., Padé summation [29]). These "High-Order Perturbation of Surfaces" (HOPS) methods have been shown to be highly efficient, accurate, and stable [26–28] within their domain of applicability, which is certainly the case for water wave simulations up to the limitations of the formulation (e.g., overturning and breaking). Importantly, in contrast to volumetric methods or BEM/BIM, the derivation, implementation, and rigorous analysis of these HOPS methods do *not* depend strongly on the problem dimension.

Perhaps due to the extreme analytical and numerical difficulties presented by the strongly nonlinear interfacial boundary conditions of the water wave free-boundary problem, the issue of lateral boundary conditions for this model has been given secondary consideration over the years. In fact, almost all progress has been reported for water waves which exhibit either periodic patterns [30–32] or decay to a flat state as the lateral coordinates approach infinity [33–35]. However, not only do such conditions prohibit the study of subharmonic (of periods longer than the base period) or noncommensurate (of periods irrationally related to the base period) wave interactions, but also, ocean waves simply look neither periodic nor decaying at infinity. For these reasons, researchers have studied solutions to model equations which exhibit quasiperiodic patterns [36–42]. Particularly, two of the authors studied the full water wave problem with solutions exhibiting quasiperiodic patterning in the sense advocated by Moser [43] and made precise in Section 2.1. They utilized a surface integral formulation specific to water waves which they addressed with a conformal mapping technique that delivered highly accurate solutions in a rapid and stable fashion. However, this scheme is inherently limited to two-dimensional configurations (one lateral and one vertical) and a three-dimensional version is highly desirable.

As mentioned in the work of Wilkening and Zhao [38], one possible approach to extending their results to higher dimensions is to follow the work of Craig and Sulem [9] and restate the water wave equations in terms of the DNO associated with Laplace's equation subject to quasiperiodic boundary conditions. For this, one requires not only a fast, stable, and accurate numerical algorithm for the simulation of these DNOs, but also a rigorous analysis justifying their convergence. In this publication, we provide, for the first time in the setting of quasiperiodic boundary conditions, *three* such HOPS algorithms, together with an analyticity theory that provides the crucial first step in a full numerical analysis.

The paper is organized as follows: We begin by stating the governing equations for the water wave problem [6] (Section 2) which is the motivation for our study, with a particular discussion of the quasiperiodic boundary conditions we employ (Section 2.1). We then define the DNO, which allows us to restate the water wave problem in terms of surface variables (Section 3) with a detailed specification of the transparent boundary conditions that permit a precise and uniform statement of the boundary conditions in the far field (Section 3.1). With this, we establish analyticity of the DNO in Section 4 which requires a change of variables (Section 4.1), a discussion of function spaces (Section 4.2), an elliptic estimate (Section 4.3), and an inductive lemma (Section 4.4) to deliver the analyticity results for the field (Section 4.5) and the DNO (Section 4.6). In Section 5, we describe our numerical algorithms, providing a brief summary of our High–Order Spectral (HOS) approach in Section 5.1. For the simulation of DNOs subject to quasiperiodic boundary conditions, we describe our novel generalizations of the methods of Operator Expansions (OE), Section 5.2, Field Expansions (FE), Section 5.3, and Transformed Field Expansions (TFE), Section 5.4. In Section 5.5, we recall the method of Padé approximation which we use to implement numerical analytic continuation. Finally, in Section 6, we present numerical results of our implementations of the OE, FE, and TFE algorithms as compared with exact solutions from the Method of Manufactured Solutions (Section 6.1). In Appendix A, we present a proof of the requisite elliptic estimate under quasiperiodic boundary conditions which is central to establishing our result.

2 | The Water Wave Problem

The classical water wave model simulates the free-surface evolution of an irrotational, inviscid, and incompressible (ideal) fluid under the influence of gravity and capillarity [6, 7]. These Euler equations are posed on the moving domain

$$S_{h,\eta} = \left\{ (x, y) \in \mathbf{R}^{n-1} \times \mathbf{R} \mid -h < y < \eta(x, t) \right\}, \quad n \in \{2, 3\},$$

where $0 < h \leq \infty$. The irrotational nature of the flow demands that the fluid velocity be the gradient of a potential, $\vec{v} = \nabla\varphi$, while incompressibility enforces that the fluid's velocity be divergence free, $\text{div}\left[\vec{v}\right] = 0$. Therefore, the velocity potential φ is harmonic,

$$\Delta_x \varphi + \partial_y^2 \varphi = 0.$$

In the case of finite depth, the bottom is assumed to be impermeable so that

$$\partial_y \varphi(x, -h) = 0,$$

while a fluid of infinite depth mandates

$$\partial_y \varphi \to 0, \qquad y \to -\infty.$$

These are supplemented with initial conditions, and the kinematic and Bernoulli conditions at the free surface,

$$\partial_t \eta + \nabla_x \eta \cdot \nabla_x \varphi - \partial_y \varphi = 0, \qquad y = \eta, \quad (2.1a)$$

$$\partial_t \varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta - \sigma \text{div}_x \left[\frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}} \right] = 0, \qquad y = \eta. \quad (2.1b)$$

All that remains is to specify the lateral boundary conditions which the velocity potential, φ, and interface, η, must satisfy. For this, we choose quasiperiodicity in a sense we now make precise.

2.1 | Quasiperiodic Functions

There are many notions of quasiperiodicity that have been advanced in the literature and we focus on the one prescribed by Moser [43]. We define a function $f(x)$ to be quasiperiodic if

$$f(x) = \tilde{f}(\alpha), \quad \alpha = Kx, \quad x \in \mathbf{R}^n, \quad \alpha \in \mathbf{R}^d, \quad K \in \mathbf{R}^{d\times n},$$

and the envelope function $\tilde{f}(\alpha)$ is *periodic* with respect to the lattice

$$\Gamma = (2\pi\mathbf{Z})^d,$$

so that

$$\tilde{f}(\alpha) = \sum_{p\in\Gamma'} \hat{f}_p e^{ip\cdot\alpha}, \quad \hat{f}_p = \frac{1}{(2\pi)^d} \int_{P(\Gamma)} \tilde{f}(\alpha) e^{-ip\cdot\alpha}\, d\alpha,$$

where

$$\Gamma' = \mathbf{Z}^d, \qquad P(\Gamma) = [0, 2\pi)^d.$$

To specify truly nonperiodic functions, we demand that $d > n$ and the rows of K are linearly independent over the integers. For example, in the case $n = 1$ and $d = 2$, we can choose

$$K = \begin{pmatrix} 1 \\ \kappa \end{pmatrix}, \quad \kappa \notin \mathbf{Q}.$$

For later use, we point out that simple calculations reveal

$$\nabla_x f(x) = K^T \nabla_\alpha \tilde{f}(\alpha), \quad \Delta_x f(x) = \text{div}_\alpha\left[K K^T \nabla_\alpha \tilde{f}(\alpha)\right].$$

Inspired by this definition of quasiperiodic functions of the lateral variable x, we extend this notion to functions which are laterally quasiperiodic but not vertically. For instance, u is said to be laterally quasiperiodic if

$$u(x, y) = \tilde{u}(\alpha, y), \quad \alpha = Kx, \quad x \in \mathbf{R}^n, \quad \alpha \in \mathbf{R}^d, \quad K \in \mathbf{R}^{d\times n},$$

and $\tilde{u}(\alpha, y)$ is *α-periodic* with respect to the lattice $\Gamma = (2\pi\mathbf{Z})^d$ so that

$$\tilde{u}(\alpha, y) = \sum_{p\in\Gamma'} \hat{u}_p(y) e^{ip\cdot\alpha}, \quad \Gamma' = \mathbf{Z}^d.$$

3 | The DNO

Following the work of Craig and Sulem [9], we restate the water wave problem in terms of the DNO. Due to the time-independent nature of the DNO, we suppress time dependence of the free interface in its definition and use the notation $y = g(x)$ to denote its parameterization, which is assumed to be quasiperiodic throughout the current work. For this, we require the following definition.

Definition 3.1. Given Dirichlet data, $\xi(x)$, a laterally quasiperiodic solution of the boundary value problem

$$\Delta_x \varphi + \partial_y^2 \varphi = 0 \qquad \text{in } S_{h,g}, \quad (3.1a)$$

$$\varphi(x, g(x)) = \xi(x), \qquad \text{at } y = g(x) \quad (3.1b)$$

subject to the condition,

$$\partial_y \varphi(x, -h) = 0, \qquad \text{if } h < \infty, \quad (3.1c)$$

$$\partial_y \varphi \to 0, \quad y \to -\infty, \qquad \text{if } h = \infty \quad (3.1d)$$

specifies the Neumann data,

$$\nu(x) = \partial_y \varphi(x, g(x)) - \nabla_x g(x) \cdot \nabla_x \varphi(x, g(x)).$$

The DNO is defined by

$$G(g) : \xi(x) \to \nu(x). \quad (3.2)$$

Given this definition, we can restate the water wave problem as the system of PDEs [9, 32],

$$\partial_t \eta = G(\eta)\xi,$$

$$\partial_t \xi = -g\eta - \frac{1}{2(1 + |\nabla_x \eta|^2)} \Big[|\nabla_x \xi|^2 - (G(\eta)\xi)^2 - 2(G(\eta)\xi)\nabla_x \xi \cdot \nabla_x \eta + |\nabla_x \xi|^2 |\nabla_x \eta|^2 - (\nabla_x \xi \cdot \nabla_x \eta)^2 \Big] + \sigma \text{div}_x \left[\frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}} \right].$$

For our purposes, it is more convenient to restate the definition of the DNO in terms of the independent variable α and the envelope functions $\{\tilde{\varphi}, \tilde{g}, \tilde{\xi}, \tilde{\nu}\}$. To do so, we lift the lower-dimensional quasiperiodic problem in the lateral direction to a periodic problem defined on a higher-dimensional torus. Therefore, the equations are defined on the new domain

$$S_{h,\tilde{g}} = \{(\alpha, y) \in P(\Gamma) \times \mathbf{R} \mid -h < y < \tilde{g}(\alpha)\}.$$

Definition 3.2. Given Dirichlet data, $\tilde{\xi}(\alpha)$, a laterally periodic solution of the boundary value problem

$$\operatorname{div}_\alpha \left[KK^T \nabla_\alpha \tilde{\varphi}(\alpha, y)\right] + \partial_y^2 \tilde{\varphi}(\alpha, y) = 0, \qquad \text{in } S_{h,\tilde{g}}, \tag{3.3a}$$

$$\tilde{\varphi}(\alpha, \tilde{g}(\alpha)) = \tilde{\xi}(\alpha), \qquad y = \tilde{g}(\alpha) \tag{3.3b}$$

subject to the condition,

$$\partial_y \tilde{\varphi}(\alpha, -h) = 0, \qquad \text{if } h < \infty, \tag{3.3c}$$

$$\partial_y \tilde{\varphi} \to 0, \qquad y \to -\infty, \qquad \text{if } h = \infty \tag{3.3d}$$

specifies the Neumann data,

$$\tilde{\nu}(\alpha) = \partial_y \tilde{\varphi}(\alpha, \tilde{g}(\alpha)) - (K^T \nabla_\alpha \tilde{g}(\alpha)) \cdot K^T \nabla_\alpha \tilde{\varphi}(\alpha, \tilde{g}(\alpha)).$$

The DNO is defined by

$$\tilde{G}(\tilde{g}) : \tilde{\xi}(\alpha) \to \tilde{\nu}(\alpha). \tag{3.4}$$

3.1 | Transparent Boundary Conditions

Regarding boundary conditions at the bottom of the fluid domain we can state a rigorous formulation for quasiperiodic waves which simultaneously accounts for a fluid of *any* depth, even infinite. We begin with the case of a fluid of infinite depth and select a value a such that $-a < -||\tilde{g}||_{L^\infty}$. Clearly, beneath the artificial boundary at $y = -a$ the bounded quasiperiodic solution of Laplace's equation is

$$\tilde{\varphi}(\alpha, y) = \sum_{p \in \Gamma'} \hat{\psi}_p e^{|K^T p|(y+a)} \mathrm{e}^{ip\cdot\alpha}.$$

From this, we have

$$\tilde{\varphi}(\alpha, -a) = \sum_{p \in \Gamma'} \hat{\psi}_p \mathrm{e}^{ip\cdot\alpha} =: \tilde{\psi}(\alpha).$$

Since

$$\partial_y \tilde{\varphi}(\alpha, y) = \sum_{p \in \Gamma'} \hat{\psi}_p |K^T p| \mathrm{e}^{|K^T p|(y+a)} \mathrm{e}^{ip\cdot\alpha},$$

we find

$$\partial_y \tilde{\varphi}(\alpha, -a) = \sum_{p \in \Gamma'} |K^T p| \hat{\psi}_p \mathrm{e}^{ip\cdot\alpha} =: |K^T D|[\tilde{\psi}(\alpha)],$$

which defines the order-one Fourier multiplier $|K^T D|$. With this we can state the infinite depth transparent boundary condition, (3.3d), as

$$\partial_y \tilde{\varphi} - |K^T D|[\tilde{\varphi}] = 0, \qquad y = -a.$$

Similarly, in finite depth, $h < \infty$, we choose a such that $-h < -a < -||\tilde{g}||_{L^\infty}$ so that beneath the artificial boundary at $y = -a$ the solution of Laplace's equation satisfying

$$\partial_y \tilde{\varphi} = 0, \qquad y = -h$$

is

$$\tilde{\varphi}(\alpha, y) = \sum_{p \in \Gamma'} \hat{\psi}_p \frac{\cosh(|K^T p|(h + y))}{\cosh(|K^T p|(h - a))} \mathrm{e}^{ip\cdot\alpha}.$$

With this, we have

$$\tilde{\varphi}(\alpha, -a) = \sum_{p \in \Gamma'} \hat{\psi}_p \mathrm{e}^{ip\cdot\alpha} =: \tilde{\psi}(\alpha),$$

and, since

$$\partial_y \tilde{\varphi}(\alpha, y) = \sum_{p \in \Gamma'} \hat{\psi}_p |K^T p| \frac{\sinh(|K^T p|(h + y))}{\cosh(|K^T p|(h - a))} \mathrm{e}^{ip\cdot\alpha},$$

we find

$$\begin{aligned} \partial_y \tilde{\varphi}(\alpha, -a) &= \sum_{p \in \Gamma'} |K^T p| \tanh(|K^T p|(h - a)) \hat{\psi}_p \mathrm{e}^{ip\cdot\alpha} \\ &=: |K^T D| \tanh((h - a)|K^T D|)[\tilde{\psi}(\alpha)], \end{aligned}$$

which defines the order-one Fourier multiplier $|K^T D| \tanh((h - a)|K^T D|)$. Thus, the transparent boundary condition in finite depth, (3.3c), reads

$$\partial_y \tilde{\varphi} - |K^T D| \tanh((h - a)|K^T D|)[\tilde{\varphi}] = 0, \qquad y = -a.$$

Therefore, if we define

$$\tilde{T} := \begin{cases} |K^T D| \tanh((h - a)|K^T D|), & h < \infty, \\ |K^T D|, & h = \infty, \end{cases}$$

then we have the uniform statement of the transparent boundary conditions, (3.3c) and (3.3d), as

$$\partial_y \tilde{\varphi} - \tilde{T}[\tilde{\varphi}] = 0, \qquad y = -a. \tag{3.5}$$

With this, we can (equivalently) restate the definition of the DNO.

Definition 3.3. Given Dirichlet data, $\tilde{\xi}(\alpha)$, a laterally periodic solution of the boundary value problem

$$\operatorname{div}_\alpha \left[KK^T \nabla_\alpha \tilde{\varphi}(\alpha, y)\right] + \partial_y^2 \tilde{\varphi}(\alpha, y) = 0, \qquad \text{in } S_{a,\tilde{g}}, \tag{3.6a}$$

$$\tilde{\varphi}(\alpha, \tilde{g}(\alpha)) = \tilde{\xi}(\alpha), \qquad y = \tilde{g}(\alpha), \tag{3.6b}$$

$$\partial_y \tilde{\varphi} - \tilde{T}[\tilde{\varphi}] = 0, \qquad y = -a, \tag{3.6c}$$

$$\tilde{\varphi}(\alpha + \gamma, y) = \tilde{\varphi}(\alpha, y), \qquad \gamma \in \Gamma, \tag{3.6d}$$

specifies the Neumann data,

$$\tilde{\nu}(\alpha) = \partial_y \tilde{\varphi}(\alpha, \tilde{g}(\alpha)) - (K^T \nabla_\alpha \tilde{g}(\alpha)) \cdot K^T \nabla_\alpha \tilde{\varphi}(\alpha, \tilde{g}(\alpha)).$$

The DNO is defined by

$$\tilde{G}(\tilde{g}) : \tilde{\xi}(\alpha) \to \tilde{\nu}(\alpha). \tag{3.7}$$

Remark 3.4. To clarify the connection between the original statement of the boundary value problem (3.3), its associated DNO (3.4), and the reformulation given in (3.6) and (3.7), we will find a *unique* solution to (3.6) (Theorem 4.12) which defines a *unique* DNO (Theorem 4.13) via (3.7). Assuming a unique solution to (3.3) exists, if (3.3) and (3.6) are presented with the *same* Dirichlet data, $\tilde{\xi}$, they will necessarily produce the *same* Neumann data, $\tilde{\nu}$ indicating that either problem will deliver the same DNO. Beyond this, both problems will deliver solutions that agree on $S_{a,\tilde{g}}$.

4 | Analyticity of the DNO

We now follow the approach of Nicholls and Reitich [26, 28] and utilize the Method of TFE to establish the analyticity with respect to boundary deformation, $\tilde{g}$, of not only the field, $\tilde{\varphi}$, but also the DNO, $\tilde{G}$. The procedure is, by now, well-established and begins with a domain-flattening change of variables. Once this has been accomplished, the candidate solution is expanded in a Taylor series in powers of the interfacial deformation, resulting in a recurrence of (inhomogeneous) linearized problems to be solved. These are recursively estimated to establish the convergence of the Taylor series for the field which then yields the analyticity of the DNO.

4.1 | Change of Variables

Consider the change of variables (known as the C-Method [44, 45] in electromagnetics or σ-coordinates [46] in oceanography)

$$\alpha' = \alpha, \quad y' = a\left(\frac{y - \tilde{g}(\alpha)}{a + \tilde{g}(\alpha)}\right),$$

and the transformed field

$$\tilde{u}(\alpha', y') = \tilde{\varphi}\left(\alpha', \left(\frac{a + \tilde{g}(\alpha')}{a}\right)y' + \tilde{g}(\alpha')\right),$$

which, upon dropping the primes for simplicity, transforms (3.6) to

$$\mathrm{div}_\alpha\left[KK^T\nabla_\alpha \tilde{u}(\alpha, y)\right] + \partial_y^2 \tilde{u}(\alpha, y) = \tilde{F}(\alpha, y; \tilde{g}, \tilde{u}), \quad -a < y < 0, \tag{4.1a}$$

$$\tilde{u}(\alpha, 0) = \tilde{\xi}(\alpha), \quad y = 0, \tag{4.1b}$$

$$\partial_y \tilde{u} - \tilde{T}[\tilde{u}] = \tilde{J}(\alpha; \tilde{g}, \tilde{u}), \quad y = -a, \tag{4.1c}$$

$$\tilde{u}(\alpha + \gamma, y) = \tilde{u}(\alpha, y), \quad \gamma \in \Gamma, \tag{4.1d}$$

from which one can produce the Neumann data

$$\tilde{\nu}(\alpha) = \partial_y \tilde{u}(\alpha, 0) + \tilde{L}(\alpha; \tilde{g}, \tilde{u}). \tag{4.1e}$$

The forms for $\tilde{F}$, $\tilde{J}$, and $\tilde{L}$ are readily derived and are all $\mathcal{O}(\tilde{g})$.

We now make the smallness assumption on $\tilde{g}$,

$$\tilde{g}(\alpha) = \varepsilon \tilde{f}(\alpha), \quad \varepsilon \ll 1, \quad \tilde{f} = \mathcal{O}(1),$$

where previous results indicate that we will be able to drop the size assumption on ε provided that it is *real* [28, 47]. With this assumption, we seek a solution of the form

$$\tilde{u} = \tilde{u}(\alpha, y; \varepsilon) = \sum_{n=0}^{\infty} \tilde{u}_n(\alpha, y)\varepsilon^n, \tag{4.2}$$

which, upon insertion into (4.1), delivers

$$\mathrm{div}_\alpha[KK^T\nabla_\alpha \tilde{u}_n(\alpha, y)] + \partial_y^2 \tilde{u}_n(\alpha, y) = \tilde{F}_n(\alpha, y), \quad -a < y < 0, \tag{4.3a}$$

$$\tilde{u}_n(\alpha, 0) = \delta_{n,0}\tilde{\xi}(\alpha), \quad y = 0, \tag{4.3b}$$

$$\partial_y \tilde{u}_n - \tilde{T}[\tilde{u}_n] = \tilde{J}_n(\alpha), \quad y = -a, \tag{4.3c}$$

$$\tilde{u}_n(\alpha + \gamma, y) = \tilde{u}_n(\alpha, y), \quad \gamma \in \Gamma, \tag{4.3d}$$

where $\delta_{n,m}$ is the Kronecker delta. One can produce the Neumann data using the solution of the above equations

$$\tilde{\nu}_n(\alpha) = \partial_y \tilde{u}_n(\alpha, 0) + \tilde{L}_n(\alpha), \tag{4.3e}$$

which gives $\tilde{G}_n(\tilde{f})[\tilde{\xi}] = \tilde{\nu}_n$. Here, we have

$$\tilde{F}_n = \mathrm{div}_\alpha[K\tilde{F}_n^\alpha] + \partial_y \tilde{F}_n^y + \tilde{F}_n^0, \tag{4.4a}$$

where

$$\begin{aligned} a^2\tilde{F}_n^\alpha &= -2a\tilde{f}(K^T\nabla_\alpha \tilde{u}_{n-1}) + a(a+y)(K^T\nabla_\alpha \tilde{f})\partial_y \tilde{u}_{n-1} \\ &\quad - (\tilde{f})^2(K^T\nabla_\alpha \tilde{u}_{n-2}) + (a+y)\tilde{f}(K^T\nabla_\alpha \tilde{f})\partial_y \tilde{u}_{n-2}, \end{aligned} \tag{4.4b}$$

and

$$\begin{aligned} a^2\tilde{F}_n^y &= a(a+y)(K^T\nabla_\alpha \tilde{f}) \cdot (K^T\nabla_\alpha \tilde{u}_{n-1}) \\ &\quad + (a+y)\tilde{f}(K^T\nabla_\alpha \tilde{f}) \cdot (K^T\nabla_\alpha \tilde{u}_{n-2}) \\ &\quad - (a+y)^2\left|K^T\nabla_\alpha \tilde{f}\right|^2 \partial_y \tilde{u}_{n-2}, \end{aligned} \tag{4.4c}$$

and

$$\begin{aligned} a^2\tilde{F}_n^0 &= a(K^T\nabla_\alpha \tilde{f}) \cdot (K^T\nabla_\alpha \tilde{u}_{n-1}) + \tilde{f}(K^T\nabla_\alpha \tilde{f}) \cdot (K^T\nabla_\alpha \tilde{u}_{n-2}) \\ &\quad - (a+y)\left|K^T\nabla_\alpha \tilde{f}\right|^2 \partial_y \tilde{u}_{n-2}, \end{aligned} \tag{4.4d}$$

and

$$a\tilde{J}_n = \tilde{f}\tilde{T}[\tilde{u}_{n-1}(\alpha, -a)], \tag{4.4e}$$

and

$$\begin{aligned} a\tilde{L}_n &= -a(K^T\nabla_\alpha \tilde{f}) \cdot (K^T\nabla_\alpha \tilde{u}_{n-1}(\alpha, 0)) - \tilde{f}\tilde{\nu}_{n-1} \\ &\quad - \tilde{f}(K^T\nabla_\alpha \tilde{f}) \cdot (K^T\nabla_\alpha \tilde{u}_{n-2}(\alpha, 0)) + a\left|K^T\nabla_\alpha \tilde{f}\right|^2 \partial_y \tilde{u}_{n-2}(\alpha, 0). \end{aligned} \tag{4.4f}$$

We point out that (4.3e) and (4.4f) give the following formula for the nth correction of the DNO (Neumann data)

$$\tilde{G}_n(\tilde{f})[\tilde{\xi}] = \partial_y \tilde{u}_n(\alpha, 0) - (K^T\nabla_\alpha \tilde{f}) \cdot (K^T\nabla_\alpha \tilde{u}_{n-1}(\alpha, 0)) - \frac{1}{a}\tilde{f}\tilde{G}_{n-1}(\tilde{f})[\tilde{\xi}]$$

$$-\frac{1}{a}\tilde{f}(K^T\nabla_\alpha \tilde{f})\cdot(K^T\nabla_\alpha \tilde{u}_{n-2}(\alpha,0))+\left|K^T\nabla_\alpha \tilde{f}\right|^2\partial_y\tilde{u}_{n-2}(\alpha,0). \tag{4.5}$$

4.2 | Function Spaces

We now establish the analyticity of the transformed field, $\tilde{u}$, and DNO, $\tilde{G}$, under quasiperiodic boundary conditions. Our proof follows Nicholls and Reitich [26], with extra attention given to handling the small divisors and the derivatives along the quasiperiodic direction in the estimates. We begin by defining the Fourier multipliers $|K^T D|^q$,

$$|K^T D|^q \tilde{\psi}(\alpha) := \sum_{p\in\Gamma'} |K^T p|^q \hat{\psi}_p e^{ip\cdot\alpha}, \qquad q\in\mathbf{R},$$

$$|K^T D|^q \tilde{w}(\alpha,y) := \sum_{p\in\Gamma'} |K^T p|^q \hat{w}_p(y) e^{ip\cdot\alpha}, \qquad q\in\mathbf{R}.$$

Next, we recall the classical interfacial Sobolev spaces

$$H^s(P(\Gamma)) = \left\{ \tilde{\xi}(\alpha)\in L^2(P(\Gamma)) \mid \left\|\tilde{\xi}\right\|_{H^s} < \infty \right\},$$

where

$$\left\|\tilde{\xi}\right\|_{H^s}^2 = \sum_{p\in\Gamma'} \langle p\rangle^{2s}\left|\hat{\xi}_p\right|^2 = \left\|\langle D\rangle^s \tilde{\xi}\right\|_{L^2(P(\Gamma))}^2, \qquad \langle p\rangle^s = (1+|p|^2)^{s/2},$$

and volumetric spaces

$$X^s(\Omega) = \left\{ \tilde{u}(\alpha,y)\in L^2(\Omega) \mid \|\tilde{u}\|_{X^s} < \infty \right\},$$

on

$$\Omega := P(\Gamma)\times(-a,0),$$

where

$$\|\tilde{u}\|_{X^s}^2 = \sum_{p\in\Gamma'} \langle p\rangle^{2s}\left\|\hat{u}_p(y)\right\|_{L^2((-a,0))}^2 = \|\langle D\rangle^s \tilde{u}\|_{L^2(\Omega)}^2.$$

Remark 4.1. With these definitions we can illustrate a curious, but crucial, property of the operators $|K^T D|^q$. If we consider the space of zero-mean functions

$$H_0^s(P(\Gamma)) = \left\{ \tilde{\xi}(\alpha)\in H^s(P(\Gamma)) \mid \hat{\xi}_0 = 0 \right\},$$

then a direct application of the Poincaré inequality yields

$$\left\|\tilde{\xi}\right\|_{s+1} \le C\left\|\nabla_\alpha \tilde{\xi}\right\|_s. \tag{4.6}$$

However, an estimate of the form (4.6) no longer holds if we replace ∇_α by $|K^T D|$. Instead one typically settles for a "nonresonance" condition of the form [43]

$$\gamma\langle p\rangle^{-r} < |K^T p|, \quad |p|\ge 1 \tag{4.7}$$

for some $\gamma, r>0$. With this, we can only realize

$$\left\|\tilde{\xi}\right\|_{s-r}^2 = \sum_{p\in\Gamma'} \langle p\rangle^{2s-2r}\left|\hat{\xi}_p\right|^2 \le \sum_{p\in\Gamma'} \gamma^{-2}|K^T p|^2\langle p\rangle^{2s}\left|\hat{\xi}_p\right|^2$$

$$= \sum_{p\in\Gamma'} \gamma^{-2}\langle p\rangle^{2s}\left||K^T p|\hat{\xi}_p\right|^2 = \gamma^{-2}\left\||K^T D|\tilde{\xi}\right\|_s^2,$$

so that

$$|K^T D|\tilde{\xi}\in H_0^s \quad\Rightarrow\quad \tilde{\xi}\in H_0^{s-r}.$$

In particular, $|K^T D|\tilde{\xi}\in H^s$ does *not* imply $\xi\in H_0^{s+1}$ as one might expect.

With the definitions of H^s, X^s, and their norms we can state and prove the following important Algebra Property [26, 48, 49].

Lemma 4.2. *Given an integer $s > d/2$ there exists a constant $M = M(s)$ such that all of the following estimates are true:*

- *If $\tilde{f}\in H^s(P(\Gamma))$ and $\tilde{\xi}\in H^s(P(\Gamma))$, then*

$$\left\|\tilde{f}\tilde{\xi}\right\|_{H^s} \le M\left\|\tilde{f}\right\|_{H^s}\left\|\tilde{\xi}\right\|_{H^s}. \tag{4.8a}$$

- *If $\tilde{f}\in H^s(P(\Gamma))$ and $\tilde{u}\in X^s(\Omega)$, then*

$$\left\|\tilde{f}\tilde{u}\right\|_{X^s} \le M\left\|\tilde{f}\right\|_{H^s}\|\tilde{u}\|_{X^s}. \tag{4.8b}$$

- *If $\tilde{g}\in H^{s+1/2}(P(\Gamma))$, $\tilde{\psi}\in H^s(P(\Gamma))$, and $|K^T D|^{1/2}\tilde{\psi}\in H^s(P(\Gamma))$ then*

$$\left\||K^T D|^{1/2}[\tilde{g}\tilde{\psi}]\right\|_{H^s} \le M\|\tilde{g}\|_{H^{s+1/2}}\left\{\|\tilde{\psi}\|_{H^s} + \left\||K^T D|^{1/2}\tilde{\psi}\right\|_{H^s}\right\}. \tag{4.8c}$$

- *If $\tilde{g}\in H^{s+1/2}(P(\Gamma))$, $\tilde{v}\in X^s(\Omega)$, and $|K^T D|^{1/2}\tilde{v}\in X^s(\Omega)$ then*

$$\left\||K^T D|^{1/2}[\tilde{g}\tilde{v}]\right\|_{X^s} \le M\|\tilde{g}\|_{H^{s+1/2}}\left\{\|\tilde{v}\|_{X^s} + \left\||K^T D|^{1/2}\tilde{v}\right\|_{X^s}\right\}. \tag{4.8d}$$

Proof. The proofs of (4.8a) and (4.8b) are standard [49], while the proof of (4.8d) is similar to that of (4.8c). Therefore, we now work to establish (4.8c) and begin with

$$\left\||K^T D|^{1/2}[\tilde{g}\tilde{\psi}]\right\|_{H^s}^2 \le \sum_{q\in\Gamma'}\langle q\rangle^{2s}|K^T q|\left\{\sum_{p\in\Gamma'}\left|\hat{g}_{q-p}\right|\left|\hat{\psi}_p\right|\right\}^2$$

$$= \sum_{q\in\Gamma'}\left\{\sum_{p\in\Gamma'}\langle q\rangle^s|K^T q|^{1/2}\left|\hat{g}_{q-p}\right|\left|\hat{\psi}_p\right|\right\}^2$$

$$\le C\sum_{q\in\Gamma'}\left\{\sum_{p\in\Gamma'}(\langle q-p\rangle^s+\langle p\rangle^s)\left(|K^T(q-p)|^{1/2}+|K^T p|^{1/2}\right)\left|\hat{g}_{q-p}\right|\left|\hat{\psi}_p\right|\right\}^2$$

$$\le CC'\sum_{q\in\Gamma'}\left\{\sum_{p\in\Gamma'}(\langle q-p\rangle^s+\langle p\rangle^s)\left(\langle q-p\rangle^{1/2}+|K^T p|^{1/2}\right)\left|\hat{g}_{q-p}\right|\left|\hat{\psi}_p\right|\right\}^2.$$

Using the Young's convolution inequality

$$\|f_1 * f_2\|_{\ell^2} \le \|f_1\|_{\ell^1}\|f_2\|_{\ell^2},$$

we can show that

$$\left\{\sum_{q\in\Gamma'}\left(\sum_{p\in\Gamma'}\langle q-p\rangle^{s+1/2}\left|\hat{g}_{q-p}\right|\left|\hat{\psi}_p\right|\right)^2\right\}^{1/2}$$

$$\leq \|\tilde{g}\|_{H^{s+1/2}}\sum_{p\in\Gamma'}\left|\hat{\psi}_p\right| \leq C\|\tilde{g}\|_{H^{s+1/2}}\|\tilde{\psi}\|_{H^s}.$$

Similarly, we can also prove that

$$\left\{\sum_{q\in\Gamma'}\left(\sum_{p\in\Gamma'}\langle q-p\rangle^{s}\left|\hat{g}_{q-p}\right|\left|K^Tp\right|^{1/2}\left|\hat{\psi}_p\right|\right)^2\right\}^{1/2} \leq C\|\tilde{g}\|_{H^s}\left\|\left|K^TD\right|^{1/2}\tilde{\psi}\right\|_{H^s},$$

$$\left\{\sum_{q\in\Gamma'}\left(\sum_{p\in\Gamma'}\langle q-p\rangle^{1/2}\left|\hat{g}_{q-p}\right|\langle p\rangle^s\left|\hat{\psi}_p\right|\right)^2\right\}^{1/2} \leq C\|\tilde{g}\|_{H^{s+1/2}}\|\tilde{\psi}\|_{H^s},$$

$$\left\{\sum_{q\in\Gamma'}\left(\sum_{p\in\Gamma'}\left|\hat{g}_{q-p}\right|\langle p\rangle^s\left|K^Tp\right|^{1/2}\left|\hat{\psi}_p\right|\right)^2\right\}^{1/2} \leq \|\tilde{g}\|_{H^s}\left\|\left|K^TD\right|^{1/2}\tilde{\psi}\right\|_{H^s}.$$

Therefore, we have

$$\left\|\left|K^TD\right|^{1/2}[\tilde{g}\tilde{\psi}]\right\|_{H^s} \leq M\|\tilde{g}\|_{H^{s+1/2}}\left(\|\tilde{\psi}\|_{H^s}+\left\|\left|K^TD\right|^{1/2}\tilde{\psi}\right\|_{H^s}\right).$$

□

Remark 4.3. At this point, we observe how our current theory will differ in an important way from the periodic case explored in Nicholls and Reitich [27]. In the latter, we were able to establish

$$\left\||D|^{1/2}[\tilde{g}\tilde{\psi}]\right\|_{H^s} \leq M\|\tilde{g}\|_{H^{s+1/2}}\left(\|\tilde{\psi}\|_{H^s}+\left\||D|^{1/2}\tilde{\psi}\right\|_{H^s}\right)$$

$$\leq 2M\|\tilde{g}\|_{H^{s+1/2}}\left\||D|^{1/2}\tilde{\psi}\right\|_{H^s}.$$

By contrast, we *cannot* bound $\|\tilde{\psi}\|_{H^s}$ by $\left\|\left|K^TD\right|^{1/2}\tilde{\psi}\right\|_{H^s}$ (see Remark 4.1) which will necessitate a more powerful elliptic estimate (Theorem 4.8) that controls not only

$$\left\{\left|K^TD\right|^{1/2}\tilde{u}, \left|K^TD\right|^{1/2}\partial_y\tilde{u}, \left|K^TD\right|^{1/2}K^T\nabla_\alpha\tilde{u}\right\}$$

but also

$$\left\{\tilde{u}, \partial_y\tilde{u}, K^T\nabla_\alpha\tilde{u}\right\}.$$

From this, it is straightforward to establish the following.

Corollary 4.4. *Given an integer $s > d/2$ there exists a constant $M = M(s)$ such that all of the following estimates are true:*

- *If $\tilde{g}, \tilde{h} \in H^{s+1/2}(P(\Gamma))$, $\tilde{\psi} \in H^s(P(\Gamma))$, and $\left|K^TD\right|^{1/2}\tilde{\psi} \in H^s(P(\Gamma))$ then*

$$\left\|\left|K^TD\right|^{1/2}[\tilde{g}\tilde{h}\tilde{\psi}]\right\|_{H^s} \leq M^2\|\tilde{g}\|_{H^{s+1/2}}\|\tilde{h}\|_{H^{s+1/2}}\left\{\|\tilde{\psi}\|_{H^s}+\left\|\left|K^TD\right|^{1/2}\tilde{\psi}\right\|_{H^s}\right\}. \tag{4.9a}$$

- *If $\tilde{g}, \tilde{h} \in H^{s+1/2}(P(\Gamma))$, $\tilde{v} \in X^s(\Omega)$, and $\left|K^TD\right|^{1/2}\tilde{v} \in X^s(\Omega)$ then*

$$\left\|\left|K^TD\right|^{1/2}[\tilde{g}\tilde{h}\tilde{v}]\right\|_{X^s} \leq M^2\|\tilde{g}\|_{H^{s+1/2}}\|\tilde{h}\|_{H^{s+1/2}}\left\{\|\tilde{v}\|_{X^s}+\left\|\left|K^TD\right|^{1/2}\tilde{v}\right\|_{X^s}\right\}. \tag{4.9b}$$

Finally, we state the following, readily proven, result for later use.

Lemma 4.5. *Given an integer $s > 0$ the following estimates are true:*

- *If $\tilde{v} \in X^s(\Omega)$, then for any $r > 0$,*

$$\|(a+y)^r\tilde{v}\|_{X^s} \leq a^r\|\tilde{v}\|_{X^s}. \tag{4.10a}$$

- *If $\left|K^TD\right|^{1/2}\tilde{v} \in X^s(\Omega)$, then for any $r > 0$,*

$$\left\|\left|K^TD\right|^{1/2}[(a+y)^r\tilde{v}]\right\|_{X^s} \leq a^r\left\|\left|K^TD\right|^{1/2}\tilde{v}\right\|_{X^s}. \tag{4.10b}$$

Remark 4.6. Equation (4.10a) holds because

$$\|(a+y)^r\tilde{v}\|_{X^s}^2 = \sum_{p\in\Gamma'}\langle p\rangle^s\left\|(a+y)^r\hat{v}_p(y)\right\|_{L^2((-a,0))}^2 \leq a^{2r}\|\tilde{v}\|_{X^s}^2.$$

The proof for Equation (4.10b) follows similarly.

4.3 | Elliptic Estimate

We now state the elliptic estimate we require concerning the generic boundary value problem

$$\operatorname{div}_\alpha\left[KK^T\nabla_\alpha\tilde{u}(\alpha,y)\right]+\partial_y^2\tilde{u}(\alpha,y) = \tilde{F}(\alpha,y), \quad -a<y<0, \tag{4.11a}$$

$$\tilde{u}(\alpha,0) = \tilde{\xi}(\alpha), \tag{4.11b}$$

$$\partial_y\tilde{u}(\alpha,-a)-\tilde{T}[\tilde{u}(\alpha,-a)] = \tilde{J}(\alpha), \tag{4.11c}$$

where

$$\tilde{F}(\alpha,y) := \operatorname{div}_\alpha[K\tilde{F}^\alpha(\alpha,y)]+\partial_y\tilde{F}^y(\alpha,y)+\tilde{F}^0(\alpha,y). \tag{4.11d}$$

For convenience, we define the maximum of a number of quantities we must control in our estimation.

Definition 4.7. Given an integer $s \geq 0$ we define the following maximum:

$$\mathcal{M}_s[\tilde{u}] := \max\left\{\|\tilde{u}\|_{X^s}, \left\|\partial_y\tilde{u}\right\|_{X^s}, \|K^T\nabla_\alpha\tilde{u}\|_{X^s},\right.$$
$$\left\|\left|K^TD\right|^{1/2}\tilde{u}\right\|_{X^s}, \left\|\left|K^TD\right|^{1/2}\partial_y\tilde{u}\right\|_{X^s}, \left\|\left|K^TD\right|^{1/2}K^T\nabla_\alpha\tilde{u}\right\|_{X^s},$$
$$\left.\left\|\partial_y\tilde{u}(\alpha,0)\right\|_{H^s}, \|K^T\nabla_\alpha\tilde{u}(\alpha,0)\|_{H^s}, \|\tilde{T}[\tilde{u}(\alpha,-a)]\|_{H^s}\right\}.$$

The elliptic estimate, proven in Appendix A, can now be stated as follows.

Theorem 4.8. *Given an integer $s > d/2$, provided that*

$$\tilde{\xi} \in H^{s+1}(P(\Gamma)), \quad \tilde{J} \in H^s(P(\Gamma)),$$

$$\tilde{F}^j, \left|K^T D\right|^{1/2} \tilde{F}^j \in X^s(\Omega), \quad \tilde{F}^j(\alpha, 0) \in H^s(P(\Gamma)), \quad j \in \{\alpha, y, 0\},$$

and

$$\tilde{F}^y(\alpha, -a) = 0,$$

there exists a unique solution $\tilde{u} \in X^s(\Omega)$ of (4.11) such that, for some $C_e > 0$,

$$\mathcal{M}_s[\tilde{u}] \le C_e \Big\{ \left\|\tilde{\xi}\right\|_{H^{s+1}} + \|\tilde{J}\|_{H^s} + \sum_{j\in\{\alpha,y,0\}} \Big(\|\tilde{F}^j\|_{X^s} + \left\|\left|K^T D\right|^{1/2} \tilde{F}^j\right\|_{X^s} + \|\tilde{F}^j(\alpha,0)\|_{H^s} \Big) \Big\}.$$

Remark 4.9. To illustrate the advance we have made in the current contribution, we point out the difference between the elliptic estimates in the periodic and quasiperiodic cases. In the (zero-mean) periodic case [27],

$$\Delta_x u + \partial_y^2 u = F \quad \Rightarrow \quad \hat{u}_p(y) = \left[\partial_y^2 + |p|^2\right]^{-1} \hat{F}_p \quad \Rightarrow \quad \|u\|_{X^{s+2}} \le C_e \|F\|_{X^s},$$

that is, u is two more orders regular than F. However, the same is *not* true in the (zero-mean) quasiperiodic case where

$$\operatorname{div}_\alpha \left[K K^T \nabla_\alpha \tilde{u}\right] + \partial_y^2 u = F \quad \Rightarrow \quad \hat{u}_p(y) = \left[\partial_y^2 + \left|K^T p\right|^2\right]^{-1} \hat{F}_p,$$

and the factor $\left|K^T p\right|$ can be arbitrarily close to zero as $|p| \to \infty$. This is why $\tilde{u}$ only belongs to X^s instead of X^{s+2} in Theorem 4.8. Moreover, $\left|K^T D\right| \tilde{u} \in H^s$ does *not* imply that $\tilde{u} \in H^{s+1}$ or even $\tilde{u} \in H^s$ (see Remark 4.1). Thus, $\|\tilde{u}\|_{X^s}$, $\|K^T \nabla_\alpha \tilde{u}\|_{X^s}$ and $\left\|\left|K^T D\right|^{1/2} \tilde{u}\right\|_{X^s}$ all appear in the elliptic estimates.

Remark 4.10. We point out that the condition $\tilde{F}^y(\alpha, -a) = 0$ is satisfied in Equation (4.4) above.

4.4 | A Recursive Lemma

To prove the analyticity of the field, we establish the following recursive estimate.

Lemma 4.11. *Given an integer $s > d/2$, if $\tilde{f} \in H^{s+3/2}(P(\Gamma))$ and*

$$\mathcal{M}_s[\tilde{u}_n] \le K_0 B^n, \quad \forall\, n < N$$

for constants $K_0, B > 0$, then we have, for all $j \in \{\alpha, y, 0\}$, the estimate

$$\max \left\{ \|\tilde{J}_N\|_{H^s}, \left\|\tilde{F}_N^j\right\|_{X^s}, \left\|\left|K^T D\right|^{1/2} \tilde{F}_N^j\right\|_{X^s}, \left\|\tilde{F}_N^j(\alpha,0)\right\|_{H^s} \right\} \le K_1 K_0 \Big(\|\tilde{f}\|_{H^{s+3/2}} B^{N-1} + \|\tilde{f}\|_{H^{s+3/2}}^2 B^{N-2} \Big)$$

for a positive constant $K_1 > 0$.

Proof. Noting that (cf. (4.4a))

$$\tilde{F}_N = \operatorname{div}_\alpha \left[K \tilde{F}_N^\alpha\right] + \partial_y \tilde{F}_N^y + \tilde{F}_N^0,$$

we focus on one representative term from each $\tilde{F}_N^j$; the proof for the other terms follows similarly. To begin we consider (cf. (4.4b)),

$$\tilde{F}_N^\alpha = \mathcal{A}_N + \cdots, \quad \mathcal{A}_N := \frac{(a+y)}{a^2} \tilde{f} (K^T \nabla_\alpha \tilde{f}) \partial_y \tilde{u}_{N-2},$$

and estimate, using (4.10b) and (4.9a),

$$\begin{aligned} \|\mathcal{A}_N\|_{X^s} &= \left\| \frac{(a+y)}{a^2} \tilde{f} (K^T \nabla_\alpha \tilde{f}) \partial_y \tilde{u}_{N-2} \right\|_{X^s} \\ &\le \frac{M^2}{a} \|\tilde{f}\|_{H^s} \|K^T \nabla_\alpha \tilde{f}\|_{H^s} \|\partial_y \tilde{u}_{N-2}\|_{X^s} \\ &\le K_1 \|\tilde{f}\|_{H^{s+3/2}}^2 K_0 B^{N-2}, \end{aligned}$$

and

$$\begin{aligned} \left\|\left|K^T D\right|^{1/2} \mathcal{A}_N\right\|_{X^s} &= \left\| \left|K^T D\right|^{1/2} \left[\frac{(a+y)}{a^2} \tilde{f} (K^T \nabla_\alpha \tilde{f}) \partial_y \tilde{u}_{N-2} \right] \right\|_{X^s} \\ &\le \frac{M^2}{a} \|\tilde{f}\|_{H^{s+1/2}} \|K^T \nabla_\alpha \tilde{f}\|_{H^{s+1/2}} \\ &\quad \times \left\{ \|\partial_y \tilde{u}_{N-2}\|_{X^s} + \left\|\left|K^T D\right|^{1/2} \partial_y \tilde{u}_{N-2}\right\|_{X^s} \right\} \\ &\le K_1 \|\tilde{f}\|_{H^{s+3/2}}^2 K_0 B^{N-2} \end{aligned}$$

for K_1 chosen appropriately. Also, since

$$\mathcal{A}_N(\alpha, 0) = \frac{1}{a} \tilde{f} (K^T \nabla_\alpha \tilde{f}) \partial_y \tilde{u}_{N-2}(\alpha, 0),$$

we have

$$\begin{aligned} \|\mathcal{A}_N(\alpha,0)\|_{H^s} &\le \left\| \frac{\tilde{f}}{a} (K^T \nabla_\alpha \tilde{f}) \partial_y \tilde{u}_{N-2}(\alpha,0) \right\|_{H^s} \\ &\le \frac{M^2}{a} \|\tilde{f}\|_{H^s} \|K^T \nabla_\alpha \tilde{f}\|_{H^s} \|\partial_y \tilde{u}_{N-2}(\alpha,0)\|_{H^s} \\ &\le K_1 \|f\|_{H^{s+3/2}}^2 K_0 B^{N-2} \end{aligned}$$

for K_1 large enough. Next we recall that (cf. (4.4c)),

$$\tilde{F}_N^y = \mathcal{B}_N + \cdots, \quad \mathcal{B}_N := -\frac{(a+y)^2}{a^2} \left|K^T \nabla_\alpha \tilde{f}\right|^2 \partial_y \tilde{u}_{N-2},$$

and compute, with (4.10b) and (4.9a),

$$\begin{aligned} \|\mathcal{B}_N\|_{X^s} &= \left\| \frac{(a+y)^2}{a^2} \left|K^T \nabla_\alpha \tilde{f}\right|^2 \partial_y \tilde{u}_{N-2} \right\|_{X^s} \\ &\le M^2 \|K^T \nabla_\alpha \tilde{f}\|_{H^s}^2 \|\partial_y \tilde{u}_{N-2}\|_{X^s} \\ &\le K_1 \|\tilde{f}\|_{H^{s+3/2}}^2 K_0 B^{N-2}, \end{aligned}$$

and

$$\begin{aligned} \left\|\left|K^T D\right|^{1/2} \mathcal{B}_N\right\|_{X^s} &= \left\| \left|K^T D\right|^{1/2} \left[\frac{(a+y)^2}{a^2} \left|K^T \nabla_\alpha \tilde{f}\right|^2 \partial_y \tilde{u}_{N-2} \right] \right\|_{X^s} \\ &\le M^2 \|K^T \nabla_\alpha \tilde{f}\|_{H^{s+1/2}}^2 \\ &\quad \times \left\{ \|\partial_y \tilde{u}_{N-2}\|_{X^s} + \left\|\left|K^T D\right|^{1/2} \partial_y \tilde{u}_{N-2}\right\|_{X^s} \right\} \\ &\le K_1 \|\tilde{f}\|_{H^{s+3/2}}^2 K_0 B^{N-2}, \end{aligned}$$

if K_1 is chosen large enough. Also, since

$$\mathcal{B}_N(\alpha,0) = -\left|K^T\nabla_\alpha\tilde{f}\right|^2\partial_y\tilde{u}_{N-2}(\alpha,0),$$

we have

$$\begin{aligned}\|\mathcal{B}_N(\alpha,0)\|_{H^s} &\le \left\|\left|K^T\nabla_\alpha\tilde{f}\right|^2\partial_y\tilde{u}_{N-2}(\alpha,0)\right\|_{H^s}\\ &\le M^2\left\|K^T\nabla_\alpha\tilde{f}\right\|_{H^s}^2\left\|\partial_y\tilde{u}_{N-2}(\alpha,0)\right\|_{H^s}\\ &\le K_1\|f\|_{H^{s+3/2}}^2 K_0B^{N-2}\end{aligned}$$

for K_1 large enough. Finally (cf. (4.4d)),

$$\tilde{F}_N^0 = \mathcal{C}_N + \cdots, \quad \mathcal{C}_N := -\frac{(a+y)}{a^2}\left|K^T\nabla_\alpha\tilde{f}\right|^2\partial_y\tilde{u}_{N-2},$$

and we estimate, using (4.10b) and (4.9a),

$$\begin{aligned}\|\mathcal{C}_N\|_{X^s} &= \left\|\frac{(a+y)}{a^2}\left|K^T\nabla_\alpha\tilde{f}\right|^2\partial_y\tilde{u}_{N-2}\right\|_{X^s}\\ &\le \frac{M^2}{a}\left\|K^T\nabla_\alpha\tilde{f}\right\|_{H^s}^2\left\|\partial_y\tilde{u}_{N-2}\right\|_{X^s}\\ &\le K_1\left\|\tilde{f}\right\|_{H^{s+3/2}}^2 K_0B^{N-2},\end{aligned}$$

and

$$\begin{aligned}\left\|\left|K^TD\right|^{1/2}\mathcal{C}_N\right\|_{X^s} &= \left\|\left|K^TD\right|^{1/2}\left[\frac{(a+y)}{a^2}\left|K^T\nabla_\alpha\tilde{f}\right|^2\partial_y\tilde{u}_{N-2}\right]\right\|_{X^s}\\ &\le \frac{M^2}{a}\left\|K^T\nabla_\alpha\tilde{f}\right\|_{H^{s+1/2}}^2\\ &\quad\times\left\{\left\|\partial_y\tilde{u}_{N-2}\right\|_{X^s} + \left\|\left|K^TD\right|^{1/2}\partial_y\tilde{u}_{N-2}\right\|_{X^s}\right\}\\ &\le K_1\left\|\tilde{f}\right\|_{H^{s+3/2}}^2 K_0B^{N-2}\end{aligned}$$

for K_1 sufficiently large. Also, since

$$\mathcal{C}_N(\alpha,0) = -\frac{1}{a}\left|K^T\nabla_\alpha\tilde{f}\right|^2\partial_y\tilde{u}_{N-2}(\alpha,0),$$

we have

$$\begin{aligned}\|\mathcal{C}_N(\alpha,0)\|_{H^s} &\le \left\|\frac{1}{a}\left|K^T\nabla_\alpha\tilde{f}\right|^2\partial_y\tilde{u}_{N-2}(\alpha,0)\right\|_{H^s}\\ &\le \frac{M^2}{a}\left\|K^T\nabla_\alpha\tilde{f}\right\|_{H^s}^2\left\|\partial_y\tilde{u}_{N-2}(\alpha,0)\right\|_{H^s}\\ &\le K_1\|f\|_{H^{s+3/2}}^2 K_0B^{N-2}\end{aligned}$$

for K_1 large enough. To close (cf. (4.4e)),

$$\tilde{J}_N = \frac{1}{a}\tilde{f}\tilde{T}[\tilde{u}_{N-1}(\alpha,-a)],$$

and we compute, using (4.8a),

$$\begin{aligned}\|\tilde{J}_N\|_{H^s} &= \left\|\frac{1}{a}\tilde{f}\tilde{T}[\tilde{u}_{N-1}(\alpha,-a)]\right\|_{H^s} \le \frac{M}{a}\|\tilde{f}\|_{H^s}\|\tilde{T}[\tilde{u}_{N-1}(\alpha,-a)]\|_{H^s}\\ &\le \frac{M}{a}\left\|\tilde{f}\right\|_{H^s}K_0B^{N-1} \le K_1\left\|\tilde{f}\right\|_{H^{s+3/2}}K_0B^{N-1}\end{aligned}$$

for K_1 big enough. □

4.5 | Analyticity of the Field

We are now in a position to establish the analyticity of the transformed field $\tilde{u}$ in a sense which we now make precise.

Theorem 4.12. *Given an integer $s > d/2$, if $\tilde{f} \in H^{s+3/2}(P(\Gamma))$ and $\tilde{\xi} \in H^{s+1}(P(\Gamma))$ there exists a unique solution*

$$\tilde{u}(\alpha,y;\varepsilon) = \sum_{n=0}^{\infty}\tilde{u}_n(\alpha,y)\varepsilon^n$$

of (4.1) satisfying

$$\mathcal{M}_s[\tilde{u}_n] \le K_0B^n, \quad \forall\, n \ge 0 \tag{4.12}$$

for any $B > C_0\left\|\tilde{f}\right\|_{H^{s+3/2}}$ and positive constants $K_0, C_0 > 0$.

Proof. We work by induction on n. At order $n = 0$, we must solve (4.3) where $\tilde{F}_0 \equiv \tilde{J}_n \equiv 0$. From Theorem 4.8, we have that

$$\mathcal{M}_s[\tilde{u}_0] \le C_e\left\|\tilde{\xi}\right\|_{H^{s+1}} =: K_0 < \infty.$$

We now assume estimate (4.12) for all $n < N$ and study $\mathcal{M}_s[\tilde{u}_N]$. For this, we invoke the elliptic estimate, Theorem 4.8, to realize that

$$\mathcal{M}_s[u_N] \le C_e \left\{\|\tilde{J}_N\|_{H^s} + \sum_{j\in\{\alpha,y,0\}}\left(\left\|\tilde{F}_N^j\right\|_{X^s} + \left\|\left|K^TD\right|^{1/2}\tilde{F}_N^j\right\|_{X^s} + \left\|\tilde{F}_N^j(\alpha,0)\right\|_{H^s}\right)\right\}.$$

From the recursive estimate in Lemma 4.11, we now deduce that

$$\mathcal{M}_s[u_N] \le C_e 10K_1K_0\left(\left\|\tilde{f}\right\|_{H^{s+3/2}}B^{N-1} + \left\|\tilde{f}\right\|_{H^{s+3/2}}B^{N-2}\right).$$

We realize

$$\mathcal{M}_s[u_N] \le K_0B^N,$$

provided that, for instance,

$$B > \max\left\{20C_eK_1, \sqrt{20C_eK_1}\right\}\left\|\tilde{f}\right\|_{H^{s+3/2}},$$

and we are done. □

4.6 | Analyticity of the DNO

At last, we can establish the analyticity of the DNO, $\tilde{G}$. More specifically, we prove the following result.

Theorem 4.13. *Given an integer $s > d/2$, if $\tilde{f} \in H^{s+3/2}(P(\Gamma))$ and $\tilde{\xi} \in H^{s+1}(P(\Gamma))$ then the series*

$$\tilde{G}(\varepsilon\tilde{f}) = \sum_{n=0}^{\infty}\tilde{G}_n(\tilde{f})\varepsilon^n \tag{4.13}$$

converges strongly as an operator from $H^{s+1}(P(\Gamma))$ to $H^s(P(\Gamma))$. More precisely,

$$\left\|\tilde{G}_n(\tilde{f})[\tilde{\xi}]\right\|_{H^s} \le K_2B^n, \quad \forall\, n \ge 0 \tag{4.14}$$

for any $B > C_0 \|\tilde{f}\|_{H^{s+3/2}}$ and positive constant $K_2 > 0$.

Proof. We work by induction in n and begin with the formula for $\tilde{G}_0$,

$$\tilde{G}_0[\tilde{\xi}] = \begin{cases} \sum_{p\in\Gamma'} |K^T p|\hat{\xi}_p \mathrm{e}^{i\alpha\cdot p}, & h = \infty, \\ \sum_{p\in\Gamma'} |K^T p| \tanh(h|K^T D|)\hat{\xi}_p \mathrm{e}^{i\alpha\cdot p}, & h < \infty. \end{cases}$$

We focus our attention on the infinite depth case ($h = \infty$) and note that the finite depth case ($h < \infty$) can be established in a similar fashion. We estimate

$$\left\|\tilde{G}_0[\tilde{\xi}]\right\|_{H^s}^2 \le \sum_{p\in\Gamma'} \langle p\rangle^{2s}|K^T p|^2\left|\hat{\xi}_p\right|^2 \le C \sum_{p\in\Gamma'} \langle p\rangle^{2(s+1)}\left|\hat{\xi}_p\right|^2 \le \left\|\tilde{\xi}\right\|_{H^{s+1}}^2 =: K_2.$$

We now assume (4.14) for all $n < N$ and, from (4.5), we estimate

$$\begin{aligned}\left\|\tilde{G}_N(\tilde{f})[\tilde{\xi}]\right\|_{H^s} \le{}& \left\|\partial_y \tilde{u}_N(\alpha,0)\right\|_{H^s} + \|(K^T\nabla_\alpha \tilde{f})\cdot(K^T\nabla_\alpha \tilde{u}_{N-1}(\alpha,0))\|_{H^s} \\ &+ \frac{1}{a}\left\|\tilde{f}\tilde{G}_{N-1}(\tilde{f})[\tilde{\xi}]\right\|_{H^s} \\ &+ \frac{1}{a}\|\tilde{f}(K^T\nabla_\alpha \tilde{f})\cdot(K^T\nabla_\alpha \tilde{u}_{N-2}(\alpha,0))\|_{H^s} \\ &+ \left\||K^T\nabla_\alpha \tilde{f}|^2\partial_y \tilde{u}_{N-2}(\alpha,0)\right\|_{H^s}.\end{aligned}$$

Now, from (4.8a) we have

$$\begin{aligned}\left\|\tilde{G}_N(\tilde{f})[\tilde{\xi}]\right\|_{H^s} \le{}& \left\|\partial_y \tilde{u}_N(\alpha,0)\right\|_{H^s} + M\|K^T\nabla_\alpha \tilde{f}\|_{H^s}\|K^T\nabla_\alpha \tilde{u}_{N-1}(\alpha,0)\|_{H^s} \\ &+ \frac{1}{a}M\|\tilde{f}\|_{H^s}\left\|\tilde{G}_{N-1}(\tilde{f})[\tilde{\xi}]\right\|_{H^s} \\ &+ \frac{1}{a}M^2\|\tilde{f}\|_{H^s}\|K^T\nabla_\alpha \tilde{f}\|_{H^s}\|K^T\nabla_\alpha \tilde{u}_{n-2}(\alpha,0)\|_{H^s} \\ &+ M^2\|K^T\nabla_\alpha \tilde{f}\|_{H^s}^2\left\|\partial_y \tilde{u}_{N-2}(\alpha,0)\right\|_{H^s}.\end{aligned}$$

From (4.12), we have

$$\begin{aligned}\left\|\tilde{G}_N(\tilde{f})[\tilde{\xi}]\right\|_{H^s} \le{}& K_0B^N + M\|\tilde{f}\|_{H^{s+3/2}}K_0B^{N-1} + \frac{1}{a}M\|\tilde{f}\|_{H^{s+3/2}}K_2B^{N-1} \\ &+ \frac{1}{a}M^2\|\tilde{f}\|_{H^{s+3/2}}\|\tilde{f}\|_{H^{s+3/2}}K_0B^{N-2} + M^2\|\tilde{f}\|_{H^{s+3/2}}^2 K_0B^{N-2} \\ \le{}& K_2B^N\end{aligned}$$

provided that each of the five terms is bounded above by $(K_2/5)B^N$. This demands that

$$K_2 \ge 5K_0, \quad B \ge M \max\left\{\frac{5K_0}{K_2}, \frac{5}{a}, \sqrt{\frac{5K_0}{aK_2}}, \sqrt{\frac{5K_0}{K_2}}\right\}\|\tilde{f}\|_{H^{s+3/2}},$$

and accommodating all of these delivers the theorem. □

Remark 4.14. We emphasize at this point that both Theorems 4.12 and 4.13 and their proofs are true for any physical dimension $n \ge 2$. In addition, in the following sections we describe results of numerical simulations in both the $n = 2$ and $n = 3$ cases, illustrating how the formulation can be readily utilized in any dimension.

5 | Numerical Algorithms

To illustrate the possibilities enabled by the analyticity theory outlined above, we now briefly describe and validate three HOPS algorithms [26–28] suggested by this. To keep our discussion focused, we use the two-dimensional geometry ($x, y \in \mathbf{R}$) as an example to explain our numerical algorithms, which feature lateral quasiperiodicity specified by the matrix $K \in \mathbf{R}^{2\times 1}$ (two base periods). Later, we also showcase much more intensive three-dimensional computations ($x \in \mathbf{R}^2$, $y \in \mathbf{R}$) with lateral quasiperiodicity determined by $K \in \mathbf{R}^{3\times 2}$ in Section 6.3.

5.1 | HOS Methods

In light of the separable geometries we consider (the fundamental period cell determined by the lattice Γ, $P(\Gamma)$, in tensor product with the interval $[-a, 0]$) we chose HOS methods to discretize each of our algorithms. The interested reader should consult one of the excellent texts on the topic (e.g., [14, 15, 50, 51]) for complete details as we now provide only a brief description of the concepts we require in this subsection.

In our schemes, we approximated doubly 2π-periodic functions by their truncated Fourier series, for example,

$$\tilde{f}(\alpha) \approx \tilde{f}^{N_\alpha}(\alpha) := \sum_{p\in\mathcal{P}_{N_\alpha}} \hat{f}_p \mathrm{e}^{ip\cdot\alpha},$$

where

$$p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, \quad N_\alpha = \begin{pmatrix} N_\alpha^1 \\ N_\alpha^2 \end{pmatrix},$$

$$\mathcal{P}_{N_\alpha} = \left\{ p \in \Gamma' \;\middle|\; -\frac{N_\alpha^m}{2} \le p_m \le \frac{N_\alpha^m}{2} - 1, \quad m = 1, 2 \right\}.$$

The gradient of such a function can be readily approximated with high accuracy by

$$\nabla_\alpha \tilde{f}(\alpha) \approx \sum_{p\in\mathcal{P}_{N_\alpha}} (ip)\hat{f}_p \mathrm{e}^{ip\cdot\alpha},$$

as can Fourier multipliers, for example,

$$m(D)[\tilde{f}(\alpha)] \approx \sum_{p\in\mathcal{P}_{N_\alpha}} m(p)\hat{f}_p \mathrm{e}^{ip\cdot\alpha}.$$

Using well-known procedures involving the fast Fourier transform (FFT), products of these functions can be simulated at the equally spaced gridpoints

$$\alpha_j^m = \frac{2\pi j_m}{N_\alpha^m}, \quad 0 \le j_m \le N_\alpha^m - 1, \quad m = 1, 2.$$

This can also be used in conjunction with the Trapezoidal Rule to approximate the Fourier coefficients, $\hat{f}_p$, with very high accuracy (in fact, spectral, if $\tilde{f}(\alpha)$ is analytic) [14, 15, 50, 51].

In relation to the recursions we have analyzed in our theory above, we also require the approximation of volumetric functions. For this, we simulate laterally doubly 2π-periodic functions of y on the interval $[-a, 0]$ by their truncated Fourier–Chebyshev series, for example,

$$\tilde{u}(\alpha, y) \approx \tilde{u}^{N_\alpha, N_y}(\alpha, y) := \sum_{p \in \mathcal{P}_{N_\alpha}} \sum_{q=0}^{N_y} \hat{u}_{p,q} T_q\left(\frac{2y+a}{a}\right) e^{ip\cdot\alpha}.$$

Again, classical techniques from the theory of Spectral Methods can be used to compute derivatives of these functions with respect to both α and y. As above, using methods involving the Fast Chebyshev Transform, products of these functions can be simulated at the Chebyshev points

$$y_r = \frac{a}{2}\left(\cos\left(\frac{\pi r}{N_y}\right) - 1\right), \quad 0 \le r \le N_y.$$

Once again, these can also be used in conjunction with quadrature formulas to approximate the Fourier–Chebyshev coefficients, $\hat{u}_{p,q}$, with high fidelity.

5.2 | The Method of OE

The first HOPS method we discuss is the classical Method of OE due to Milder [52–55] and Craig and Sulem [9]. In the infinite depth case ($h = \infty$), the description of this approach begins by defining the function

$$\tilde{\varphi}_p(\alpha, y) := e^{|K^T p| y} e^{ip\cdot\alpha}$$

for some $p \in \mathbf{Z}^2$. This function satisfies the quasiperiodic Laplace's equation, (3.6a), and the boundary conditions, (3.6c) and (3.6d), so that we can insert it into the definition of the quasiperiodic DNO, (3.4), to realize the true statement

$$\tilde{G}(\tilde{g})[\tilde{\varphi}_p(\alpha, \tilde{g})] = \partial_y \tilde{\varphi}_p(\alpha, \tilde{g}) - (K^T \nabla_\alpha \tilde{g}) \cdot (K^T \nabla_\alpha \tilde{\varphi}(\alpha, \tilde{g})).$$

Setting $\tilde{g}(\alpha) = \varepsilon \tilde{f}(\alpha)$ and using the analyticity of the DNO with respect to ε, we expand to find

$$\left(\sum_{n=0}^{\infty} \tilde{G}_n(\tilde{f})\varepsilon^n\right)\left[\sum_{m=0}^{\infty} \tilde{F}_m |K^T p|^m e^{ip\cdot\alpha}\varepsilon^m\right] = \sum_{n=0}^{\infty} \tilde{F}_n |K^T p|^{n+1} e^{ip\cdot\alpha}\varepsilon^n - (\varepsilon K^T \nabla_\alpha \tilde{f})\left(\sum_{n=0}^{\infty} \tilde{F}_n (iK^T p)|K^T p|^n e^{ip\cdot\alpha}\varepsilon^n\right),$$

where

$$\tilde{F}_n(\alpha) := \frac{\tilde{f}(\alpha)^n}{n!}.$$

At order $\mathcal{O}(\varepsilon^0)$, we find

$$\tilde{G}_0\left[e^{ip\cdot\alpha}\right] = |K^T p| e^{ip\cdot\alpha},$$

while at order $\mathcal{O}(\varepsilon^n)$ for $n > 0$ we find

$$\tilde{G}_n(\tilde{f})\left[e^{ip\cdot\alpha}\right] = \tilde{F}_n |K^T p|^{n+1} e^{ip\cdot\alpha} - (K^T \nabla_\alpha \tilde{f})\tilde{F}_{n-1}(iK^T p)|K^T p|^{n-1} e^{ip\cdot\alpha} - \sum_{\ell=0}^{n-1} \tilde{G}_\ell(\tilde{f})\left[\tilde{F}_{n-\ell}|K^T p|^{n-\ell} e^{ip\cdot\alpha}\right].$$

Now, if we express the generic function $\tilde{\xi}$ in the Fourier series

$$\tilde{\xi}(\alpha) = \sum_{p \in \Gamma'} \hat{\xi}_p e^{ip\cdot\alpha},$$

then we can conclude that

$$\tilde{G}_0[\tilde{\xi}] = \tilde{G}_0\left[\sum_{p\in\Gamma'} \hat{\xi}_p e^{ip\cdot\alpha}\right] = \sum_{p\in\Gamma'} \hat{\xi}_p \tilde{G}_0\left[e^{ip\cdot\alpha}\right] = \sum_{p\in\Gamma'} |K^T p| \hat{\xi}_p e^{ip\cdot\alpha} =: |K^T D|\tilde{\xi}, \tag{5.1}$$

which we use to define the order-one Fourier multiplier $|K^T D|$. In a similar fashion, we express, for $n > 0$,

$$\tilde{G}_n(\tilde{f})[\tilde{\xi}] = \tilde{F}_n |K^T D|^{n+1}\tilde{\xi} - (K^T \nabla_\alpha \tilde{f})\tilde{F}_{n-1}(iK^T D)|K^T D|^{n-1}\tilde{\xi} - \sum_{\ell=0}^{n-1} \tilde{G}_\ell(\tilde{f})\left[\tilde{F}_{n-\ell}|K^T D|^{n-\ell}\tilde{\xi}\right],$$

and we point out the *recursive* nature of this formula: $\tilde{G}_\ell$ must be recomputed for all $0 \le \ell \le n-1$ at each perturbation order n.

It was noted in [56, 57] that the self-adjointness of the DNO could be used to advantage in producing a *rapid* version of these recursions. Recalling that the adjoint operation satisfies $(AB)^* = B^* A^*$ for any two linear operators A and B, we can express $\tilde{G}_n = \tilde{G}_n^*$ as

$$\tilde{G}_n(\tilde{f})[\tilde{\xi}] = |K^T D|^{n+1}\left[\tilde{F}_n \tilde{\xi}\right] - |K^T D|^{n-1}(iK^T D)\left[(K^T \nabla_\alpha \tilde{f})\tilde{F}_{n-1}\tilde{\xi}\right] - \sum_{\ell=0}^{n-1} |K^T D|^{n-\ell}\left[\tilde{F}_{n-\ell}\tilde{G}_\ell(\tilde{f})[\tilde{\xi}]\right], \tag{5.2}$$

which is much faster than the formula above as the $\tilde{G}_\ell[\tilde{\xi}]$ may simply be stored from one perturbation order to the next rather than recomputed from scratch.

The OE algorithm is a Fourier HOS discretization of the formulas (5.1) and (5.2) where we begin by considering $\tilde{g}(\alpha) = \varepsilon \tilde{f}(\alpha)$, expanding

$$\tilde{G}(\alpha; \varepsilon) = \sum_{n=0}^{\infty} \tilde{G}_n \varepsilon^n,$$

and approximating

$$\tilde{G}(\alpha; \varepsilon) \approx \tilde{G}^N(\alpha; \varepsilon) := \sum_{n=0}^{N} \tilde{G}_n(\alpha)\varepsilon^n,$$

where

$$\tilde{G}_n(\alpha) \approx \tilde{G}_n^{N_\alpha}(\alpha) := \sum_{p\in\mathcal{P}_{N_\alpha}} \hat{G}_{p,n} e^{ip\cdot\alpha}. \tag{5.3}$$

Given

$$\tilde{\xi}^{N_\alpha}(\alpha) = \sum_{p\in\mathcal{P}_{N_\alpha}} \hat{\xi}_p e^{ip\cdot\alpha},$$

we then insert these forms into (5.1) and (5.2), and demand these be true for $0 \le n \le N$. Using the approach outlined in Section 5.1 the coefficients $\{\hat{G}_{p,n}\}$ are readily recovered enabling the approximation $\tilde{\nu}^{N,N_\alpha}$.

5.3 | The Method of FE

The second HOPS method we discuss is Bruno and Reitich's Method of FE [58–60]. To describe this approach, we again consider $\tilde{g}(\alpha) = \varepsilon \tilde{f}(\alpha)$ and note that the solution of (3.6) will depend upon ε; we assume that this dependence is *analytic* so that

$$\tilde{\varphi}(\alpha, y; \varepsilon) = \sum_{n=0}^{\infty} \tilde{\varphi}_n(\alpha, y)\varepsilon^n.$$

It can be shown that these $\tilde{\varphi}_n$ satisfy

$$\operatorname{div}_\alpha \left[KK^T \nabla_\alpha \tilde{\varphi}_n(\alpha, y)\right] + \partial_y^2 \tilde{\varphi}_n(\alpha, y) = 0, \qquad -a < y < 0, \tag{5.4a}$$

$$\tilde{\varphi}_n(\alpha, 0) = \delta_{n,0}\tilde{\xi}(\alpha) + Q_n(\alpha), \qquad y = 0, \tag{5.4b}$$

$$\partial_y \tilde{\varphi}_n - \tilde{T}[\tilde{\varphi}_n] = 0, \qquad y = -a, \tag{5.4c}$$

$$\tilde{\varphi}_n(\alpha + \gamma, y) = \tilde{\varphi}_n(\alpha, y), \qquad \gamma \in \Gamma, \tag{5.4d}$$

where

$$Q_n(\alpha) = -\sum_{\ell=0}^{n-1} \tilde{F}_{n-\ell}\partial_y^{n-\ell}\tilde{\varphi}_\ell(\alpha, 0). \tag{5.4e}$$

In the infinite depth case ($h = \infty$), the solutions of (5.4a), (5.4c), and (5.4d) can be expressed as

$$\tilde{\varphi}_n(\alpha, y) = \sum_{p \in \Gamma'} a_{n,p} e^{|K^T p| y} e^{ip\cdot\alpha},$$

while (5.4b) and (5.4e) can be used to discover the $\{a_{n,p}\}$. More specifically,

$$a_{n,p} = \delta_{n,0}\hat{\xi}_p - \sum_{q \in \Gamma'} \sum_{\ell=0}^{n-1} \hat{F}_{n-\ell,p-q} \left|K^T q\right|^{n-\ell} a_{\ell,q}, \tag{5.5}$$

which constitute the FE recursions.

The FE approach is a Fourier HOS approximation of the recursions (5.5) which recovers the $\{a_{n,p}\}$ given the Fourier coefficients $\{\hat{\xi}_p, \hat{f}_p\}$. With these, it is not difficult to approximate the $\tilde{G}_n$ which gives a simulation of the form (5.3).

5.4 | The Method of TFE

The final HOPS scheme we discuss is Nicholls and Reitich's Method of TFE [26–28, 61, 62]. Simply put, this algorithm is a HOS [14, 15, 50, 51] discretization and approximation of the transformed problem (4.3) which we utilized in our theoretical demonstrations earlier. In a bit more detail, we consider $\tilde{g}(\alpha) = \varepsilon \tilde{f}(\alpha)$ and use our *rigorous* demonstration of the *analyticity* of the solution to write

$$\tilde{u}(\alpha, y; \varepsilon) = \sum_{n=0}^{\infty} \tilde{u}_n(\alpha, y)\varepsilon^n.$$

We begin by approximating

$$\tilde{u}(\alpha; y; \varepsilon) \approx \tilde{u}^N(\alpha, y; \varepsilon) := \sum_{n=0}^{N} \tilde{u}_n(\alpha, y)\varepsilon^n, \tag{5.6}$$

and then represent

$$\tilde{u}_n(\alpha, y) \approx \sum_{p \in \mathcal{P}_N} \sum_{q=0}^{N_y} \hat{u}_{p,q,n} e^{ip\cdot\alpha} T_q\left(\frac{2y + a}{a}\right).$$

We then insert this form into (4.3) and utilize the collocation approach as described in Section 5.1. Using well-known procedures involving the FFT and the Fast Cheybshev Transform [14, 15, 50, 51] the coefficients $\{\hat{u}_{p,q,n}\}$ are readily recovered. These, in turn, can be used to approximate the DNO, $\tilde{G}$.

5.5 | Padé Approximation

An important question is how the Taylor series, for example, (5.6), in ε is summed, for instance, the approximation of $\hat{u}_{p,q}(\varepsilon)$ by

$$\hat{u}^N_{p,q}(\varepsilon) := \sum_{n=0}^{N} \hat{u}_{p,q,n}\varepsilon^n.$$

As we have seen in [28, 59, 63], Padé approximation [29] has been used in conjunction with HOPS methods with great success and we recommend its use here. Padé approximation estimates the truncated Taylor series $\hat{u}^N_{p,q}(\varepsilon)$ by the rational function

$$[L/M](\varepsilon) := \frac{a^L(\varepsilon)}{b^M(\varepsilon)} = \frac{\sum_{\ell=0}^{L} a_\ell \varepsilon^\ell}{1 + \sum_{m=1}^{M} b_m \varepsilon^m}, \quad L + M = N,$$

and

$$[L/M](\varepsilon) = \hat{u}^N_{p,q}(\varepsilon) + \mathcal{O}(\varepsilon^{L+M+1});$$

classical formulas for the coefficients $\{a_\ell, b_m\}$ can be found in [29], though we utilized the sophisticated implementation of Gonnet et al. [64] in our numerics which follow. This method has stunning properties of enhanced convergence, and we refer the interested reader to Section 2.2 of Baker and Graves–Morris [29] and the insightful calculations of Section 8.3 of Bender and Orszag [65] for a thorough discussion of the capabilities and limitations of Padé approximants.

6 | Numerical Results

We now present numerical results which not only demonstrate the validity of our implementations of the three algorithms presented above (OE, FE, and TFE) but also illuminate their characteristics and behavior. With each of these, we see that for small and smooth interfaces (e.g., $\tilde{f}$ analytic and $\varepsilon \ll 1$) all three algorithms deliver highly accurate solutions in a stable and efficient manner. However, for large and rough profiles (e.g., $\tilde{f}$

Lipschitz and $\varepsilon = \mathcal{O}(1)$), the enhanced stability properties of the TFE method [27, 62] and superior accuracy of Padé summation become extremely important to realize accurate solutions.

6.1 | The Method of Manufactured Solutions

To test the validity of our implementation, we used the Method of Manufactured Solutions [66–68]. To summarize this, consider the general system of PDEs subject to generic boundary conditions

$$\mathcal{P}v = 0, \qquad \text{in } \Omega,$$
$$\mathcal{B}v = 0, \qquad \text{at } \partial\Omega.$$

It is typically just as easy to implement a numerical algorithm to solve the nonhomogeneous version of this set of equations

$$\mathcal{P}v = \mathcal{F}, \qquad \text{in } \Omega,$$
$$\mathcal{B}v = \mathcal{J}, \qquad \text{at } \partial\Omega.$$

To validate our code, we began with the "manufactured solution," $\tilde{v}$, and set

$$\mathcal{F}_{\tilde{v}} := \mathcal{P}\tilde{v}, \quad \mathcal{J}_{\tilde{v}} := \mathcal{J}\tilde{v}.$$

Thus, given the pair $\{\mathcal{F}_{\tilde{v}}, \mathcal{J}_{\tilde{v}}\}$ we had an *exact* solution of the nonhomogeneous problem, namely, $\tilde{v}$. While this does not prove an implementation to be correct, if the function $\tilde{v}$ is chosen to imitate the behavior of anticipated solutions (e.g., satisfying the boundary conditions exactly) then this gives us confidence in our algorithm.

6.2 | Two-Dimensional Simulations

For a two-dimensional fluid of infinite depth ($h = \infty$), we considered the laterally doubly 2π-periodic function

$$\tilde{\varphi}^{\text{exact}}(\alpha, y) = A_q e^{|K^T q| y} e^{i q \cdot \alpha}, \quad K \in \mathbf{R}^{2\times 1}, \quad q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \in \mathbf{Z}^2,$$

from which we readily computed

$$\tilde{\xi}^{\text{exact}}(\alpha) = \tilde{\varphi}^{\text{exact}}(\alpha, \tilde{g}(\alpha)), \tag{6.1a}$$

$$\tilde{\nu}^{\text{exact}}(\alpha) = \partial_y \tilde{\varphi}^{\text{exact}}(\alpha, \tilde{g}(\alpha)) - \left(K^T \nabla_\alpha \tilde{g}(\alpha)\right) \cdot K^T \nabla_\alpha \tilde{\varphi}^{\text{exact}}(\alpha, \tilde{g}(\alpha)), \tag{6.1b}$$

and, from these,

$$\varphi^{\text{exact}}(x, y) = \tilde{\varphi}^{\text{exact}}(Kx, y), \quad \xi^{\text{exact}}(x) = \tilde{\xi}^{\text{exact}}(Kx), \quad \nu^{\text{exact}}(x) = \tilde{\nu}^{\text{exact}}(Kx).$$

We chose the following physical parameters:

$$A_q = -3, \quad q = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad K = \begin{pmatrix} 1 \\ \kappa \end{pmatrix}, \quad \kappa = \sqrt{2}, \tag{6.2}$$

and the numerical parameters

$$N_\alpha = \begin{pmatrix} 64 \\ 64 \end{pmatrix}, \quad N_y = 16, \quad N = 16, \quad a = \frac{1}{10}. \tag{6.3}$$

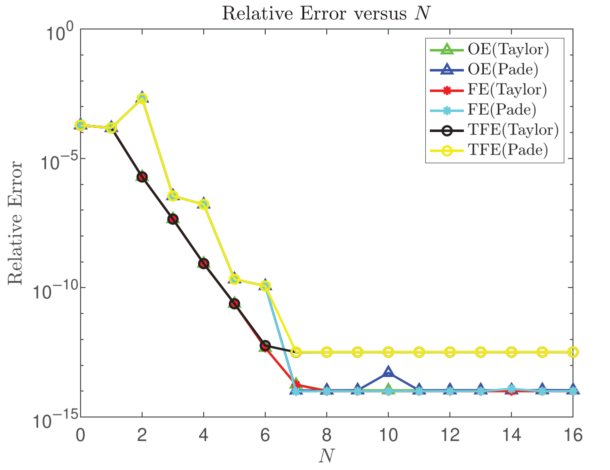

FIGURE 1 | Plot of relative error, (6.4), for a small perturbation ($\varepsilon = 0.02$) in the surface Neumann data for the three HOPS algorithms (OE, FE, and TFE) using both Taylor and Padé summation. Physical parameters were (6.2) and numerical discretization was (6.3).

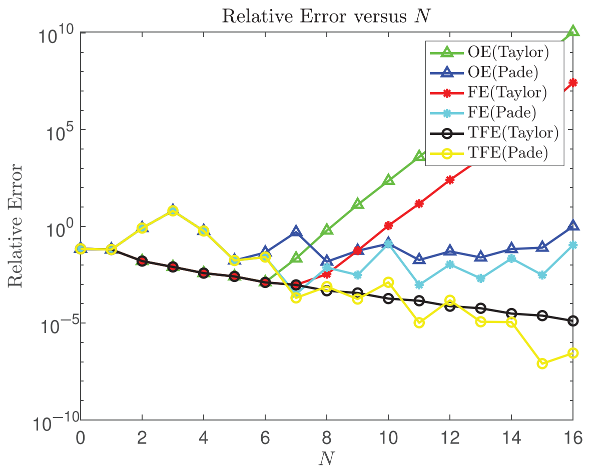

FIGURE 2 | Plot of relative error, (6.4), for a medium perturbation ($\varepsilon = 0.5$) in the surface Neumann data for the three HOPS algorithms (OE, FE, and TFE) using both Taylor and Padé summation. Physical parameters were (6.2) and numerical discretization was (6.3).

To illuminate the behavior of our scheme, we considered

$$\tilde{g}(\alpha) = \varepsilon \tilde{f}(\alpha), \quad \tilde{f}(\alpha) = \cos(\alpha_1)\sin(\alpha_2),$$

so that $g(x) = \varepsilon f(x)$, and studied three choices

$$\varepsilon = 0.02, 0.5, 1.$$

For this, we supplied the "exact" input data, $\tilde{\xi}^{\text{exact}}$, from (6.1) to our HOPS algorithm and compared the output of this, $\tilde{\nu}^{\text{approx}}$, with the "exact" output, $\tilde{\nu}^{\text{exact}}$, by computing the relative error

$$\text{Error}_{\text{rel}} := \frac{\left|\left|\tilde{\nu}^{\text{exact}} - \tilde{\nu}^{\text{approx}}\right|\right|_{L^\infty}}{\left|\left|\tilde{\nu}^{\text{exact}}\right|\right|_{L^\infty}}. \tag{6.4}$$

To evaluate our implementation and demonstrate the behavior of each of the three schemes, we report our results in Figures 1, 2, and 3. More specifically, Figure 1 ($\varepsilon = 0.02$) shows both the rapid and stable decay of the relative error for this small perturbation

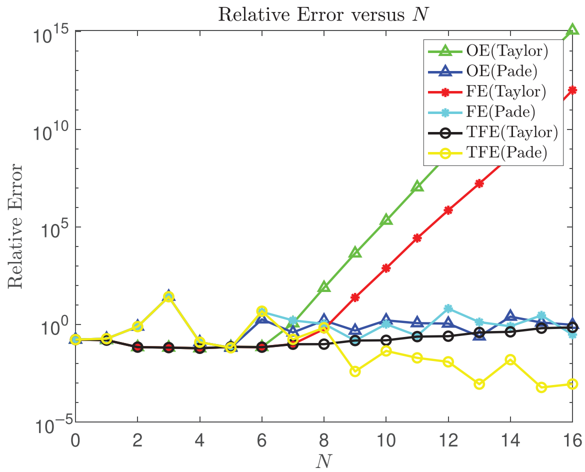

FIGURE 3 | Plot of relative error, (6.4), for a large perturbation ($\varepsilon = 1$) in the surface Neumann data for the three HOPS algorithms (OE, FE, and TFE) using both Taylor and Padé summation. Physical parameters were (6.2) and numerical discretization was (6.3).

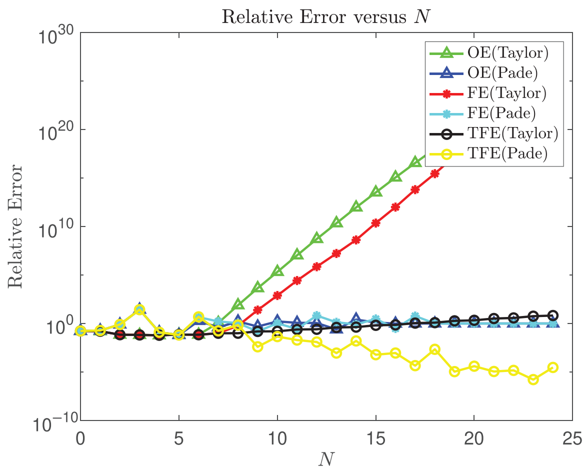

FIGURE 4 | Plot of relative error, (6.4), for a large perturbation ($\varepsilon = 1$) in the surface Neumann data for the three HOPS algorithms (OE, FE, and TFE) using both Taylor and Padé summation. Physical parameters were (6.2) and numerical discretization was (6.3) with N increased to 24.

as N is increased for all three algorithms. By contrast, Figure 2 ($\varepsilon = 0.5$) shows the extremely beneficial effects of using the TFE algorithm for this much larger deformation, particularly in concert with Padé summation. Finally, in Figure 3 ($\varepsilon = 1$) we see the *necessity* of using both the TFE method and Padé summation to realize any accuracy for such a huge perturbation. In Figure 4, we show that the TFE algorithm supplemented with Padé approximation continues to deliver enhanced convergence.

Our observations are very much in line with previous ones made about the OE, FE, and TFE methods [61, 62]. In particular, for small and smooth perturbations (e.g., Figure 1) a small number of perturbation orders, N, are sufficient to deliver errors on the order of machine precision. For n small all three methods provide robust estimates of $\tilde{G}_n$ and, therefore, excellent performance is observed for all three. By contrast, larger and rougher perturbations (e.g., Figures 2, 3, and 4) require much larger values of N for accurate approximation. Here, the *ill-*

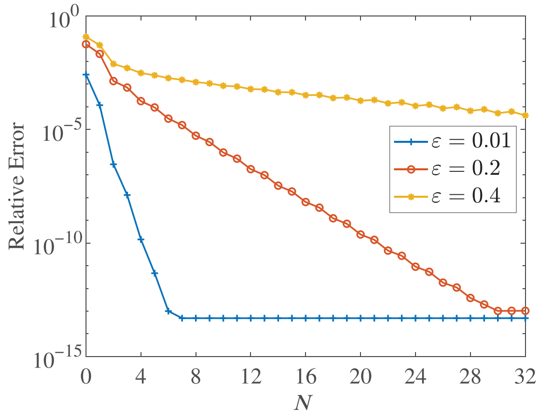

FIGURE 5 | Plot of relative error, (6.4), in the surface Neumann data for the TFE algorithms using Taylor summation for perturbation sizes $\varepsilon = 0.01, 0.2, 0.4$. Physical parameters were (6.5) and numerical discretization was (6.6).

conditioned nature of the OE and FE recursion formulas for $\tilde{G}_n$ not only prevents the accurate simulation of the DNO beyond some minimum accuracy, but also induces divergent behavior of the approximation as N becomes large. We refer the interested reader to [62] for an extensive discussion of these issues in the context of the Helmholtz equation relevant to electromagnetic scattering.

Remark 6.1. We have made a very particular choice of boundary shape function $\tilde{f}$ (a product of trigonometric functions) in order to emphasize the quasiperiodic nature of our deformation. However, any sufficiently smooth shape function can be considered and one can ponder the effects on our numerical simulations. As noted in previous work [28], while convergence is observed as N is increased for any permissible deformation, the *rate* of convergence is strongly affected by its smoothness. In fact, Theorem 4.13 demands that $B > C_0 \left\|\tilde{f}\right\|_{H^{s+3/2}}$ while the radius of convergence of our expansion is proportional to B^{-1}. As the smoothness of $\tilde{f}$ deteriorates, the rate of convergence will also slow down (see [28]).

6.3 | Three-Dimensional Simulations

We begin a description of our three-dimensional numerical simulations with the infinite depth case ($h = \infty$) where we chose the following triply 2π-periodic function

$$\tilde{\varphi}^{\text{exact}}(\alpha, y) = \left\{A_q e^{iq\cdot\alpha} + \bar{A}_q e^{-iq\cdot\alpha}\right\} e^{|K^T q| y}, \quad K \in \mathbf{R}^{3\times 2}, \quad q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \in \mathbf{Z}^3.$$

As before, with this we can generate all of $\{\tilde{\xi}^{\text{exact}}, \tilde{\nu}^{\text{exact}}\}$ and $\{\varphi^{\text{exact}}, \xi^{\text{exact}}, \nu^{\text{exact}}\}$. We chose the physical parameters

$$A_q = -1, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \tag{6.5}$$

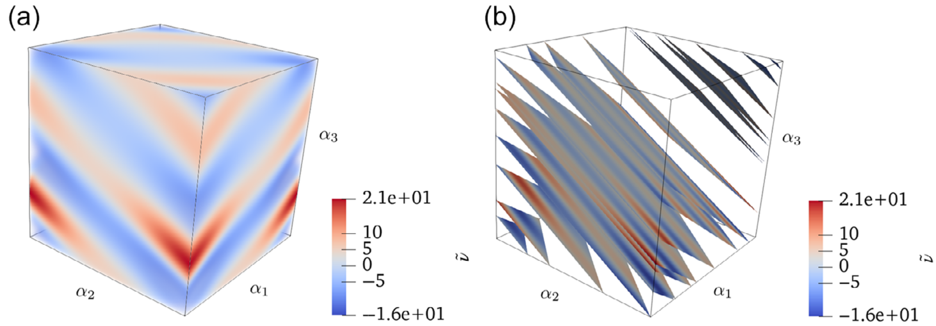

FIGURE 6 | Plot of the computed Neumann data, $\tilde{u}$, for $\varepsilon = 0.2$. (a) The full field; (b) A slice of the full field with a plane that has normal vector $(1/\sqrt{2}, -1/\sqrt{3}, -1)^T$.

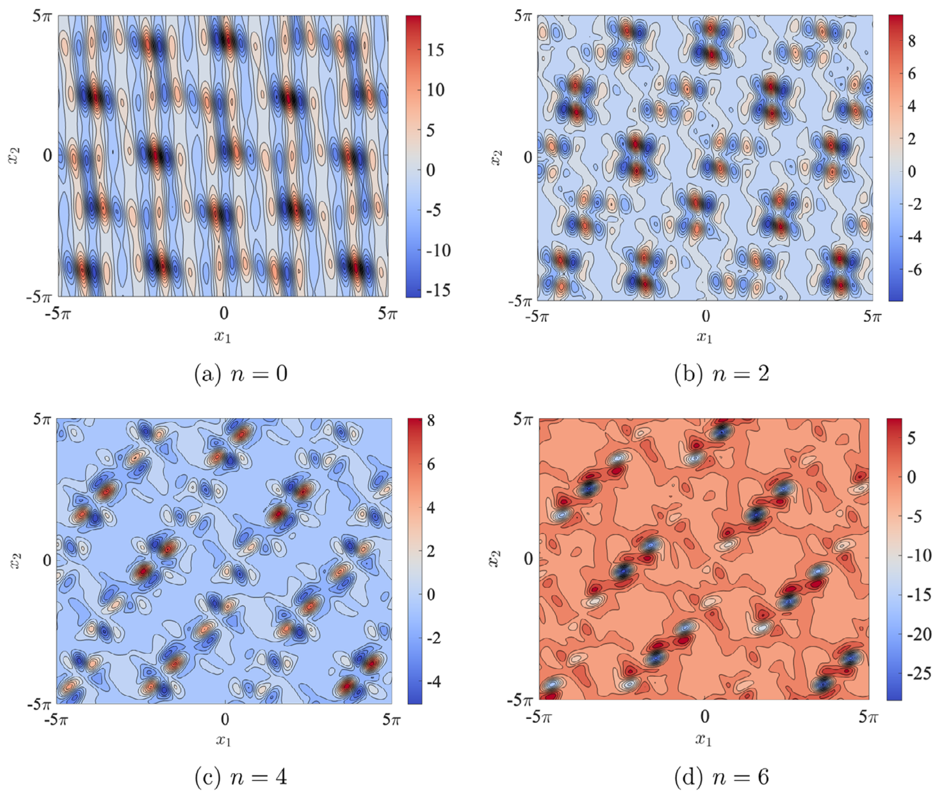

FIGURE 7 | Contour plots of the Taylor corrections $\tilde{\nu}_n$ for $n = 0$ (panel (a)), $n = 2$ (panel (b)), $n = 4$ (panel (c)), and $n = 6$ (panel (d)).

and the numerical parameters

$$N_\alpha = \begin{pmatrix} 64 \\ 64 \\ 64 \end{pmatrix}, \quad N_y = 64, \quad N = 32, \quad a = 2. \tag{6.6}$$

Finally, we considered the profile

$$\tilde{g}(\alpha) = \varepsilon \tilde{f}(\alpha), \quad \tilde{f}(\alpha) = \cos(\alpha_1) + \cos(\alpha_2) + \sin(\alpha_3).$$

We summarize our results in Figure 5 which focuses on the TFE algorithm utilizing Taylor summation alone. Here, for the small perturbation $\varepsilon = 0.01$, we see stable and rapid convergence to nearly machine zero after merely six perturbation orders. As ε increases, the convergence rate slows down. More specifically, in the moderate deformation case ($\varepsilon = 0.2$) it takes nearly 30 orders for the algorithm to achieve machine precision.

In Figure 6, on the left, we show a plot of our simulation in the case $\varepsilon = 0.2$ of the Neumann data, $\tilde{\nu}(\alpha)$, and on the right we depict a slice of this Neumann data with a plane that has normal vector $(1/\sqrt{2}, -1/\sqrt{3}, -1)^T$.

This simulation produced Taylor corrections $\tilde{\nu}_n$ and we display in Figure 7 contour plots of $\tilde{\nu}_0$ (panel (a)), $\tilde{\nu}_2$ (panel (b)), $\tilde{\nu}_4$ (panel (c)), and $\tilde{\nu}_6$ (panel (d)), each of which clearly exhibit nonperiodic patterns.

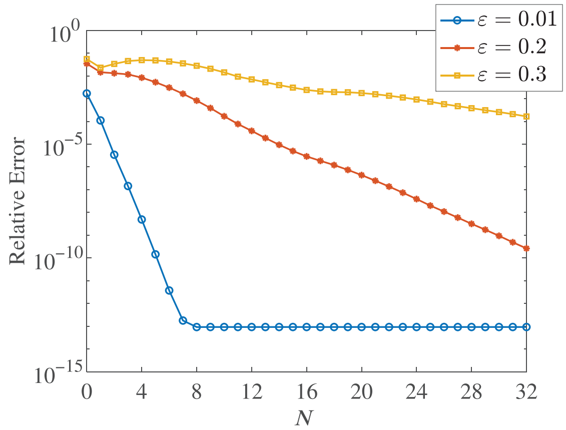

FIGURE 8 | Plot of relative error, (6.4), in the surface Neumann data for the TFE algorithms using Taylor summation for perturbation sizes $\varepsilon = 0.01, 0.2, 0.3$. Physical parameters were (6.7) and numerical discretization was (6.8).

We conclude with a description of our three-dimensional numerical simulations in the finite depth case ($h = 1$) where we chose the following triply 2π-periodic function:

$$\tilde{\varphi}^{\text{exact}}(\alpha, y) = \left\{ A_q e^{iq\cdot\alpha} + \bar{A}_q e^{-iq\cdot\alpha} \right\} \frac{\cosh(|K^T q|(h + y))}{\cosh(|K^T q| h)},$$

$$K \in \mathbf{R}^{3\times 2}, \quad q = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \in \mathbf{Z}^3.$$

Once again, with this we can generate all of $\{\tilde{\xi}^{\text{exact}}, \tilde{\nu}^{\text{exact}}\}$ and $\{\varphi^{\text{exact}}, \xi^{\text{exact}}, \nu^{\text{exact}}\}$. We chose the physical parameters

$$A_q = -1, \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}, \quad q = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \tag{6.7}$$

and the numerical parameters

$$N_\alpha = \begin{pmatrix} 64 \\ 64 \\ 64 \end{pmatrix}, \quad N_y = 64, \quad N = 32, \quad a = 1. \tag{6.8}$$

In addition, we considered the profile

$$\tilde{g}(\alpha) = \varepsilon \tilde{f}(\alpha), \quad \tilde{f}(\alpha) = \cos(\alpha_1) + \cos(\alpha_2) + \sin(\alpha_3).$$

In Figure 8, we display results from our simulations using the TFE algorithm with Taylor summation. As before, we see that for the small perturbation, $\varepsilon = 0.01$, we have stable and rapid convergence to nearly machine zero after merely eight perturbation orders.

Acknowledgments

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Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Appendix A: Proof of the Elliptic Estimate

In this Appendix, we take up the proof of Theorem 4.8 which we restate here for completeness. Recall that we consider the generic boundary value problem, (4.11),

$$\begin{aligned}
&\mathrm{div}_\alpha\left[KK^T\nabla_\alpha\tilde{u}(\alpha,y)\right]+\partial_y^2\tilde{u}(\alpha,y)=\tilde{F}(\alpha,y), \qquad -a<y<0,\\
&\tilde{u}(\alpha,0)=\tilde{\xi}(\alpha),\\
&\partial_y\tilde{u}(\alpha,-a)-\tilde{T}[\tilde{u}(\alpha,-a)]=\tilde{J}(\alpha),
\end{aligned}$$

where

$$\tilde{F}(\alpha,y):=\mathrm{div}_\alpha[K\tilde{F}^\alpha(\alpha,y)]+\partial_y\tilde{F}^y(\alpha,y)+\tilde{F}^0(\alpha,y).$$

Recalling the maximum,

$$\begin{aligned}
\mathcal{M}_s[\tilde{u}]:=\max\Big\{&\|\tilde{u}\|_{X^s},\left\|\partial_y\tilde{u}\right\|_{X^s},\|K^T\nabla_\alpha\tilde{u}\|_{X^s},\\
&\left\||K^TD|^{1/2}\tilde{u}\right\|_{X^s},\left\||K^TD|^{1/2}\partial_y\tilde{u}\right\|_{X^s},\left\||K^TD|^{1/2}K^T\nabla_\alpha\tilde{u}\right\|_{X^s},\\
&\left\|\partial_y\tilde{u}(\alpha,0)\right\|_{H^s},\|K^T\nabla_\alpha\tilde{u}(\alpha,0)\|_{H^s},\|\tilde{T}[\tilde{u}(\alpha,-a)]\|_{H^s}\Big\},
\end{aligned}$$

the elliptic estimate we require (Theorem 4.8) can be stated as follows.

Theorem A.1. *Given an integer $s\geq 0$, provided that*

$$\tilde{\xi}\in H^{s+1}(P(\Gamma)),\quad \tilde{J}\in H^s(P(\Gamma)),$$

$$\tilde{F}^j,|K^TD|^{1/2}\tilde{F}^j\in X^s(\Omega),\quad \tilde{F}^j(\alpha,0)\in H^s(P(\Gamma)),\quad j\in\{\alpha,y,0\},$$

and

$$\tilde{F}^y(\alpha,-a)=0,$$

there exists a unique solution $\tilde{u}\in X^s(\Omega)$ of (4.11) such that, for some $C_e>0$,

$$\begin{aligned}
\mathcal{M}_s[\tilde{u}]\leq C_e\Big\{&\left\|\tilde{\xi}\right\|_{H^{s+1}}+\|\tilde{J}\|_{H^s}\\
&+\sum_{j\in\{\alpha,y,0\}}\left(\|\tilde{F}^j\|_{X^s}+\left\||K^TD|^{1/2}\tilde{F}^j\right\|_{X^s}+\|\tilde{F}^j(\alpha,0)\|_{H^s}\right)\Big\}.
\end{aligned}$$

Proof. We will focus on the case $h=\infty$ and note that the proof for the case $h<\infty$ follows similarly. We expand the data and solution in Fourier series as,

$$\{\tilde{u},\tilde{F}^j\}(\alpha,y)=\sum_{p\in\Gamma'}\left\{\hat{u}_p(y),\hat{F}_p^j(y)\right\}e^{ip\cdot\alpha},\quad \{\tilde{\xi},\tilde{J}\}(\alpha)=\sum_{p\in\Gamma'}\left\{\hat{\xi}_p,\hat{J}_p\right\}e^{ip\cdot\alpha},$$

giving,

$$\begin{aligned}
&\partial_y^2\hat{u}_p(y)-|K^Tp|^2\hat{u}_p(y)=(iK^Tp)\cdot\hat{F}_p^\alpha(y)+\partial_y\hat{F}_p^y(y)+\hat{F}_p^0(y), \quad -a<y<0,\\
&\hat{u}_p(0)=\hat{\xi}_p,\\
&\partial_y\hat{u}_p(-a)-|K^Tp|\hat{u}_p(-a)=\hat{J}_p.
\end{aligned}$$

To complete our proof, we work with the exact solutions of these which we now derive and estimate explicitly.

a. Inhomogeneous BCs: We begin with the case $\hat{F}^j\equiv 0$ and write

$$\hat{u}_p(y)=\hat{\xi}_pE_p(y)+\hat{J}_pS_p(y), \tag{A1}$$

where

$$E_p(y):=\exp(|K^Tp|y),\quad S_p(y):=\begin{cases}y, & p=0,\\ \frac{\sinh(|K^Tp|y)}{|K^Tp|\exp(|K^Tp|a)}, & p\neq 0.\end{cases}$$

From this, we compute

$$\partial_y\hat{u}_p(y)=|K^Tp|\hat{\xi}_pE_p(y)+\hat{J}_pC_p(y),$$

where

$$C_p(y):=\begin{cases}1, & p=0,\\ \frac{\cosh(|K^Tp|y)}{\exp(|K^Tp|a)}, & p\neq 0.\end{cases}$$

Computing the L^2 norms of $E_p(y)$, $S_p(y)$, and $C_p(y)$, we obtain

$$\left\|E_p\right\|_{L^2}^2=\frac{1-\exp(-2|K^Tp|a)}{2|K^Tp|},$$

$$\left\|S_p\right\|_{L^2}^2=\begin{cases}a^3/3, & p=0,\\ \frac{-2a|K^Tp|+\sinh(2a|K^Tp|)}{4\exp(2a|K^Tp|)|K^Tp|^3}, & p\neq 0,\end{cases}$$

$$\left\|C_p\right\|_{L^2}^2=\begin{cases}a, & p=0,\\ \frac{2a|K^Tp|+\sinh(2a|K^Tp|)}{4\exp(2a|K^Tp|)|K^Tp|}, & p\neq 0,\end{cases}$$

which have the following limits when $p\neq 0$: As $|K^Tp|\to 0$ we have

$$\left\|E_p\right\|_{L^2}^2\to a,\quad \left\|S_p\right\|_{L^2}^2\to\frac{a^3}{3},\quad \left\|C_p\right\|_{L^2}^2\to a,$$

and, as $|K^Tp|\to\infty$ we find

$$\left\|E_p\right\|_{L^2}^2\to\frac{1}{2}|K^Tp|^{-1},\quad \left\|S_p\right\|_{L^2}^2\to\frac{1}{4}|K^Tp|^{-3},\quad \left\|C_p\right\|_{L^2}^2\to\frac{1}{4}|K^Tp|^{-1}.$$

From these, through interpolation it is not difficult to establish the following result.

Lemma A.2.

$$\begin{aligned}
&|K^Tp|^m\left\|E_p\right\|_{L^2}^2<C, && 0\leq m\leq 1,\quad \forall\,p\in\Gamma',\\
&|K^Tp|^m\left\|S_p\right\|_{L^2}^2<C, && 0\leq m\leq 3,\quad \forall\,p\in\Gamma',\\
&|K^Tp|^m\left\|C_p\right\|_{L^2}^2<C, && 0\leq m\leq 1,\quad \forall\,p\in\Gamma'.
\end{aligned}$$

In addition, we note that

$$\hat{u}_p(0) = \hat{\xi}_p, \quad \partial_y \hat{u}_p(0) = |K^T p|\hat{\xi}_p + \hat{J}_p \exp(-|K^T p|a),$$

$$\hat{u}_p(-a) = \begin{cases} \hat{\xi}_0 - a\hat{J}_0, & p = 0, \\ \hat{\xi}_p \exp(-|K^T p|a) - \hat{J}_p \frac{(1-\exp(-2|K^T p|a))}{2|K^T p|}, & p \neq 0. \end{cases}$$

We now estimate the nine terms in $\mathcal{M}_s(\tilde{u})$.

i. Field:

$$\|\tilde{u}\|_{X^s}^2 = \sum_{p\in\Gamma'} \langle p\rangle^{2s} \left\|\hat{\xi}_p E_p(y) + \hat{J}_p S_p(y)\right\|_{L^2}^2$$
$$\leq \sum_{p\in\Gamma'} 2\langle p\rangle^{2s} \left(\left|\hat{\xi}_p\right|^2 \left\|E_p(y)\right\|_{L^2}^2 + \left|\hat{J}_p\right|^2 \left\|S_p(y)\right\|_{L^2}^2 \right)$$
$$\leq 2C\left(\left\|\tilde{\xi}\right\|_{H^s}^2 + \|\tilde{J}\|_{H^s}^2 \right),$$

where the last inequality follows from Lemma A.2. In a similar manner, we can show that

$$\left\| \left|K^T D\right|^{1/2} \tilde{u} \right\|_{X^s} \leq 2C\left(\left\|\tilde{\xi}\right\|_{H^s}^2 + \|\tilde{J}\|_{H^s}^2 \right).$$

ii. Vertical derivative:

$$\left\|\partial_y \tilde{u}\right\|_{X^s}^2 = \sum_{p\in\Gamma'} \langle p\rangle^{2s} \left\| |K^T p| \hat{\xi}_p E_p(y) + \hat{J}_p C_p(y)\right\|_{L^2}^2$$
$$\leq \sum_{p\in\Gamma'} 2\langle p\rangle^{2s} \left(|K^T p| \left|\hat{\xi}_p\right|^2 |K^T p| \left\|E_p(y)\right\|_{L^2}^2 + \left|\hat{J}_p\right|^2 \left\|C_p(y)\right\|_{L^2}^2 \right)$$
$$\leq 2CC'\left(\left\|\tilde{\xi}\right\|_{H^{s+1/2}}^2 + \|\tilde{J}\|_{H^s}^2 \right),$$

where the last inequality follows from Lemma A.2 and $|K^T p| \leq C' \, |p|$ for some $C' \geq 1$ and for any $p \in \Gamma'$. Similarly, one can show that

$$\left\| \left|K^T D\right|^{1/2} \partial_y \tilde{u} \right\|_{X^s}^2 \leq 2CC'^2\left(\left\|\tilde{\xi}\right\|_{H^{s+1}}^2 + \|\tilde{J}\|_{H^s}^2 \right),$$
$$\left\|\partial_y \tilde{u}(0)\right\|_{H^s}^2 \leq 2C'^2\left(\left\|\tilde{\xi}\right\|_{H^{s+1}}^2 + \|\tilde{J}\|_{H^s}^2 \right).$$

iii. Horizontal derivative:

$$\|K^T \nabla_\alpha \tilde{u}\|_{X^s}^2 = \sum_{p\in\Gamma'} \langle p\rangle^{2s} |K^T p|^2 \left\|\hat{\xi}_p E_p(y) + \hat{J}_p S_p(y)\right\|_{L^2}^2$$
$$\leq \sum_{p\in\Gamma'} 2\langle p\rangle^{2s} \left(|K^T p| \left|\hat{\xi}_p\right|^2 |K^T p| \left\|E_p(y)\right\|_{L^2}^2 \right.$$
$$\left. + \left|\hat{J}_p\right|^2 |K^T p|^2 \left\|S_p(y)\right\|_{L^2}^2 \right)$$
$$\leq 2CC'\left(\left\|\tilde{\xi}\right\|_{H^{s+1/2}}^2 + \|\tilde{J}\|_{H^s}^2 \right).$$

Similarly, one can prove that

$$\left\| \left|K^T D\right|^{1/2} K^T \nabla_\alpha \tilde{u} \right\|_{X^s}^2 \leq 2CC'^2\left(\left\|\tilde{\xi}\right\|_{H^{s+1}}^2 + \|\tilde{J}\|_{H^s}^2 \right),$$

$$\|K^T \nabla_\alpha \tilde{u}(0)\|_{H^s}^2 \leq C'^2 \left\|\tilde{\xi}\right\|_{H^{s+1}}^2.$$

iv. Dirichlet–Neumann operators (DNO) at the lower boundary: Moving to $y = -a$, we have

$$\|\tilde{T}[\tilde{u}(-a)]\|_{H^s}^2 = \sum_{p\in\Gamma'} \langle p\rangle^{2s} \left| \hat{\xi}_p |K^T p| \exp(-|K^T p|a) \right.$$
$$\left. - \frac{1}{2}\hat{J}_p(1 - \exp(-2|K^T p|a)) \right|^2$$
$$\leq \sum_{p\in\Gamma'} 2\langle p\rangle^{2s} \left(\left|K^T p\right|^2 \left|\hat{\xi}_p\right|^2 + \left|\hat{J}_p\right|^2 \right)$$
$$\leq 2C'^2 \left(\left\|\tilde{\xi}\right\|_{H^{s+1}}^2 + \|\tilde{J}\|_{H^s}^2 \right).$$

b. Inhomogeneous PDE ($\tilde{F}^0 \not\equiv 0$): We move to the case $\hat{\xi}_p \equiv \hat{J}_p \equiv 0$ and $\hat{F}^j \equiv 0$, $j \in \{\alpha, y\}$. For $\tilde{F} = \tilde{F}^0$, we have

$$\hat{u}_p(y) = \exp(|K^T p|a) \left\{ \int_y^0 E_p(y) S_p(t) \hat{F}_p(t)\, dt + \int_{-a}^y E_p(t) S_p(y) \hat{F}_p(t)\, dt \right\}.$$

In Appendix B, we prove the following estimates of this function and its derivatives.

Lemma A.3.

$$\left|K^T p\right|^m \left\|\hat{u}_p\right\|_{L^2} \leq C\left\|\hat{F}_p\right\|_{L^2}, \qquad 0 \leq m \leq 2, \qquad \forall\, p \in \Gamma',$$
$$\left|K^T p\right|^m \left\|\partial_y \hat{u}_p\right\|_{L^2} \leq C\left\|\hat{F}_p\right\|_{L^2}, \qquad 0 \leq m \leq 1, \qquad \forall\, p \in \Gamma',$$
$$\left|K^T p\right|^m \left|\partial_y \hat{u}_p(0)\right| \leq C\left\|\hat{F}_p\right\|_{L^2}, \qquad 0 \leq m \leq 1/2, \qquad \forall\, p \in \Gamma',$$
$$\left|K^T p\right|^m \left|\hat{u}_p(-a)\right| \leq C\left\|\hat{F}_p\right\|_{L^2}, \qquad 0 \leq m \leq 3/2, \qquad \forall\, p \in \Gamma'.$$

Using Lemma A.3 we can prove the following estimates in a rather straightforward way.

i. Field:

$$\max\left\{ \|\tilde{u}\|_{X^s}, \left\| \left|K^T D\right|^{1/2} \tilde{u} \right\|_{X^s} \right\} \leq C\|\tilde{F}^0\|_{X^s}.$$

ii. Vertical derivative:

$$\max\left\{ \left\|\partial_y \tilde{u}\right\|_{X^s}, \left\| \left|K^T D\right|^{1/2} \partial_y \tilde{u} \right\|_{X^s}, \left\|\partial_y \tilde{u}(0)\right\|_{H^s}^2 \right\} \leq C\|\tilde{F}^0\|_{X^s}.$$

iii. Horizontal derivative:

$$\max\left\{ \|K^T \nabla_\alpha \tilde{u}\|_{X^s}, \left\| \left|K^T D\right|^{1/2} K^T \nabla_\alpha \tilde{u} \right\|_{X^s}, \|K^T \nabla_\alpha \tilde{u}(0)\|_{H^s} \right\} \leq C\|\tilde{F}^0\|_{X^s}.$$

iv. DNO at the lower boundary:

$$\|\tilde{T}[\tilde{u}(-a)]\|_{H^s} = \||K^T D|\tilde{u}(-a)\|_{H^s} \leq C\|\tilde{F}^0\|_{X^s}.$$

c. Inhomogeneous PDE ($\tilde{F}^\alpha \not\equiv 0$): We move to the case $\hat{\xi}_p \equiv \hat{J}_p \equiv 0$ and $\hat{F}^j \equiv 0$, $j \in \{0, y\}$. Setting $\tilde{F} = \tilde{F}^\alpha$, we have

$$\hat{u}_p(y) = \exp(|K^T p|a) \left\{ \int_y^0 E_p(y) S_p(t) (iK^T p) \hat{F}_p(t)\, dt \right.$$
$$\left. + \int_{-a}^y E_p(t) S_p(y) (iK^T p) \hat{F}_p(t)\, dt \right\}.$$

By simply replacing $\hat{F}_p$ by $(iK^T p)\hat{F}_p$ in Lemma A.3, we obtain the following estimates.

Lemma A.4.

$$|K^T p|^m \left\|\hat{u}_p\right\|_{L^2} \leq C\left\|\hat{F}_p\right\|_{L^2}, \qquad 0 \leq m \leq 1, \qquad \forall\, p \in \Gamma',$$

$$\left\|\partial_y \hat{u}_p\right\|_{L^2} \leq C\left\|\hat{F}_p\right\|_{L^2}, \qquad \forall\, p \in \Gamma',$$

$$\left|\partial_y \hat{u}_p(0)\right| \leq C|K^T p|^{1/2}\left\|\hat{F}_p\right\|_{L^2}, \qquad \forall\, p \in \Gamma',$$

$$|K^T p|^m \left|\hat{u}_p(-a)\right| \leq C\left\|\hat{F}_p\right\|_{L^2}, \qquad 0 \leq m \leq 1/2, \qquad \forall\, p \in \Gamma'.$$

With this, it is not difficult to show the following.

i. Field:

$$\max\left\{\|\tilde{u}\|_{X^s}, \left\||K^T D|^{1/2}\tilde{u}\right\|_{X^s}\right\} \leq C\|\tilde{F}^\alpha\|_{X^s}.$$

ii. Vertical derivative:

$$\left\|\partial_y \tilde{u}\right\|_{X^s} \leq C\|\tilde{F}^\alpha\|_{X^s},$$

$$\max\left\{\left\||K^T D|^{1/2}\partial_y \tilde{u}\right\|_{X^s}, \left\|\partial_y \tilde{u}(0)\right\|_{H^s}^2\right\} \leq C\left\||K^T D|^{1/2}\tilde{F}^\alpha\right\|_{X^s}.$$

iii. Horizontal derivative:

$$\|K^T \nabla_\alpha \tilde{u}\|_{X^s} \leq C\|\tilde{F}^\alpha\|_{X^s},$$

$$\left\{\left\||K^T D|^{1/2} K^T \nabla_\alpha \tilde{u}\right\|_{X^s}, \|K^T \nabla_\alpha \tilde{u}(0)\|_{X^s}\right\} \leq C\left\||K^T D|^{1/2}\tilde{F}^\alpha\right\|_{X^s}.$$

iv. DNO at the lower boundary:

$$\|\tilde{T}[\tilde{u}(-a)]\|_{H^s} = \||K^T D|\tilde{u}(-a)\|_{H^s} \leq C\left\||K^T D|^{1/2}\tilde{F}^\alpha\right\|_{X^s}.$$

d. Inhomogeneous PDE ($\tilde{F}^y \not\equiv 0$): We move to the case $\hat{\xi}_p \equiv \hat{J}_p \equiv 0$ and $\hat{F}^j \equiv 0$, $j \in \{0, \alpha\}$. For $\tilde{F} = \tilde{F}^y$, we have

$$\hat{u}_p(y) = \exp(|K^T p|a)\left\{\int_y^0 E_p(y)S_p(t)\hat{F}'_p(t)\,dt + \int_{-a}^y E_p(t)S_p(y)\hat{F}'_p(t)\,dt\right\}.$$

Writing this as

$$\hat{u}_p(y) = I_1[\partial_y \hat{F}_p](y) - I_2[\partial_y \hat{F}_p](y) + I_3[\partial_y \hat{F}_p](y) - I_4[\partial_y \hat{F}_p](y),$$

where I_i for $i = 1, 2, 3, 4$ are given by (B1). Using Integration-by-Parts, together with $\hat{F}^y_p(-a) = 0$, we discover

$$\hat{u}_p(y) = |K^T p|\left\{-I_1[\hat{F}_p](y) - I_2[\hat{F}_p](y) - I_3[\hat{F}_p](y) + I_4[\hat{F}_p](y)\right\}.$$

From this, we compute

$$\partial_y \hat{u}_p(y) = |K^T p|^2\left\{-I_1[\hat{F}_p](y) - I_2[\hat{F}_p](y) - I_3[\hat{F}_p](y) - I_4[\hat{F}_p](y)\right\} + \hat{F}_p(y),$$

which gives

$$\hat{u}_p(0) = 0, \quad \hat{u}_p(-a) = |K^T p|\left\{-I_1[\hat{F}_p](-a) - I_2[\hat{F}_p](-a)\right\},$$
$$\partial_y \hat{u}_p(0) = -2|K^T p|^2 I_3[\hat{F}_p](0) + \hat{F}_p(0).$$

Following the proof of Lemma A.3, we can show the following bounds.

Lemma A.5.

$$|K^T p|^m \left\|\hat{u}_p\right\|_{L^2} \leq C\left\|\hat{F}_p\right\|_{L^2}, \qquad 0 \leq m \leq 1, \qquad \forall\, p \in \Gamma',$$

$$\left\|\partial_y \hat{u}_p\right\|_{L^2} \leq C\left\|\hat{F}_p\right\|_{L^2}, \qquad \forall\, p \in \Gamma',$$

$$\left|\partial_y \hat{u}_p(0)\right| \leq C|K^T p|^{1/2}\left\|\hat{F}_p\right\|_{L^2}, \qquad \forall\, p \in \Gamma',$$

$$|K^T p|^m \left|\hat{u}_p(-a)\right| \leq C\left\|\hat{F}_p\right\|_{L^2}, \qquad 0 \leq m \leq 1/2, \qquad \forall\, p \in \Gamma'.$$

With this we obtain the following estimates:

i. Field:

$$\max\left\{\|\tilde{u}\|_{X^s}, \left\||K^T D|^{1/2}\tilde{u}\right\|_{X^s}\right\} \leq C\|\tilde{F}^y\|_{X^s}.$$

ii. Vertical derivative:

$$\left\|\partial_y \tilde{u}\right\|_{X^s} \leq C\|\tilde{F}^y\|_{X^s},$$

$$\max\left\{\left\||K^T D|^{1/2}\partial_y \tilde{u}\right\|_{X^s}, \left\|\partial_y \tilde{u}(0)\right\|_{H^s}^2\right\} \leq C\left\||K^T D|^{1/2}\tilde{F}^y\right\|_{X^s}.$$

iii. Horizontal derivative:

$$\|K^T \nabla_\alpha \tilde{u}\|_{X^s} \leq C\|\tilde{F}^y\|_{X^s},$$

$$\max\left\{\left\||K^T D|^{1/2} K^T \nabla_\alpha \tilde{u}\right\|_{X^s}, \|K^T \nabla_\alpha \tilde{u}(0)\|_{X^s}\right\} \leq C\left\||K^T D|^{1/2}\tilde{F}^y\right\|_{X^s}.$$

iv. DNO at the lower boundary:

$$\|\tilde{T}[\tilde{u}(-a)]\|_{H^s} = \||K^T D|\tilde{u}(-a)\|_{H^s} \leq C\left\||K^T D|^{1/2}\tilde{F}^y\right\|_{X^s}.$$

□

Appendix B: Proof of Lemma A.3

We now provide the full proof of Lemma A.3.

Proof. The proof for the case when $p = 0$ is straightforward and we focus on the case when $p \neq 0$. To simplify our developments, we write

$$\hat{u}_p(y) = I_1[\hat{F}_p](y) - I_2[\hat{F}_p](y) + I_3[\hat{F}_p](y) - I_4[\hat{F}_p](y),$$

where

$$\begin{aligned}
I_1[\hat{F}_p](y) &:= \frac{\exp(|K^T p|y)}{2|K^T p|}\int_y^0 \exp(|K^T p|t)\hat{F}_p(t)\,dt,\\
I_2[\hat{F}_p](y) &:= \frac{\exp(|K^T p|y)}{2|K^T p|}\int_y^0 \exp(-|K^T p|t)\hat{F}_p(t)\,dt,\\
I_3[\hat{F}_p](y) &:= \frac{\exp(|K^T p|y)}{2|K^T p|}\int_{-a}^y \exp(|K^T p|t)\hat{F}_p(t)\,dt,\\
I_4[\hat{F}_p](y) &:= \frac{\exp(-|K^T p|y)}{2|K^T p|}\int_{-a}^y \exp(|K^T p|t)\hat{F}_p(t)\,dt.
\end{aligned} \tag{B1}$$

From this, we obtain

$$\partial_y \hat{u}_p(y) = |K^T p|\left\{I_1[\hat{F}_p](y) - I_2[\hat{F}_p](y) + I_3[\hat{F}_p](y) + I_4[\hat{F}_p](y)\right\},$$
$$\hat{u}_p(0) = 0, \quad \hat{u}_p(-a) = I_1[\hat{F}_p](-a) - I_2[\hat{F}_p](-a),$$
$$\partial_y \hat{u}_p(0) = 2|K^T p|\left\{I_3[\hat{F}_p](0)\right\}.$$

Computing the L^2 norm of $I_1[\hat{F}_p](y) - I_2[\hat{F}_p](y)$, we obtain

$$\left\| I_1[\hat{F}_p](y) - I_2[\hat{F}_p](y) \right\|_{L^2}^2 = \int_{-a}^{0} \exp(|K^T p| 2a)\, dy \left| \int_y^0 E_p(y) S_p(t) \hat{F}_p(t)\, dt \right|^2$$

$$= \frac{1}{|K^T p|^2} \int_{-a}^{0} \exp(2|K^T p| y)\, dy \left(\int_y^0 \sinh(|K^T p| t) \hat{F}_p(t)\, dt \right)^2$$

$$\leq \frac{1}{|K^T p|^2} \int_{-a}^{0} \exp(2|K^T p| y)\, dy \int_y^0 \sinh^2(|K^T p| t)\, dt \int_y^0 |\hat{F}_p(t)|^2\, dt$$

$$\leq \frac{a}{|K^T p|^2} \left\| \hat{F}_p \right\|_{L^2}^2 \int_{-a}^{0} \exp(2|K^T p| y) \sinh^2(|K^T p| y)\, dy,$$

since

$$\int_y^0 \sinh^2(|K^T p| t)\, dt \leq \sinh^2(|K^T p| y) \int_y^0 dt \leq a \sinh^2(|K^T p| y).$$

Continuing

$$\left\| I_1[\hat{F}_p](y) - I_2[\hat{F}_p](y) \right\|_{L^2}^2 \leq a f\left(|K^T p|\right) \left\| \hat{F}_p \right\|_{L^2}^2 \leq C \left\| \hat{F}_p \right\|_{L^2}^2,$$

where

$$f\left(|K^T p|\right) = -\frac{1}{16|K^T p|^3} \left(\exp(-4|K^T p| a)\right) - 4\exp(-2|K^T p| a) - 4a|K^T p| + 3 \Big)$$

$$\to \begin{cases} \frac{a^3}{3}, & |K^T p| \to 0, \\ \frac{a}{4|K^T p|^2}, & |K^T p| \to \infty. \end{cases}$$

Similarly, one can prove that

$$\left\| I_3[\hat{F}_p](y) - I_4[\hat{F}_p](y) \right\|_{L^2}^2 \leq C \left\| \hat{F}_p \right\|_{L^2}^2,$$

thus, we have

$$\left\| \hat{u}_p \right\|_{L^2} \leq C \left\| \hat{F}_p \right\|_{L^2}. \tag{B2}$$

Next, we are going to show that

$$|K^T p|^2 \left\| \hat{u}_p \right\|_{L^2} \leq C \left\| \hat{F}_p \right\|_{L^2}.$$

Computing the squared L^2 norm of $|K^T p|^2 I_1[\hat{F}_p]$, we have

$$|K^T p|^4 \left\| I_1[\hat{F}_p] \right\|_{L^2}^2 = \frac{|K^T p|^2}{4} \int_{-a}^{0} \exp(2|K^T p| y)\, dy \left| \int_y^0 \exp(|K^T p| t) \hat{F}_p(t)\, dt \right|^2$$

$$\leq \frac{|K^T p|^2}{4} \int_{-a}^{0} \exp(2|K^T p| y)\, dy \int_y^0 \exp(2|K^T p| t)\, dt \int_y^0 |\hat{F}_p(t)|^2\, dt$$

$$\leq \frac{|K^T p|^2}{4} \left\| \hat{F}_p \right\|_{L^2}^2 \int_{-a}^{0} \exp(2|K^T p| y)\, dy \int_{-a}^{0} \exp(2|K^T p| t)\, dt$$

$$\leq \frac{1}{16} \left\| \hat{F}_p \right\|_{L^2}^2 \left(1 - \exp(-2|K^T p| a)\right)^2$$

$$\leq C \left\| \hat{F}_p \right\|_{L^2}^2.$$

An estimation of L^2 norm of $\left|K^T p\right|^2 I_2[\hat{F}_p]$ proceeds as follows:

$$
\begin{aligned}
\left|K^T p\right|^2 \left\|I_2[\hat{F}_p]\right\|_{L^2} &= \frac{\left|K^T p\right|}{2}\left(\int_{-a}^{0} dy \left|\int_{y}^{0} \exp\left(\left|K^T p\right|(y-t)\right)\hat{F}_p(t)\, dt\right|^2\right)^{1/2} \\
&\le \frac{\left|K^T p\right|}{2}\left(\int_{-a}^{0} dy \left(\int_{-a}^{0} \exp\left(\left|K^T p\right|(y-t)\right)\left|\hat{F}_p(t)\right| dt\right)^2\right)^{1/2} \\
&\le \frac{\left|K^T p\right|}{2}\int_{-a}^{0} \exp(\left|K^T p\right| y)\, dy \left(\int_{-a}^{0} \left|\hat{F}_p(y)\right|^2 dy\right)^{1/2} \\
&= \frac{1}{2}(1-\exp(-\left|K^T p\right| a))\left\|\hat{F}_p\right\|_{L^2} \\
&\le C\left\|\hat{F}_p\right\|_{L^2}.
\end{aligned}
$$

Similarly, we can show that

$$
\left|K^T p\right|^2 \left\|I_j[\hat{F}_p]\right\|_{L^2} \le C\left\|\hat{F}_p\right\|_{L^2}, \quad j = 3, 4.
$$

Therefore, we completed the proof of

$$
\left|K^T p\right|^2 \left\|\hat{u}_p\right\|_{L^2} \le C\left\|\hat{F}_p\right\|_{L^2}. \tag{B3}
$$

Interpolating (B2) and (B3), we obtain the first inequality in Lemma A.3. The second inequality in Lemma A.3 can be proved similarly.

At $y = 0$, we have

$$
\begin{aligned}
\left|\partial_y \hat{u}_p(0)\right| &= \left|\int_{-a}^{0} \exp(\left|K^T p\right| t)\hat{F}_p(t)\, dt\right| \\
&\le \left(\int_{-a}^{0} \exp(2\left|K^T p\right| t)\, dt\right)^{1/2} \left\|\hat{F}_p\right\|_{L^2} \\
&= \left(\frac{1-\exp(-2\left|K^T p\right| a)}{2\left|K^T p\right|}\right)^{1/2} \left\|\hat{F}_p\right\|_{L^2}.
\end{aligned}
$$

Therefore, we obtain

$$
\left|\partial_y \hat{u}_p(0)\right| \le C\left\|\hat{F}_p\right\|_{L^2}, \qquad \left|K^T p\right|^{1/2}\left|\partial_y \hat{u}_p(0)\right| \le C\left\|\hat{F}_p\right\|_{L^2}.
$$

Through interpolation, we finish the proof of the third inequality in Lemma A.3. We use the following two inequalities to prove the last inequality in Lemma A.3. At $y = -a$, we have

$$
\begin{aligned}
\left|\hat{u}_p(-a)\right| &\le \frac{\exp(-\left|K^T p\right| a)}{\left|K^T p\right|}\int_{-a}^{0} \left|\sinh(\left|K^T p\right| t)\hat{F}_p(t)\right| dt \\
&\le \frac{\exp(-\left|K^T p\right| a)}{\left|K^T p\right|}\int_{-a}^{0} \sinh(\left|K^T p\right| a)\left|\hat{F}_p(t)\right| dt \\
&\le \frac{1-\exp(-2\left|K^T p\right| a)}{2\left|K^T p\right|}\int_{-a}^{0} |\hat{F}_p(t)|\, dt,
\end{aligned}
$$

since $\left|\sinh(\left|K^T p\right| t)\right| \le \sinh(\left|K^T p\right| a)$ on $[-a, 0]$. Continuing,

$$
\left|\hat{u}_p(-a)\right| \le \frac{1-\exp(-2\left|K^T p\right| a)}{2\left|K^T p\right|}\sqrt{a}\left\|\hat{F}_p\right\|_{L^2} \le C\left\|\hat{F}_p\right\|_{L^2}.
$$

Finally, since $\sinh(y) \le \exp(-y)$ for $y < 0$,

$$\begin{aligned}
\left|K^T p\right|^{3/2}\left|\hat{u}_p(-a)\right| &\le \left|K^T p\right|^{1/2} \exp(-\left|K^T p\right|a) \int_{-a}^{0} \exp(-\left|K^T p\right|t)\left|\hat{F}_p(t)\right| \, dt \\
&\le \left|K^T p\right|^{1/2} \exp(-\left|K^T p\right|a)\left(\int_{-a}^{0} \exp(-2\left|K^T p\right|t) \, dt\right)^{1/2} \left\|\hat{F}_p\right\|_{L^2} \\
&\le C\left\|\hat{F}_p\right\|_{L^2}.
\end{aligned}$$

□