J. math. fluid mech. 5 (2003) 92–118
1422-6928/03/010092-27


**Journal of Mathematical Fluid Mechanics**

# Solitary Wave Interactions of the Euler–Poisson Equations

Mariana Haragus*, David P. Nicholls† and David H. Sattinger‡

**Abstract.** We study solitary wave interactions in the Euler–Poisson equations modeling ion acoustic plasmas and their approximation by KdV $n$-solitons. Numerical experiments are performed and solutions compared to appropriately scaled KdV $n$-solitons. While largely correct qualitatively the soliton solutions did not accurately capture the scattering shifts experienced by the solitary waves. We propose correcting this discrepancy by carrying out the singular perturbation scheme which produces the KdV equation at lowest order to higher order. The foundation for this program is laid and preliminary results are presented.

**Mathematics Subject Classification (2000).** 35Q51, 35Q53.

**Keywords.** KdV equation, plasma equations, resonant soliton interactions.

## 1. Introduction

Over the past 150 years one of the successes in the application of mathematics to science and engineering has been in the simulation of nonlinear, dispersive evolutionary physical processes by integrable partial differential equations. For example, the Korteweg–de Vries (KdV) and Nonlinear Schrödinger (NLS) equations have been used as a models in applications as diverse as ocean wave dynamics, plasma physics, and nonlinear optics. Of particular significance in the success of these models is the fact that they are integrable and possess, among other things, soliton solutions which propagate without change in form and interact "elastically," i.e. the waveforms emerge from interactions with the same speed and shape, suffering only a scattering shift as evidence of the collision.

A natural question to ask is how well the multi-soliton solutions approximate the true solutions they are meant to model. In the setting of gravity waves on the surface of an ideal fluid, modelled by the Euler equations, several answers have been proposed. Craig [3] has shown that suitably scaled solutions of the

*Research partially supported by the Ministère de la Recherche de la Technologie, ACI jeunes chercheurs.
†Research supported by NSF Grant DMS-0196452.
‡Research supported by NSF Grant DMS-9971249.

Euler equations are "close" to solutions of the relevant KdV equation for a finite, though potentially quite long, period of time. More recently Schneider & Wayne [12] have shown that suitably scaled and sufficiently small solutions of the water wave problem in the long wave limit split up into two wave packets, one moving to the right and one to the left, which evolve independently as solutions of two KdV equations on a finite but again potentially quite long time interval. Their analysis includes soliton solutions and allows for studying the interaction of solitary waves in certain cases. However, the scattering anomalies in our numerical experiments (see § 3) imply that the scattering shifts experienced by the solitary waves are not accurately described by the KdV equation itself.

Zou & Su [13] calculated numerically the second and third order corrections to the KdV equation for the Euler equations. They found that the interaction is elastic at second order, but that a dispersive tail appears at third order. Fenton & Rienecker [5] carried out a numerical study of interacting solitary waves for the Euler equations and compared the numerical results with solutions of the Korteweg–de Vries equation. They noted [5]:

> Results support . . . applicability of the Korteweg–de Vries equation . . . since the waves during interaction are long and low. However, some deviations from the theoretical predictions were observed: the overtaking wave grew significantly at the expense of the slower wave, and the predicted phase shift was only roughly described by the theory.

In fact, the inelastic nature of the interaction has been reported by others studying solitary wave interactions for non-integrable dispersive systems [1], [2], [7], [8].

Among the class of nonlinear, dispersive, evolutionary physical models admitting an approximation by the KdV equation, one of the simplest is modelled by the Euler–Poisson equations of ion acoustic plasma physics. These equations share many of the same features as the Euler equations which model the evolution of the free surface of an ideal fluid such as existence of solitary waves and a wave of maximum speed, crest instability of solitary waves, Hamiltonian structure and certain symmetries. However, they lack many of the complications of the Euler equations such as the presence of the non-local Dirichlet–Neumann operator (or one of its analogues) [4] for a formulation which can be naturally related to the KdV equation. While this operator can be analyzed it does render the Euler equations significantly more complicated than the Euler–Poisson equations and for this reason we restrict our current investigations to the latter equation.

A further instance of the inelastic behavior mentioned above, but in the context of plasma waves, was noticed by Li & Sattinger [9] in a numerical simulation of interaction of low amplitude solitary waves (of different amplitude) of the Euler–Poisson equations of plasma physics. They compared their numerically computed solution with an appropriately scaled 2-soliton solution of the KdV equation and found a dispersive tail approximately 0.00666% of the magnitude of the smaller wave, consistent with the dispersive tail at third order found by Zou & Su [13] for water waves. However, Li & Sattinger's data did not confirm all of the findings of Fenton & Rienecker [5] in the case of the water waves. While it did show that the plasma waves are displaced relative to the KdV waves, the faster waves did not gain significant amplitude at the expense of the slower wave. In light of these findings the authors' aim in this research project is to understand the discrepancies between predictions of the KdV $n$-soliton solution and observations in numerical simulations on the full Euler–Poisson equations, and to produce, if possible, an enhanced model which more accurately captures effects such as the scattering shift.

The message of this paper is two-fold. First, in the setting of the Euler–Poisson equations of plasma physics, the KdV $n$-soliton solution provides qualitatively excellent results (producing solitary waves which interact essentially elastically even well outside of their purported realm of validity) while being quantitatively inaccurate in capturing certain effects (particularly the scattering shift apparent after a collision). Second, we present some preliminary results on constructing "higher order corrections" to the KdV equation which will accurately capture the quantitative inaccuracies produced by the KdV $n$-soliton solution.

The paper is organized as follows: Section 2 discusses the Euler–Poisson equations, the derivation of the KdV approximation, the numerical method developed by Li & Sattinger [9], and a convergence study indicating the accuracy, stability, and robustness of the method. In Section 3 we present results of numerical experiments on two- and three-pulse interactions in the Euler–Poisson equations and their approximation by KdV two- and three-solitons. In Section 4 we discuss some of our preliminary theoretical results. In particular in Section 4.1 we present a discussion of the time-scale of interaction for two solitary waves and in Section 4.2 we discuss the next-order correction to the KdV equation in a singular perturbation expansion for the Euler–Poisson equations. Modulational effects on these model equations *during* solitary wave interactions, and the relation to the results of Zou & Su [13] are discussed in Section 4.3. Finally, in Appendix A we present some results on two-soliton solutions of the KdV equation which are used extensively in Section 4.2.

## 2. Numerical simulation of the Euler–Poisson equations

In this section we present the equations governing the evolution of an ion acoustic plasma, i.e. the Euler–Poisson equations (§ 2.1), and present the first step in a singular perturbation expansion of the Euler–Poisson equations which gives the KdV equation at lowest order (§ 2.2). We also briefly describe a numerical method due to Li & Sattinger [9] for simulating solutions and present some numerical convergence studies which give us confidence that our numerical solutions converge to true solutions of the Euler–Poisson equations (§ 2.3).

### 2.1. Euler–Poisson equations

The Euler–Poisson equations for ion acoustic waves in plasmas are

$$n_t + (nv)_x = 0, \qquad v_t + \left(\frac{v^2}{2} + \varphi\right)_x = 0, \qquad \varphi_{xx} - e^{\varphi} + n = 0 \tag{2.1}$$

where $\varphi$, $n$, and $v$ are respectively the electric potential, ion density and ion velocity. This system supports solitary waves which travel with constant speed [11] (see [9] for a detailed proof of existence). They are solutions of the form $(1+\tilde{n}, v, \varphi)(x-ct)$, where $c$ is the propagation speed, which decay to zero as $\xi = (x-ct) \to \pm\infty$. Such solutions satisfy the system

$$\tilde{n} = \frac{v}{c-v}, \qquad v = c - \sqrt{c^2 - 2\varphi}, \tag{2.2a}$$

$$\varphi'' = e^{\varphi} - \frac{c}{\sqrt{c^2 - 2\varphi}}. \tag{2.2b}$$

Solitary waves exist for supersonic speeds $c > 1$ and there is a wave of maximum speed traveling at a rate $\bar{c} \approx 1.5852$. With this quantity one can define a *Mach number* by

$$M(c) = \frac{c-1}{\bar{c}-1} \approx \frac{c-1}{.5852}, \tag{2.3}$$

which will appear in several of our numerical experiments.

### 2.2. KdV approximation

We briefly review the derivation of the KdV equation as the leading term in a singular perturbation scheme for the Euler–Poisson equations. By introducing a rescaling of the space and time variables by $x' = \varepsilon x$, $t' = \varepsilon^3 t$, where $\varepsilon$ is the relevant smallness parameter, one obtains the singular perturbation problem

$$\begin{aligned} \varepsilon^2 n_{t'} + (nv)_{x'} &= 0 \\ \varepsilon^2 v_{t'} + \left(\frac{v^2}{2} + \varphi\right)_{x'} &= 0 \\ -\varepsilon^2 \varphi_{x'x'} + e^{\varphi} &= n. \end{aligned}$$

In the following, we shall drop the primes from the variables. As only even powers of $\varepsilon$ appear in these equations, we formally expand all quantities in powers of $\varepsilon^2$,

$$\begin{aligned} n &= 1 + \varepsilon^2 n_1 + \varepsilon^4 n_2 + \dots \\ v &= -1 + \varepsilon^2 v_1 + \varepsilon^4 v_2 + \dots \\ \varphi &= \varepsilon^2 \varphi_1 + \varepsilon^4 \varphi_2 + \dots . \end{aligned}$$

This corresponds to expanding around a density of $n = 1$ and a velocity at infinity of $v = -1$, corresponding to a Galilean frame moving with speed 1.

At order $\varepsilon^2$ we obtain, by differentiating the third equation with respect to $x$,

$$(n_1 - v_1)_x = 0 \qquad (v_1 - \varphi_1)_x = 0 \qquad (\varphi_1 - n_1)_x = 0. \tag{2.4}$$

At order $\varepsilon^4$ we get

$$(n_2 - v_2)_x = n_{1,t} + (n_1 v_1)_x \tag{2.5a}$$

$$(v_2 - \varphi_2)_x = v_{1,t} + \frac{1}{2}(v_1^2)_x \tag{2.5b}$$

$$(\varphi_2 - n_2)_x = \varphi_{1,xxx} - \varphi_1 \varphi_{1,x}. \tag{2.5c}$$

In fact at each step, $k$, we get a system of equations of the form

$$(n_k - v_k)_x = f_k^{(1)}, \quad (v_k - \varphi_k)_x = f_k^{(2)}, \quad (\varphi_k - n_k)_x = f_k^{(3)}. \tag{2.6}$$

The operator on the left side of (2.6) has a null space of the form $n_k = v_k = \varphi_k$, and the solvability condition for the system is

$$f_k^{(1)} + f_k^{(2)} + f_k^{(3)} = 0. \tag{2.7}$$

The general solution of this system vanishing as $x \to -\infty$ is

$$v_k(x,t) = \varphi_k + \int_{-\infty}^{x} f_k^{(2)}(x',t)\,dx' \tag{2.8a}$$

$$n_k(x,t) = \varphi_k - \int_{-\infty}^{x} f_k^{(3)}(x',t)\,dx', \tag{2.8b}$$

where $\varphi = \varphi(x,t)$ is to determined so that the solvability condition at the next order is satisfied.

At the lowest order (2.4) & (2.8) require that $n_1 = v_1 = \varphi_1$, while at next order (2.5) and the solvability condition (2.7) require that $\varphi_1$ satisfy

$$u_t + \frac{1}{2}u_{xxx} + u u_x = 0, \tag{2.9}$$

the KdV equation. The extension of this expansion to next order is the topic of § 4.2.

Of great significance to the rest of the paper is integrability of (2.9) and the *explicit* 2-soliton solution given by

$$u(x,t) = 6\frac{d^2}{dx^2}\log\tau, \qquad \tau = \det\left\|\delta_{jk} + \frac{e^{-(\theta_j+\theta_k)}}{\omega_j + \omega_k}\right\|, \tag{2.10}$$

where

$$\theta_j = \omega_j(x - 2\omega_j^2 t - \alpha_j), \qquad j = 1, 2. \tag{2.11}$$

The 2-soliton solution is a four parameter family of solutions of (2.9); the parameters $\alpha_j$ are the relative phases of the two waves and the speeds of the waves are given by $2\omega_j^2$. The determinant in (2.10) is called the *tau function* in the literature.

### 2.3. Numerical method and convergence

In numerical simulations the spatial boundary condition of decay at infinity is replaced by the condition that $n$, $v$, and $\varphi$ are periodic of period $L$, where $L$ is suitably large, and the support of these functions is well within the interval $[0, L]$. The numerical method we have utilized for simulations of (2.1) (with the periodic boundary condition previously mentioned) is due to Li & Sattinger and discussed in detail in [9]. To summarize, the method is a Fourier collocation method in space and an implicit first order finite difference method in time. The spatial discretization is chosen for its spectral accuracy while the time stepping strategy is chosen for its simplicity and stability. A raised cosine filter is used periodically (in time) in the spatial variable to counteract the effects of aliasing errors.

In order to build confidence in the reliability of our scheme we performed a numerical convergence study on the problem of two solitary plasma waves colliding at relatively low Mach number. In particular, we simulated the evolution of two superposed solitary wave solutions of (2.1) with speeds $c_1 = 1.1$ and $c_2 = 1.3$ on an interval of length $L = 100$ for a time interval of length $T = 490$. Equations (2.1) were solved with $c = 1.2$ so that the solutions simply exchanged positions, and discretizations were made in space and time parameterized by the number of collocation points $N_x$ and the number of time steps $N_t$. To help control aliasing errors, the raised cosine filter was applied after every 100 units of time.

The results of this study for fixed $N_t = 49000$ ($\Delta t = 0.01$) and varying $N_x$ are given in Table 1 where the "exact solution" is given by the results of a simulation with $N_x = 16,384$ ($\Delta x = 0.006103515625$). The results for varying $N_t$ and fixed

| $N_x$ | $\Delta x$ | Error in $\varphi$ (Discrete $L^2$) |
|---|---|---|
| 256 | 0.390625 | 0.011467786068109997 |
| 512 | 0.1953125 | 0.002870303277881501 |
| 1024 | 0.09765625 | $3.427685711428991e \times 10^{-5}$ |
| 2048 | 0.048828125 | $1.715995772970149 \times 10^{-7}$ |
| 4096 | 0.0244140625 | $8.53361553374557 \times 10^{-9}$ |
| 8192 | 0.01220703125 | $5.299269878231338 \times 10^{-9}$ |

TABLE 1. Convergence study of two solitary wave interaction at speeds $c_1 = 1.1$ and $c_2 = 1.3$ with fixed $N_t = 49000$ ($\Delta t = 0.01$) and variable $N_x$.

$N_x = 1024$ ($\Delta x = 0.09765625$) are given in Table 2, where the "exact solution" is given by the approximation with $N_t = 196,000$ ($\Delta t = 0.0025$). In each case we see clearly that as the discretization is increased the approximate solution is converging to the "exact solution." Furthermore, as $N_x$ is increased the rate of convergence is spectral, as one would expect from a Fourier collocation method on a periodic interval, and as $N_t$ is increased the rate of convergence is roughly linear as expected from a first order scheme.

| $N_t$ | $\Delta t$ | Error in $\varphi$ (Discrete $L^2$) |
|---|---|---|
| 6,125 | 0.08 | 0.1493931266324674 |
| 12,250 | 0.04 | 0.1200847953350079 |
| 24,500 | 0.02 | 0.07506573844755135 |
| 49,000 | 0.01 | 0.03645025098749217 |
| 98,000 | 0.005 | 0.01266804809229405 |

TABLE 2. Convergence study of two solitary wave interaction at speeds $c_1 = 1.1$ and $c_2 = 1.3$ with fixed $N_x = 1024$ ($\Delta x = 0.09765625$) and variable $N_t$.

## 3. Numerical results

We conducted several numerical experiments regarding the interaction of solitary wave solutions of the Euler–Poisson equations (2.1) in order to gain some insight into the strengths and weaknesses of the KdV approximation to these equations. In all experiments the individual traveling waves were constructed by a numerical continuation method applied to (2.1) for steady solutions, i.e. $n_t = v_t = \varphi_t = 0$, where the parameter was the speed $c$ and an initial guess of a properly scaled KdV soliton was used to step off the trivial branch of solutions ($n = v = \varphi \equiv 0$). To simulate two or more solitary waves interacting, two or more of these solutions were simply added together and, provided that the pulses were sufficiently separated in space, our results were excellent. In the experiments below, the relationship between the absolute wave speed $c_j$ of pulse $j$ in the rest frame to the solitary wave speed parameter $\omega_j$ is $c_j = 1 + 2\omega_j^2$.

In the first experiment we compared a high Mach number ($M = 0.8544$) solitary wave solution of the Euler–Poisson equation with a KdV soliton moving with the same speed. In Figure 1 we see the results of this computation and notice that the plasma wave (dashed line) is considerably more peaked than the KdV wave of equal velocity (solid line). The KdV solitary wave is $6\omega^2\text{sech}^2\omega x$, where $c = 1 + 2\omega^2$.

In the next two experiments we study two- and three-pulse interactions in the Euler–Poisson equations and compare them with two- and three-soliton solutions of KdV with speeds that match those of the plasma pulses. In Figures 2 & 3 we see the results after matching KdV two- and three-solitons, respectively, to the initial data of the plasma using the initial location of the wave peaks and the initial speeds. The experiments clearly confirm the displacement of the plasma waves relative to the corresponding KdV waves after the interaction, as noted by Fenton & Rienecker [5] in the context of water waves.

Note that in Figure 2 the smaller plasma wave is completely absorbed by the larger one in the course of the interaction, while in the interaction between waves of speeds 1.05 and 1.1, depicted in the sequence in Figure 4 from Li & Sattinger [9], momentum is exchanged between the faster and slower waves. This phenomenon is well known for the 2-soliton solutions of the KdV equation: when the wave speeds are close, there is a momentum transfer, while in the case of a large difference between the wave speeds, the slower wave is overtaken and absorbed by the faster wave.

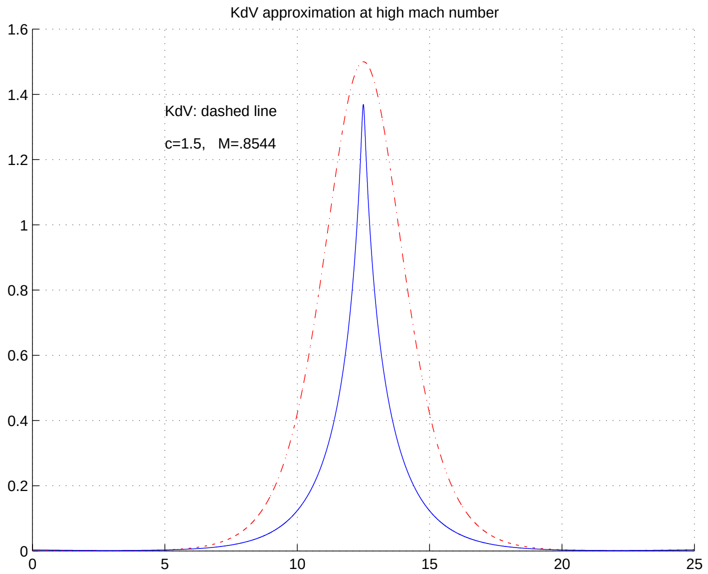



FIG. 1. Comparison of the KdV soliton (dashed) with the $\varphi$ wave (solid) of the ion acoustic plasma equations at $M = .8544$, $c = 1.5$.

The concept of scattering shift for the KdV equation is based on the fact that the emerging waves travel at the same speeds as the incoming waves, and the waves lose no energy in the course of the interaction. The trailing dispersive waves of very low magnitude found in the numerical experiments of Li & Sattinger [9] and predicted by Zou & Su [13] imply that energy is lost from the traveling waves in the course of the interaction, so that the total energy of the emerging waves is slightly less than that of the incoming waves. If this were the case, then the emerging waves would travel at slightly different speeds than the incoming solitary waves, and the distance between the plasma wave and the corresponding KdV wave after the interaction would grow linearly in time, as indicated by the plots of the scattering anomalies in Figure 5. In this plot, the $\delta_j$ are difference between $j^{th}$ KdV wave and the corresponding plasma wave. The anomalies develop on a fast time scale at the time of the interaction. For large times, $\delta_2$ appears to increase linearly, suggesting that the faster plasma wave falls away from the KdV wave at a constant speed; while the decay in $\delta_1$ suggests that the slower plasma wave has gained speed.

If the waves are asymptotic to solitary waves after they emerge from the interaction, then Figure 5 suggests that the faster plasma wave loses speed in the interaction, while the slower plasma wave picks up speed in the interaction. Thus, there is a small momentum transfer from the faster to the slower wave during the interaction. In addition, the presence of a trailing dispersive wave indicates

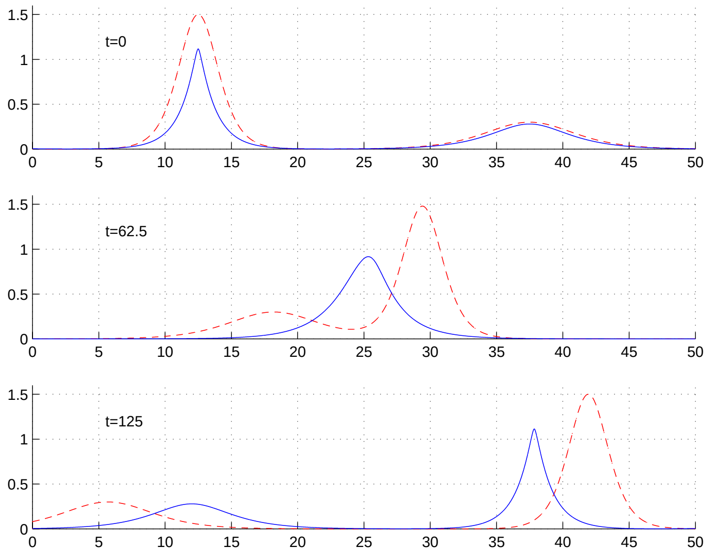



FIG. 2. Comparison of the KdV 2-soliton solution (dashed) with the $\varphi$ wave (solid) of the ion acoustic plasma equations at initial, intermediate, and final configurations. $(c_1 = 1.1, c_2 = 1.5)$

that some energy is lost from the waves in the course of the interaction. Table 3 gives the numerical values of the speeds of the waves, measured from the data; in this table $c_j$ is the speed of each individual pulse computed as a solitary wave, $v_0$ and $A_0$ are the initial (pre-collision) speed and amplitude, while $v_f$ and $A_f$ are the final (post-collision) speed and amplitude. The data show that the emitted

| Run | $c_j$ | $v_0$ | $A_0$ | $v_f$ | $A_f$ |
|---|---|---|---|---|---|
| 1 | 1.05 | 1.0501 | .1446 | 1.0499 | .1447 |
| | 1.10 | 1.0999 | .2795 | 1.0999 | .2789 |
| 2 | 1.1 | 1.0999 | .2795 | 1.1003 | .2790 |
| | 1.3 | 1.2997 | .7422 | 1.2964 | .7341 |
| 3 | 1.1 | 1.1000 | .2795 | 1.100 | 0.2797 |
| | 1.4 | 1.3996 | .9383 | 1.3939 | 0.9271 |
| 4 | 1.1 | 1.0998 | .2795 | 1.1003 | 0.2790 |
| | 1.5 | 1.5000 | 1.1165 | 1.4987 | 1.1144 |

TABLE 3. Numerical experiments on interacting two-pulse solutions of the Euler–Poisson equations: $c_j$ are the theoretical solitary wave speeds; $v_0$ and $v_f$ their observed speeds before and after the interaction; $A_0$ and $A_f$ their observed amplitudes before and after the interaction.

waves are very close in speed to the initial waves, while the plots of the scattering anomalies indicate that the faster wave loses speed during the interaction. The most significant qualitative difference between the KdV approximation and the solutions of the Euler–Poisson equations themselves lies in the displacement of the

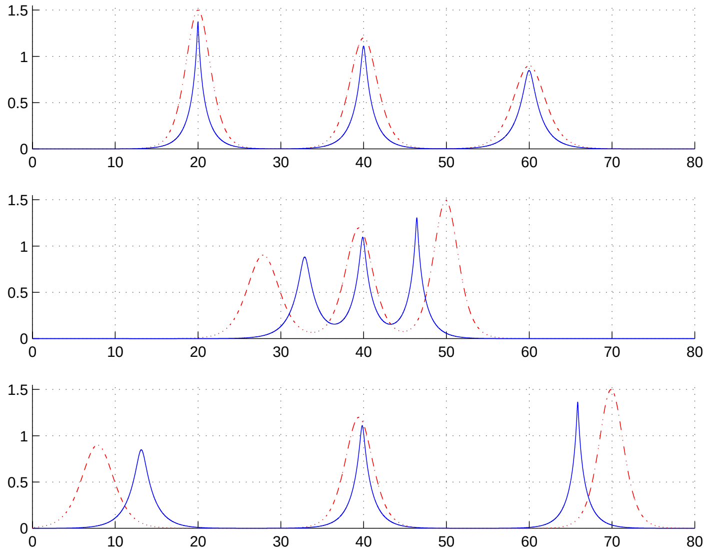



FIG. 3. Comparison of the KdV 3-soliton solution (dashed) with the $\varphi$ wave (solid) of the ion acoustic plasma equations at initial, intermediate, and final configurations. $(c_1 = 1.3, c_2 = 1.4, c_3 = 1.5)$

emerging waves from those of the 2-soliton solution of the KdV equation.

In a completely elastic interaction the waves retain their original shape, speed, and amplitude. Thus a rough measure of the elasticity of the interaction can be obtained by translating one of the waves, say the faster one, back to its original position and comparing it with the corresponding wave prior to the interaction. In the experiment discussed in Li & Sattinger [9] there was no difference graphically when this was done with the larger wave. A quantitative measure of the elasticity of the interaction can be obtained by defining a coefficient of elasticity of the interaction, as follows:

$$e = 1 - \frac{||v_0 - v_{ft}||_1}{||v_0||_1}, \qquad ||v||_1^2 = \int v_x^2 + v^2 \, dx.$$

Here, $v_0$ is the initial wave form, while $v_{ft}$ is the final waveform, translated back to its original position. By this measure the solitary wave interactions for the Euler–Poisson equations are highly elastic, even at high Mach numbers, in the sense that the faster wave regains its initial shape after the interaction, as shown in Table 4. The experimental runs in Table 4 are the same as those in Table 3.

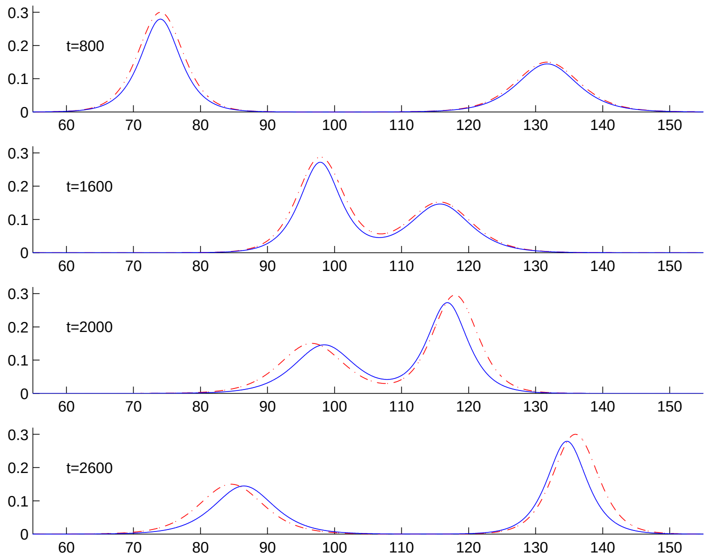



FIG. 4. Comparison of the KdV 2-soliton solution (dashed) with the $\varphi$ wave (solid) of the ion acoustic plasma equations at initial, two intermediate, and final configurations [9]. ($c_1 = 1.05, c_2 = 1.1$)

| Run | $M_1$ | $M_2$ | $e$ | $\Delta x$ |
|---|---|---|---|---|
| 1 | 0.0854 | 0.1709 | 0.9862 | 0.0245 |
| 2 | 0.1709 | 0.5126 | 0.9647 | 0.0122 |
| 3 | 0.1709 | 0.6835 | 0.9664 | 0.0037 |
| 4 | 0.1709 | 0.8544 | 0.9739 | 0.0031 |

TABLE 4. Elasticity coefficients $e$ of interacting two-pulse solutions of the Euler–Poisson equations with Mach numbers $M_1$ and $M_2$ (cf. Table 3); $\Delta x$ is the grid spacing.

## 4. Preliminary theoretical results

It is evident from the numerical experiments of § 3 that the KdV approximation of the Euler–Poisson equations (2.9) cannot provide complete details about solitary wave interactions alone. In this section we present our preliminary results regarding efforts in this direction, particularly commenting on the results of Schneider & Wayne [12] and Zou & Su [13], and our own progress on carrying out the perturbation expansion begun in Section 2.2 to higher order.

### 4.1. Time-scale of solitary wave interactions

In Eulerian variables, the Euler equations of water waves have a quantity $\eta(x,t)$, the elevation of the free surface from the undisturbed state, which, like the quan-

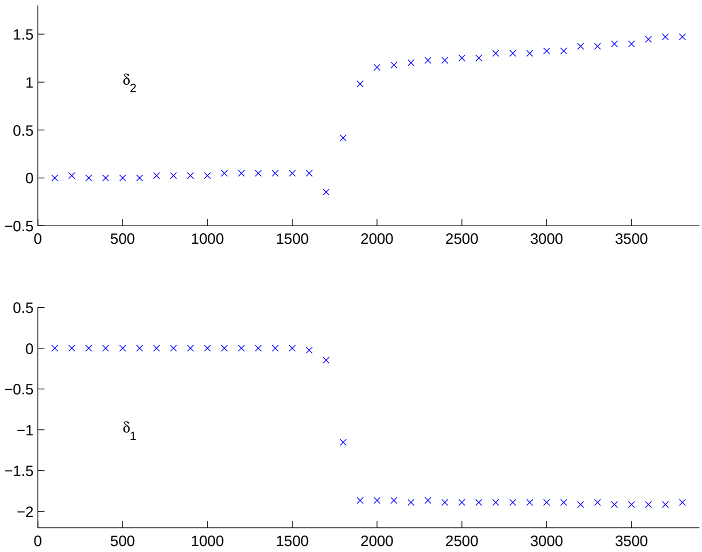



Fig. 5. Plots of the scattering anomalies. $\delta_j$ are difference between $j^{th}$ KdV wave and the corresponding plasma wave.

tities $n, v, \varphi$ from the Euler–Poisson equations, can be modelled at lowest order by the KdV equation. In a recent paper Schneider & Wayne [12] have rigorously studied the approximation of solutions of the Euler equations equations by solutions of the KdV equations. One of the corollaries in their work can be paraphrased in the following way.

Corollary 4.1 (Schneider & Wayne, Corollary 1.5 [12]). *Fix $s \geq 4$. For all $T_0 > 0$ there exist $C_2, \varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the following is true. If $\eta$ is sufficiently smooth (related to weighted $H^{s+6}$ and $H^{s+11}$ Sobolev spaces) then*

$$\sup_{t\in[0,T_0/\varepsilon^3]} \left\| \eta - \varepsilon^2 A_1(\varepsilon(\cdot - t), \varepsilon^3 t) + \varepsilon^2 A_2(\varepsilon(\cdot + t), \varepsilon^3 t) \right\|_{X^{s-3/2}} \leq C_2 \varepsilon^{5/2},$$

*where $X^{s-3/2}$ is a weighted Hölder space, and $A_1$ and $A_2$ satisfy right- and left-moving KdV equations.*

While this result has only been established for the Euler equations of water waves, it is easy to believe that the same sort of general result is also true for the Euler–Poisson equations. However, the occurrence of scattering anomalies in the numerical experiments of § 3 implies that the KdV equation by itself does not capture all of the relevant phenomena for interacting solitary waves. Of crucial importance is the time-scale on which interactions take place in the Euler–Poisson (or Euler) equations. To this end, we establish in Theorem 4.3 that the time-scale of interaction for two traveling waves is proportional to not only $\varepsilon^{-3}$ but also to a second parameter, the difference between the speeds of the two interacting waves. Independent of the size of $\varepsilon$, by making the speeds of the interacting waves close, one can make the time of interaction arbitrarily large, possibly larger than the time-scale covered by Schneider & Wayne.

The problem we consider is somewhat ambiguous, since solitary waves are not compactly supported, thus the waves never separate completely. To make the discussion concrete, we make the following definition:

**Definition 4.2.** Two solitary waves are separated to threshold $r\%$ $(0 < r < 100)$ if the solution contains two distinct pulses, and, somewhere, denoted $\alpha_r$, between the maxima of these two pulses the magnitude of the solution is $(r/100)A$ where $A$ is the $L^\infty$ norm of the solution.

In light of Definition 4.2 we are able to establish the following theorem.

**Theorem 4.3.** *The time required for two solitary waves of the KdV equation to completely interact, given a threshold of* $r\%$, *is given by*

$$T = \frac{\alpha_r}{\omega_1\omega_2(\omega_2 - \omega_1)} - \frac{1}{2\omega_1\omega_2(\omega_2 + \omega_1)} \log\left[\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}\right], \tag{4.1}$$

*where* $\operatorname{sech}^2 \alpha_r = 0.01r$.

*Proof.* The exact formula for the length of time required for the interaction of two solitary waves of the KdV equation is obtained by first calculating the time required for two free waves to interact, and then correcting for the scattering shift.

The relative velocity of two free waves with speeds $2\omega_j^2$, is $2(\omega_2^2 - \omega_1^2)$. The relative distance traveled by the overtaking wave during the course of the interaction is the sum of the widths of the two waves. If $l$ denotes the width of the solitary wave of speed $2\omega^2$, and we use $r\%$ as the threshold, we get $\omega l/2 = \alpha_r$, where $\operatorname{sech}^2 \alpha_r = .01r$. The total relative distance the faster wave must travel to completely overtake and pass the slower wave is therefore

$$d = l_1 + l_2 = 2\alpha_r\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right).$$

The time required for the interaction of two free waves is

$$\frac{2\alpha_r\left(\dfrac{1}{\omega_2} + \dfrac{1}{\omega_1}\right)}{2(\omega_2^2 - \omega_1^2)} = \frac{\alpha_r}{\omega_1\omega_2(\omega_2 - \omega_1)}.$$

During the interaction, the faster wave is shifted forward, and the slower wave is shifted backward, by the amounts

$$\frac{1}{\omega_2}\log\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}, \qquad \frac{1}{\omega_1}\log\frac{\omega_2 + \omega_1}{\omega_2 - \omega_1},$$

respectively. The boost due to the scattering of the waves effectively reduces the net distance the interacting waves must travel by

$$\left(\frac{1}{\omega_2} - \frac{1}{\omega_1}\right) \log \frac{\omega_2 + \omega_1}{\omega_2 - \omega_1}.$$

Dividing this expression by the relative velocity and simplifying, we obtain the second term in (4.1).

According to Schneider & Wayne [12], solutions of the Euler equations, and presumably the Euler–Poisson equations, are close to the scaled 2-soliton solution $\varepsilon^2 u(\varepsilon x, \varepsilon^3 t)$, $u$ being the 2-soliton solution of KdV, on a time scale of order $\varepsilon^{-3}$. In this scaling the speeds $\omega_1$ and $\omega_2$ of the KdV solitons scale as $\varepsilon\omega_1$ and $\varepsilon\omega_2$, and the interaction time $T$ in (4.1) therefore scales as $\varepsilon^{-3}$. This shows that the time scale of the interaction is the same as that for the validity of the KdV approximation. However, Eqn. (4.1) further shows that the interaction time goes to infinity as $\omega_1 \to \omega_2$, as one would expect. Thus the time required for the complete interaction of two solitary waves depends not only on the small parameter the theory, $\varepsilon$, but also on a second parameter $\delta = \omega_1 - \omega_2$. Therefore one cannot assert, without further analysis, that the result obtained in [12] is always sufficient to see the interaction of two solitary waves for sufficiently small $\varepsilon$.

### 4.2. The perturbation scheme

As we have seen in § 2.2 the KdV equation is the leading term in a singular perturbation scheme for the Euler–Poisson equations. In this section we derive a method for obtaining a next-order correction to the KdV equation, and obtain an analytical expression for the second order term to the 2-soliton solution.

Recall from § 2.2 that at each order, $k$, in the perturbation expansion for approximating the Euler–Poisson equations we are required to solve a system of equations (2.6) of the form

$$(n_k - v_k)_x = f_k^{(1)}, \quad (v_k - \varphi_k)_x = f_k^{(2)}, \quad (\varphi_k - n_k)_x = f_k^{(3)}, \tag{4.2}$$

where the solvability condition (2.7) is given by $f_k^{(1)} + f_k^{(2)} + f_k^{(3)} = 0$ and the general solutions for $n_k$ and $v_k$ in terms of $\varphi_k$ are given by (2.8). The solvability condition at order four gives the KdV equation (2.9) for $\varphi_1$.

At order $\varepsilon^6$ we have

$$(n_3 - v_3)_x = n_{2,t} + (n_1 v_2 + n_2 v_1)_x \tag{4.3a}$$

$$(v_3 - \varphi_3)_x = v_{2,t} + (v_1 v_2)_x \tag{4.3b}$$

$$(\varphi_3 - n_3)_x = \varphi_{2,xxx} - (\varphi_1\varphi_2)_x - \frac{1}{6}(\varphi_1^3)_x. \tag{4.3c}$$

The solvability condition at this order is obtained by setting the sum of the right hand sides equal to zero, as before. This leads to

$$\mathcal{L}\varphi_2 = D\mathcal{F}(\varphi_1), \tag{4.4}$$

where $\mathcal{L}$ is the linearized KdV operator at $\varphi_1$

$$\mathcal{L}w = w_t + \frac{1}{2}w_{xxx} + (\varphi_1 w)_x$$

and

$$\begin{aligned} D\mathcal{F}(\varphi_1) &= \frac{5}{4}\varphi_1\varphi_{1,xxx} + \frac{1}{2}(\varphi_{1,x}^2)_x + \frac{3}{4}\varphi_{1,xxt} \\ &= \frac{\partial}{\partial x}\left[\frac{5}{4}\varphi_1\varphi_{1,xx} - \frac{1}{8}\varphi_{1,x}^2 - \frac{3}{8}(\varphi_{1,xx} + \varphi_1^2)_{xx}\right] \\ &= \frac{\partial}{\partial x}\left[\frac{1}{2}\varphi_1\varphi_{1,xx} - \frac{7}{8}\varphi_{1,x}^2 - \frac{3}{8}\varphi_{1,xxxx}\right]. \end{aligned}$$

Thus,

$$\mathcal{F}(\varphi_1) = \frac{1}{2}\varphi_1\varphi_{1,xx} - \frac{7}{8}\varphi_{1,x}^2 - \frac{3}{8}\varphi_{1,xxxx}. \tag{4.5}$$

Using the analytical results derived in Appendix A we solve this equation explicitly when $\varphi_1$ is a 2-soliton solution of KdV (similar results hold for the $n$-solitons but we do not consider them here). From (A.14b) and (A.13) we find

$$\mathcal{F}(\varphi_1) = \frac{7}{4}\varphi_1\varphi_{1,xx} - \frac{1}{4}\varphi_{1,x}^2 + \frac{5}{12}\varphi_1^3 - 72\sum_{j=1}^{2}\omega_j^5 F_j, \tag{4.6}$$

where $F_j$ is a squared eigenfunction of the Schrödinger operator associated with the KdV equation (see Appendix A). The derivative of the first three terms in (4.6) is

$$\frac{7}{4}\varphi_1\varphi_{1,xxx} + \frac{5}{8}\left(\varphi_{1,x}^2\right)_x + \frac{5}{4}\varphi_1^2\varphi_{1,x}. \tag{4.7}$$

From (A.13) with $j = 2$ and (A.14a) we get the identity

$$\varphi_1\varphi_{1,xxx} = 48\varphi_1(\omega_1^3 DF_1 + \omega_2^3 DF_2) - 2\varphi_1^2\varphi_{1,x}, \tag{4.8}$$

and the three terms in (4.7) reduce to

$$\frac{9}{4}\varphi_1^2\varphi_{1,x} + \frac{5}{4}\varphi_{1,x}\varphi_{1,xx} + 84\varphi_1(\omega_1^3 DF_1 + \omega_2^3 DF_2) \tag{4.9}$$

Putting all these identities together, we see that the right hand side $h = D\mathcal{F}(\varphi_1)$ of (4.4) can be written as

$$h = \frac{5}{4}\varphi_{1,x}\varphi_{1,xx} - \frac{9}{4}\varphi_1^2\varphi_{1,x} + \sum_{j=1}^{2}\left(84\omega_j^3\varphi_1 - 72\omega_j^5\right)DF_j. \tag{4.10}$$

Using (A.22), (A.23), (A.21), and (A.24) we find, after some computations (cf. also [13]), that the second order correction to the 2-soliton solution is

$$\varphi_2 = -\frac{9}{4}\varphi_1^2 - 4\varphi_{1,xx} + \sum_{j=1}^{2}\big(84\omega_j^3 E_j - 72\omega_j^5 tDF_j\big), \tag{4.11}$$

where

$$E_j = \omega_1\left(\frac{\partial F_1}{\partial\omega_j}\right)_{\theta_j} + \omega_2\left(\frac{\partial F_2}{\partial\omega_j}\right)_{\theta_j}.$$

To this particular solution we may add any linear combination of homogeneous solutions of the linearized KdV equation:

$$\sum_{j=1}^{2}\frac{\partial\varphi_1}{\partial\alpha_j}\Delta\alpha_j + \frac{\partial\varphi_1}{\partial\omega_j}\Delta\omega_j, \tag{4.12}$$

(see Theorem A.3). Presumably, the coefficients $\Delta\alpha_j$ and $\Delta\omega_j$ are uniquely determined at the next order of the perturbation scheme by making the equations at next order solvable; but the computations at third order are quite complicated.

The secular terms $tDF_j$ in (4.11) are resonance terms due to the appearance of $DF_j$ in the right side of (4.4). These terms can be eliminated by modulating the 2-soliton solution; this will be discussed in § 4.3. Notice that in the second order correction to the 1-soliton solution these secular terms can be eliminated by restricting to traveling waves. The second order approximation to the solitary wave is

$$\Sigma = \Sigma_1 + \Sigma_2, \tag{4.13}$$

where $\Sigma_1$ is the KdV solitary wave

$$\Sigma_1(\theta) = 6\,\omega^2\text{sech}^2(\theta), \qquad \theta = \omega(x - 2\omega^2 t - \alpha),$$

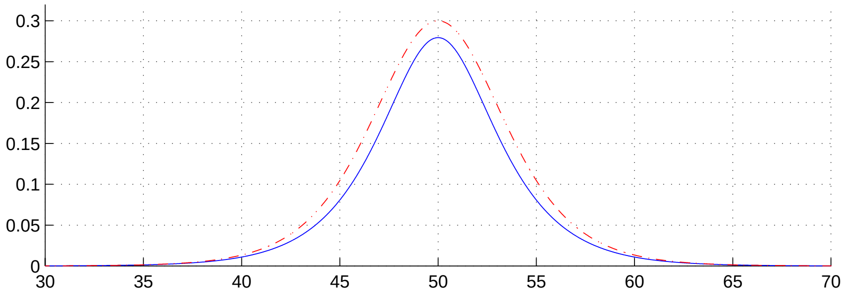
and $\Sigma_2$ is a solution of (4.4) for $\varphi_1(x,t) = \Sigma_1$. In this case a solution can be found by transforming to a moving coordinate system; then (4.4) reduces to a third order ordinary differential equation. Integrating once, we obtain

$$\Sigma_2'' + (12\text{sech}^2\theta - 4)\Sigma_2 = 9\omega^4\big(36\,\text{sech}^4\theta - 4\,\text{sech}^2\theta - 43\,\text{sech}^6\theta\big). \tag{4.14}$$

The explicit solution to this equation was obtained by the method of variation of parameters, with assistance from Maple,

$$\Sigma_2(\theta) = \frac{9}{4}\omega^4\,[20 - 23\cosh\,2\theta + 4\theta\sinh\,2\theta]\,\text{sech}^4(\theta). \tag{4.15}$$

We see that there are no secular terms when only one solitary wave is present, and it is not necessary to introduce an amplitude wave speed correction, as in [13]. The first and second order approximations are depicted in Figure 6.



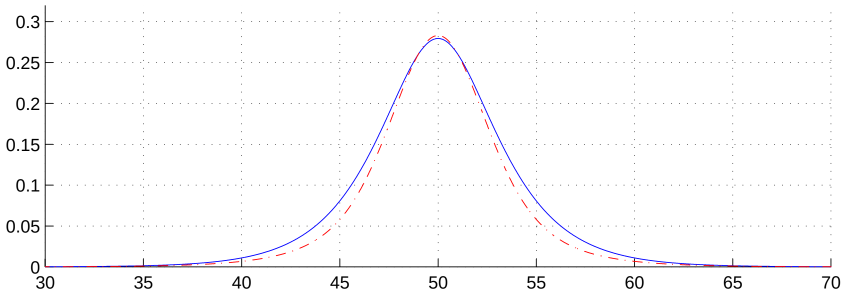




FIG. 6. Upper: The solitary $\varphi$-wave solution of the ion acoustic plasma equation (solid) and the KdV solitary wave (dashed). Lower: $\varphi$ wave (solid) vs. the second KdV approximation (dashed). The wave speed is $c = 1.1$, $\omega = 0.2231$.

### 4.3. Modulation and resonant interactions

We saw in the last section that the second order term, given by (4.11), has secular terms which grow in time for interacting solitary waves. These terms must be eliminated in order to get a uniform approximation at second order. In this section we formally set $\varepsilon = 1$ and use instead $\omega_1$ and $\omega_2$ as small parameters in the theory. For example, the solitary wave $6\omega^2 \mathrm{sech}^2\theta$ is second order in $\omega$. By Theorem A.7 the 2-soliton solution $\varphi_1$ is second order in $\omega_1$ and $\omega_2$. Since $\omega_1 < \omega_2$ we use $\omega_2$ as a measure of relative size of terms in the expansion.

Returning to the formal expansion of § 4.2, we write the first two terms as $\varphi = \varphi_1 + \varphi_2$, where $\mathcal{L}(\varphi_2) = D\mathcal{F}(\varphi_1)$. By Theorem A.3

$$\frac{\partial \varphi_1}{\partial \alpha_j} = -12\omega_j DF_j.$$

Hence by (4.11) and (4.12)

$$\varphi_2 = \widetilde{\varphi}_2 + N_2, \tag{4.16}$$

where

$$N_2 = \sum_{j=1}^{2} \frac{\partial \varphi_1}{\partial \alpha_j}(\Delta\alpha_j + 6\omega_j^4\, t) + \frac{\partial \varphi_1}{\partial \omega_j}(\Delta\omega_j)$$

and

$$\widetilde{\varphi}_2 = -\frac{9}{4}\varphi_1^2 - 4\varphi_{1,xx} + 84\sum_{j=1}^{2}\omega_j^3 E_j.$$

**Lemma 4.4.** *For fixed* $\lambda = \omega_1/\omega_2$, $\widetilde{\varphi}_2$ *is* $O(\omega_2^4)$, *uniformly in* $x$ *and* $t$. *In order that* $N_2$ *be* $O(\omega_2^4)$ *it is sufficient that*

$$\Delta\alpha_j = O(\omega_j), \qquad \Delta\omega_j = O(\omega_j^2), \qquad |t| = O(\omega_2^{-3}),$$

*and that*

$$|x - 6\omega_j^2 t - \alpha_j| = O(1).$$

*Proof.* By Theorem A.7, $\varphi_1$ is $O(\omega_2^2)$ and differentiation with respect to $x$ is $O(\omega_2)$. It follows that $\varphi_1^2$ and $\varphi_{1,xx}$ are $O(\omega_2^4)$. Again by Theorem A.7, $E_j$ is uniformly $O(\omega_2)$ hence the terms $\omega_j^3 E_j$ are $O(\omega_2^4)$, uniformly in $x$ and $t$.

We have

$$\frac{\partial \varphi_1}{\partial \alpha_j} = \frac{\partial \varphi_1}{\partial \theta_j}\frac{\partial \theta_j}{\partial \alpha_j} = -\omega_j \frac{\partial \varphi_1}{\partial \theta_j},$$

$$\frac{\partial \varphi_1}{\partial \omega_j} = \frac{\partial \varphi_1}{\partial \theta_j}\frac{\partial \theta_j}{\partial \omega_j} = \frac{\partial \varphi_1}{\partial \theta_j}(x - 6\omega_j^2 t - \alpha_j).$$

Since derivatives with respect to $\theta_j$ are order 1, and $\varphi_1$ is order $\omega_2^2$, the statements concerning the order of $N_2$ follow.

When the conditions of Lemma 4.4 are satisfied, the second order approximation can be written

$$\varphi = \varphi_1 + (\widetilde{\varphi}_2 + N_2) = (\varphi_1 + N_2) + \widetilde{\varphi}_2.$$

Let $\widetilde{\varphi}_1 = \varphi_1(\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\omega}_1, \widetilde{\omega}_2)$, where

$$\widetilde{\omega}_j = \omega_j + \Delta\omega_j, \qquad \widetilde{\alpha}_j = \alpha_j + \Delta\alpha_j + 6\omega_j^4 t,$$

and

$$\widetilde{\theta}_j = \widetilde{\omega}_j(x - 2\widetilde{\omega}_j^2 t - \widetilde{\alpha}_j) = \widetilde{\omega}_j(x - (2\widetilde{\omega}_j^2 + 6\omega_j^4)t - \alpha_j - \Delta\alpha_j).$$

Then

$$\widetilde{\varphi}_1 = \varphi_1 + N_2 + O(\omega_2^6),$$

where $\widetilde{\varphi}_1$ is the modulated 2-soliton solution and $\varphi_1 = \varphi_1(\theta_1, \theta_2, \omega_1, \omega_2)$ is the original 2-soliton solution.

**Theorem 4.5.** *When the conditions of Lemma 4.4 are satisfied, the formal approximation to the $\varphi$ wave of the plasma equations is given to second order by $\widetilde{\varphi} = \widetilde{\varphi}_1 + \widetilde{\varphi}_2$.*

There are three time regimes in the 2-soliton interaction, which must be treated separately. In the first and last, the two solitons are separated and do not interact. Thus, $\varphi_1$ may be approximated by the sum of the two solitary waves. In this case, the second order correction is given by sums of solitary waves and their second order corrections (4.15). Prior to the interaction the initial wave speeds are given by $2\omega_j^2$. After the interaction, the phases and speeds are somewhat different due to the interaction.

In the vicinity of the interaction, we use the modulated two-soliton interaction $\widetilde{\varphi}$ of Theorem 4.5 constructed above. A solution of this type was proposed in Zou & Su [13] for the entire interaction, $-\infty < t < \infty$, but this approximation is not justified for all time, since the conditions of Lemma 4.4 are met only over time intervals of order $O(\omega_2^{-3})$. For the experiment in [9], $\omega_2 = .2231$ and $\omega_2^{-3} \approx 90$, whereas the two solitary waves are found to overlap during the time interval $1500 \leq t \leq 2300$. The terms $\partial\varphi_1/\partial\omega_j$ in the nullspace of $\mathcal{L}$ were not included in the second order approximation in Zou & Su [13]. On the other hand, Figure 2 shows that the faster plasma wave drops behind the KdV wave during the interaction, hence we must have

$$2\widetilde{\omega}_2^2 + 6\omega_2^4 < 2\omega_2^2.$$

This implies that $\widetilde{\omega}_2 < \omega_2$, and hence $\Delta\omega_2 < 0$, and some correction to the $\omega_j$ is necessary. The theoretical values of $\Delta\omega_j$ and $\Delta\alpha_j$ cannot be obtained by considering second order terms alone. They presumably are determined by casting out resonances at third order, but the computations at third order are too complicated to be obtained analytically [13].

## A. Theoretical results regarding KdV 2-solitons

### A.1. Eigenfunctions of the Schrödinger operator

The KdV equation which occurs in the approximation of the Euler–Poisson equations takes the form (cf. (2.9))

$$u_t + \frac{1}{2}u_{xxx} + uu_x = 0. \tag{A.1}$$

Equation (A.1) can be written in the operator form $\dot{L} = [B, L]$, where

$$L = D^2 + \frac{u}{3}, \qquad B = -2D^3 - \frac{1}{2}(uD + Du).$$

The wave functions satisfy the pair of equations

$$L\psi + k^2\psi = 0, \qquad \psi_t - B\psi = 0, \tag{A.2}$$

and (A.1) is the compatibility condition for this over-determined system. We denote by $\varphi_+(x,t,k)$ and $\psi_+(x,t,k)$ the solutions of (A.2) with the asymptotic behaviors

$$\varphi_+(x,t,k) \sim e^{-ik(x-2k^2t)}, \qquad x \to -\infty,$$

$$\psi_+(x,t,k) \sim e^{ik(x-2k^2t)}, \qquad x \to \infty.$$

The wave function $\psi_+$ tends to zero exponentially as $x \to \infty$ for $\mathrm{Im}(k) > 0$; while $\varphi_+$ tends to zero exponentially as $x \to -\infty$ for $\mathrm{Im}(k) > 0$. The eigenvalues of the Schrödinger equation $L\psi + k^2\psi = 0$ are those values $k = i\omega_j$ $(\omega_j > 0)$ for which $\varphi_+(x,t,i\omega_j) = c_j\psi_+(x,t,i\omega_j)$ for some constant $c_j$, called the *coupling coefficient*. The parameters $\omega_j$ are precisely those which appear in (2.11). We denote the corresponding eigenfunctions, which decay exponentially as $x \to \pm\infty$, by $\psi_j$.

The eigenfunctions $\psi_j$ can be obtained by solving a linear system of algebraic equations, which are obtained as a finite dimensional reduction of the Gel'fand–Levitan integral equation for inverse scattering [6]. For the 2-soliton solution the eigenfunctions are obtained in closed form, as follows. Let

$$D_{jk} = \delta_{jk} + \frac{e^{-(\theta_j+\theta_k)}}{\omega_j + \omega_k}, \qquad E = \begin{pmatrix} e^{-\theta_1} \\ e^{-\theta_2} \end{pmatrix}.$$

By Cramer's rule

$$\psi_k = -\frac{D_k}{\tau},$$

where $\tau = \tau(x,t)$ is the determinant of the matrix $D = ||D_{jk}||$, and $D_k$ is the determinant of the matrix obtained by replacing the $k^{th}$ column of $D$ by $E$, $k = 1, 2$.

The two wave functions are given by

$$D_1 = e^{-\theta_1} - e^{-(\theta_1+2\theta_2)}\frac{\omega_2-\omega_1}{2\omega_2(\omega_2+\omega_1)} = 2e^{-(\theta_1+\theta_2+\beta_2)}\sinh(\theta_2+\beta_2), \qquad \text{(A.3)}$$

$$D_2 = e^{-\theta_2} + e^{-2\theta_1-\theta_2}\frac{\omega_2-\omega_1}{2\omega_1(\omega_2+\omega_1)} = 2e^{-(\theta_1+\theta_2+\beta_1)}\cosh(\theta_1+\beta_1), \qquad \text{(A.4)}$$

where

$$\beta_j = -\frac{1}{2}\log\frac{\omega_2-\omega_1}{2\omega_j(\omega_2+\omega_1)};$$

and

$$\tau = 1 + \frac{e^{-2\theta_1}}{2\omega_1} + \frac{e^{-2\theta_2}}{2\omega_2} + \frac{e^{-2(\theta_1+\theta_2)}}{4\omega_1\omega_2}\left(\frac{\omega_1-\omega_2}{\omega_1+\omega_2}\right)^2. \qquad \text{(A.5)}$$

Thus

$$\psi_1 = -\frac{2e^{-\beta_2}\sinh(\theta_2+\beta_2)}{\kappa}, \qquad \psi_2 = -\frac{2e^{-\beta_1}\cosh(\theta_1+\beta_1)}{\kappa}, \qquad \text{(A.6)}$$

where

$$\kappa = e^{\theta_1+\theta_2}\tau = e^{\theta_1+\theta_2} + \frac{e^{-(\theta_1+\theta_2)}}{4\omega_1\omega_2}\left(\frac{\omega_1-\omega_2}{\omega_1+\omega_2}\right)^2 + \frac{e^{\theta_2-\theta_1}}{2\omega_1} + \frac{e^{\theta_1-\theta_2}}{2\omega_2}. \tag{A.7}$$

For $k = i\omega_j$ there is a second solution $\varphi_j$ of the equations (A.2) which grows exponentially as $x \to \pm\infty$. (Since the potential $u$ decays exponentially as $x \to \pm\infty$, there is only one exponentially decaying solution. The second solution grows exponentially.)

**Lemma A.1.** *The second solution $\varphi_j$ of the Schrödinger equation in* (A.2) *can be obtained as*

$$\varphi_j = \frac{\partial}{\partial k}(\varphi_+ - c_j\psi_+)\Big|_{k=i\omega_j}, \tag{A.8}$$

*where $c_j$ is the coupling coefficient. The solution $\varphi_j$ grows linearly in $x$ and $t$.*

*Proof.* The linear growth of $\varphi_j$ in $x$ and $t$ is due to the differentiation of the wave functions $\varphi_+(x,t,k)$ and $\psi_+(x,t,k)$ with respect to the spectral parameter $k$. Hence $G_j$ contains secular terms which grow linearly in $x$ and $t$.

Differentiating the first equation in (A.2) with respect to $k$, we find that the partial derivatives of $\varphi_+$ and $\psi_+$ with respect to the spectral parameter $k$ satisfy

$$(L+k^2)\frac{\partial\varphi_+}{\partial k} + 2k\varphi_+ = 0, \qquad (L+k^2)\frac{\partial\psi_+}{\partial k} + 2k\psi_+ = 0.$$

Now set $k = i\omega_j$ in each of these equations, multiply the second equation by $c_j$ and subtract it from the first equation. Since $\varphi_+(x,t,i\omega_j) = c_j\psi_+(x,t,i\omega_j)$, we find that $L\varphi_j - \omega_j^2\varphi_j = 0$, where $\varphi_j$ is given in (A.8).

Since the two soliton solution $u$ decays exponentially as $x \to \pm\infty$, only one solution of (A.2) decays exponentially at both $\pm\infty$. Since $\varphi_j$ is linearly independent of $\psi_j$, it necessarily grows exponentially.

**Remark A.2.** The coupling coefficients are given by:

$$c_j = \frac{1}{2}e^{2\omega_j\alpha_j}\frac{\omega_1-\omega_2}{\omega_j(\omega_1+\omega_2)}.$$

### A.2. Solutions of the linearized KdV equation

The pair of functions $F_j = \psi_j^2$, $G_j = \psi_j\varphi_j$, called the *squared eigenfunctions*, satisfy the equations

$$[D^3 + \frac{2}{3}(uD + Du\cdot) - 4\omega_j^2 D]F_j = 0, \tag{A.9}$$

$$\frac{\partial}{\partial t}F_j + \frac{1}{2}D^3F_j + uDF_j = 0, \tag{A.10}$$

where $u$ is the corresponding solution of the KdV equation [6]. Equation (A.10) is called the *associated linear equation.* Differentiating (A.10) with respect to $x$, we find that $DF_j$ and $DG_j$ satisfy the linearized KdV equation

$$\mathcal{L}DF_j = \mathcal{L}DG_j = 0, \tag{A.11}$$

where

$$\mathcal{L}w = w_t + \frac{1}{2}w_{xxx} + (uw)_x \tag{A.12}$$

is the KdV operator, linearized at $u$.

The derivatives of the 2-soliton solution with respect to the four parameters $\alpha_1, \alpha_2, \omega_1, \omega_2$ are also solutions of (A.12). This follows by differentiating the KdV equation itself for the 2-soliton solution with respect to each of these four parameters. The relation between these two sets of solutions is given in the following theorem.

**Theorem A.3.** *The sets*

$$\left\{\frac{\partial u}{\partial \alpha_1}, \frac{\partial u}{\partial \alpha_2}, \frac{\partial u}{\partial \omega_1}, \frac{\partial u}{\partial \omega_2}\right\}, \qquad \{DF_1, DF_2, DG_1, DG_2\},$$

*are both solutions of the linearized KdV equation. The relationship between them is given by*

$$\frac{\partial u}{\partial a_j} = \sum_{k=1}^{4} H_{jk} DF_k,$$

*where* $F_3 = G_1$, $F_4 = G_2$, $a = (\alpha_1, \alpha_2, \omega_1, \omega_2)$, *and*

$$H_{jk} = -\,12\omega_j e^{2\omega_j\alpha_j}\delta_{jk}, \qquad j = 1, 2, \quad 1 \le k \le 4;$$

$$H_{jk} = 24\left(\frac{\omega_1+\omega_2}{\omega_1-\omega_2}\right)^2 \omega_{j-2}^2 e^{-2\omega_{j-2}\alpha_{j-2}}\delta_{jk}, \qquad j, k = 3, 4;$$

$$H_{31} = \frac{6e^{2\omega_1\alpha_1}}{\omega_1(\omega_2^2-\omega_1^2)}\Big[(2\alpha_1\omega_1-1)(\omega_1^2-\omega_2^2)+2\omega_1\omega_2\Big];$$

$$H_{42} = \frac{6e^{2\omega_2\alpha_2}}{\omega_2(\omega_1^2-\omega_2^2)}\Big[(2\alpha_2\omega_2-1)(\omega_2^2-\omega_1^2)+2\omega_1\omega_2\Big];$$

$$H_{32} = \frac{12\omega_2 e^{2\omega_2\alpha_2}}{\omega_2^2-\omega_1^2}, \qquad H_{41} = \frac{12\omega_1 e^{2\omega_1\alpha_1}}{\omega_1^2-\omega_2^2}.$$

*Proof.* These relations were determined by extensive computations using the software package Maple. One may simplify the relationships by setting the phase constants $\alpha_1 = \alpha_2 = 0$.

### A.3. Identities

The KdV equation has also the structure of an infinite dimensional Hamiltonian system, and can be written in the form

$$u_t = \frac{d}{dx}\frac{\delta H_2}{\delta u}, \qquad H_2(u) = \int_{-\infty}^{\infty} \frac{u_x^2}{4} - \frac{u^3}{6}\,dx,$$

the functional $H_2$ being the Hamiltonian for the system. There are, moreover, an infinite number of Hamiltonians in involution with $H_2$ with respect to the Gardner–Poisson bracket, [10].

When the solution of the KdV equation is a multi-soliton, the gradients of this hierarchy of Hamiltonians are linear combinations of the squared eigenfunctions [6]. (In Theorem 3.5 of [6] the $A_n$ in are obtained as solutions the Lenard recursion relation, equation (3.20) in that article, but these are the gradients of the densities, not the densities themselves, as stated in the article.) With the present scaling,

$$\frac{\delta H_j}{\delta u} = (-2)^{j-1} 12 \sum_{k=1}^{2} \omega_k^{2j-1} F_k. \tag{A.13}$$

**Remark A.4.** The normalization of the squared eigenfunctions in (A.13) is that obtained by solving the linear system of algebraic equations which comes from the Gel'fand–Levitan equation of inverse scattering [6].

The gradients of the first three conservation laws are:

$$\frac{\delta H_1}{\delta u} = u, \qquad \frac{\delta H_2}{\delta u} = -\frac{1}{2}(u_{xx} + u^2), \tag{A.14a}$$

$$\frac{\delta H_3}{\delta u} = \frac{1}{4}u_{xxxx} + \frac{5}{6}uu_{xx} + \frac{5}{12}u_x^2 + \frac{5}{18}u^3. \tag{A.14b}$$

In particular, taking $j = 1$ in (A.13), we obtain a representation of the 2-soliton solution itself as a sum of the squared eigenfunctions:

$$u = 12 \sum_{k=1}^{2} \omega_k F_k. \tag{A.15}$$

We derive a number of identities which are used in the perturbation theory of § 4.2. Since $uDF_j = D(uF_j) - F_j Du$, it follows from (A.15) that

$$\mathcal{L}F_j = F_j Du, \qquad \mathcal{L}G_j = G_j Du. \tag{A.16}$$

By a direct computation, using (A.9), (A.10), (A.12), and (A.16), we find

$$\mathcal{L}\big[D\big((x - 6\omega_j^2 t)F_j\big)\big] = \mathcal{L}\big[(x - 6\omega_j^2 t)DF_j + F_j\big] = -uDF_j. \tag{A.17}$$

By (A.15) we find, for $\alpha_j = 0$,

$$\begin{aligned}\frac{\partial u}{\partial \omega_j} =& 12F_j + 12\left(\omega_1 \frac{\partial F_1}{\partial \theta_j} + \omega_2 \frac{\partial F_2}{\partial \theta_j}\right)(x - 6\omega_j^2 t) + 12E_j \\ =& 12\big[F_j + \frac{1}{12}(x - 6\omega_j^2 t)\frac{\partial u}{\partial \theta_j} + E_j\big],\end{aligned} \tag{A.18}$$

where

$$E_j = \omega_1 \left(\frac{\partial F_1}{\partial \omega_j}\right)_{\theta_j} + \omega_2 \left(\frac{\partial F_2}{\partial \omega_j}\right)_{\theta_j}. \tag{A.19}$$

Here $(\ )_{\theta_j}$ denotes partial differentiation with $\theta_j$ held constant.

**Lemma A.5.** *We have*

$$\frac{\partial u}{\partial \theta_j} = 12DF_j,$$

*where $u$, the 2-soliton solution, is regarded as a function of $\theta_j$, $\omega_j$.*

*Proof.* By Theorem A.3,

$$\frac{\partial u}{\partial \alpha_j} = -12\omega_j DF_j,$$

at $\alpha_j = 0$. On the other hand,

$$\frac{\partial u}{\partial \alpha_j} = \frac{\partial u}{\partial \theta_j}\frac{\partial \theta_j}{\partial \alpha_j} = -\omega_j \frac{\partial u}{\partial \theta_j},$$

and the result follows.

As a consequence of Lemma A.5, (A.18) can be written

$$\begin{aligned}\frac{\partial u}{\partial \omega_j} =& 12\big[F_j + (x - 6\omega_j^2 t)DF_j + E_j\big] \\ =& 12\big[D\big((x - 6\omega_j^2 t)F_j\big) + E_j\big].\end{aligned} \tag{A.20}$$

**Lemma A.6.** *The identity*

$$\mathcal{L}(E_j) = uDF_j \tag{A.21}$$

*holds for any 2-soliton solution $u$. The identities*

$$\mathcal{L}u_{xx} = -(u_x^2)_x = -2u_x u_{xx}, \tag{A.22}$$

$$\mathcal{L}u^2 = 3u_x u_{xx} + u^2 u_x, \tag{A.23}$$

$$\mathcal{L}(tDF_j) = DF_j + t\mathcal{L}(DF_j) = DF_j. \tag{A.24}$$

*hold for any solution $u$ of the KdV equation.*

*Proof.* By (A.17), (A.20) and Theorem A.3 we find

$$0 = \mathcal{L}\left(\frac{\partial u}{\partial \omega_j}\right) = 12\mathcal{L}\big[D\big((x - 6\omega_j^2 t)F_j\big) + E_j\big]$$

$$= 12\big[\mathcal{L}(E_j) - uDF_j\big].$$

This proves (A.21). Equation (A.22) is obtained by differentiating the KdV equation twice with respect to $x$. Equation (A.23) is obtained by a direct computation, using the KdV equation for $u$. Finally, (A.24) follows from (A.11).

## A.4. Magnitude estimates

We conclude this appendix with some order of magnitude estimates for the various functions introduced. The parameters $\omega_1$ and $\omega_2$ are the small parameters of the KdV theory. Since $0 < \omega_1 < \omega_2$, we use $\omega_2$ as a measure of the order of magnitude.

**Theorem A.7.** *The 2-soliton solution* (2.10) *is $O(\omega_2^2)$ uniformly in $x$ and $t$; the squared eigenfunctions $F_j$ are each $O(\omega_j)$; and the $E_j$ defined in* (A.19) *each satisfy $0 \le E_j \le C_j(\lambda)\omega_j$, uniformly in $x$ and $t$, where $\lambda = \omega_2/\omega_1$. The constants $C_j$ tend to infinity as $\lambda$ tends to 0 or 1.*

*Proof.* By the chain rule,

$$\frac{\partial}{\partial x} = \sum_{j=1}^{2} \omega_j \frac{\partial}{\partial \theta_j}.$$

Moreover, the derivatives of $\log \tau$ with respect to $\theta_1$ and $\theta_2$ are of order 1, since they are ratios of exponential functions of $\theta_1$ and $\theta_2$. (Recall also that $\tau \ge 1$.) The second derivative with respect to $x$ therefore is a sum of terms of order 1 with coefficients $\omega_1^2$, $\omega_2^2$, $\omega_1\omega_2$. Since $\omega_1 < \omega_2$, all these terms are $O(\omega_2^2)$, and so $u$ is $O(\omega_2^2)$, uniformly in $x$ and $t$.

By (A.3) and (A.7)

$$\psi_1 = -\frac{D_1}{\tau} = \frac{Be^{-\theta_2} - e^{\theta_2}}{\kappa},$$

where

$$B = \frac{\omega_2 - \omega_1}{2\omega_2(\omega_2 + \omega_1)}.$$

Since the entries of $\kappa$ are positive,

$$\frac{e^{\theta_2}}{\kappa} \leq \frac{e^{\theta_2}}{e^{\theta_1+\theta_2} + \dfrac{e^{\theta_2-\theta_1}}{2\omega_1}} = \frac{1}{e^{\theta_1} + \dfrac{e^{-\theta_1}}{2\omega_1}}$$

$$\leq \sqrt{\frac{\omega_1}{2}} \operatorname{sech}\left(\theta_1 + \tfrac{1}{2}\log 2\omega_1\right) \leq \sqrt{\frac{\omega_1}{2}}.$$

By the same arguments, $Be^{-\theta_2}/\kappa$ is bounded above by $\sqrt{2\omega_1}$, so

$$-\sqrt{\frac{\omega_1}{2}} \leq \psi_1 \leq \sqrt{2\omega_1}$$

and $0 \leq F_1 \leq 2\omega_1$. Similarly, $0 \leq F_2 \leq 2\omega_2$ for all $x$ and $t$.

By (A.6),

$$\left(\frac{\partial F_j}{\partial \omega_k}\right)_{\theta_k} = -2F_j \frac{1}{\kappa}\left(\frac{\partial \kappa}{\partial \omega_k}\right)_{\theta_k}.$$

We have

$$\left(\frac{\partial \kappa}{\partial \omega_1}\right)_{\theta_1} = e^{-(\theta_1+\theta_2)} \frac{\partial}{\partial \omega_1} \frac{1}{4\omega_1\omega_2}\left(\frac{\omega_1-\omega_2}{\omega_1+\omega_2}\right)^2 - \frac{1}{2\omega_1^2} e^{\theta_2-\theta_1}$$

$$= -e^{-(\theta_1+\theta_2)}\left[\frac{1}{4\omega_1^2\omega_2}\left(\frac{\omega_1-\omega_2}{\omega_1+\omega_2}\right)^2 + \frac{1}{\omega_1}\frac{\omega_2-\omega_1}{(\omega_2+\omega_1)^3}\right] - \frac{1}{2\omega_1^2} e^{\theta_2-\theta_1}.$$

By the reasoning above, we find, after some calculations,

$$0 \leq -\frac{1}{\kappa}\left(\frac{\partial \kappa}{\partial \omega_1}\right)_{\theta_1} \leq \frac{2}{\omega_1} + \frac{4\omega_2}{\omega_2^2 - \omega_1^2}.$$

We find

$$E_1 = (2\omega_1 F_1 + 2\omega_2 F_2)\left[-\frac{1}{\kappa}\left(\frac{\partial \kappa}{\partial \omega_1}\right)_{\theta_1}\right]$$

$$\leq 4(\omega_1^2 + \omega_2^2)\left(\frac{2}{\omega_1} + \frac{4\omega_2}{\omega_2^2 - \omega_1^2}\right). \tag{A.25}$$

Setting $\omega_1 = \lambda\omega_2$ we obtain $0 \leq E_1 \leq C_1(\lambda)\omega_1$, the result stated in the theorem. Similarly, $0 \leq E_2 \leq C_2(\lambda)\omega_2$, uniformly in $x$ and $t$, where $C_2(\lambda) = C_1(\lambda^{-1})$.

**Remark A.8.** In the experiment by Li and Sattinger [9], $\omega_1^2 = .025$, $\omega_2^2 = .05$, and $\lambda = \sqrt{2}$. The quantity on the right hand side of (A.25) is approximately 14.1.

## References


[1] J. L. Bona, W. G. Pritchard and L. R. Scott, Solitary wave interactions, *Phys. Fluids* **23** (1980), 438–441.

[2] J. L. Bona, W. G. Pritchard and L. R. Scott, An evaluation of a model equation for water waves, *Phil. Trans. Royal Soc. London A.* **302** (1981), 457–510.

[3] W. Craig, An existence theory for water waves and the Boussinesq and Korteweg–de Vries scaling limits, *Commun. Partial Diff. Eq.* **10** (1985), 787–1003.

[4] W. Craig and C. Sulem, Numerical Simulation of Gravity Waves, *J. Comp. Phys.* **108** (1993), 73–83.

[5] J. D. Fenton and M. M. Rienecker, A Fourier method for solving nonlinear water-wave problems: application to solitary wave interactions, *J. Fluid Mech.* **118** (1982), 411–443.

[6] C. S. Gardner, J. M. Greene, M. Kruskal and R. M. Miura, Korteweg–de Vries equation and generalizations, VI. Methods for exact solutions, *Comm. Pure and Appl. Math.* **27** (1974), 97–133.

[7] Y. Kodama, On solitary wave interaction, *Phys. Letts. A* **123** (1987), 276–282.

[8] K. Konno, T. Mitsuhashi and Y. H. Ichikawa, Dynamical processes of the dressed ion acoustic solitons, *J. Phys. Soc. Japan* **43** (1977), 669–674.

[9] Y. Li and D. H. Sattinger, Solitary wave interactions in the ion acoustic plasma equations, *Journal of Mathematical Fluid Mechanics* **1** (1999), 117–130.

[10] S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, *Theory of Solitons*, Consultants Bureau, New York and London, 1984.

[11] R. Y. Sagdeev, *Cooperative phenomena and shock waves in collisionless plasmas*, 23–91, Consultants Bureau, New York, 1966.

[12] G. Schneider and E. Wayne, The long wave limit for the water wave problem. I. The case of zero surface tension, *Comm. Pure and Applied Math.* **53** (2000), 1475–1535.

[13] A. Zou and C.-H. Su, Overtaking collision between two solitary waves, *Phys. Fluids* **29** (1986), 2113–2123.



M. Haragus
Department of Mathematics
Université de Bordeaux I
F–33405 Bordeaux
France

D. P. Nicholls
Department of Mathematics
University of Notre Dame
Notre Dame, Indiana 46556
USA

D. H. Sattinger
Department of Mathematics
Utah State University
Logan, Utah 84322
USA


(accepted: April 4, 2002)