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Spectral stability of traveling water waves: Eigenvalue collision, singularities, and direct numerical simulation

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ABSTRACT

In a recent paper (Nicholls (2009) [1]) the author conjectured upon the connection between the onset of dynamic spectral instability of periodic traveling water waves, and singularities present in Taylor series representations of spectral data for the linearized water wave equations. More specifically, he proposed that the onset of instability is always coincident with encountering the smallest singularity in these Taylor series. In this paper we study this connection via a new Direct Numerical Simulation algorithm derived from the surface formulation of the water wave problem due to Zakharov (1968) [5] and Craig & Sulem (1993) [6]. We find compelling evidence that the conjecture is true in the case of deep (as compared to Benjamin & Feir's (1967) [7] critical depth $h \approx 1.363$) water, but false for shallow depths as it significantly underpredicts the onset of instability. The utility of the singularity identification strategy advocated in [1], while somewhat lessened in the shallow water case, is nonetheless upheld due to its ability to reliably identify a lower bound of stability and its extremely favorable computational complexity.

1. Introduction

The water wave equations govern the movement of a large body of water primarily under the influence of gravity (e.g., the ocean) and therefore arise in a wide array of engineering applications. From pollutant transport and the motion of sandbars, to tsunami propagation and the design of open-ocean oil rigs, the water wave equations are a central model in fluid mechanics. Among the many motions permitted by these equations, the traveling wave solutions are of great interest due to their ability to transport energy and momentum over great distances in the ocean. Of course not all of these traveling waveforms are dynamically stable and it is of crucial importance to identify those that are as these will be the only ones observed in practice.

In a recent publication [1] the author endeavored upon a study of the spectral stability of periodic traveling water waves on a two-dimensional (one vertical and one horizontal) fluid. Spectral stability refers to the fact that the eigenvalues (spectrum) of the water wave operator linearized about the traveling wave are considered rather than a full linear or even nonlinear stability analysis. The author approached the problem from a rather different point of view than the direct method applied by Longuet-Higgins [2,3] (see also the survey article of Dias & Kharif [4] for a full description of results along these lines). Rather than simply substituting a computed traveling wave into the linearized water wave problem and appealing to a numerical eigensolver, the author used the fact that traveling waves come in analytic branches to show that, generically, the spectral data can also be parametrized analytically. With this point of view, the author followed the "motion" of the spectrum in the complex plane as a wave height/steepness parameter was increased until divergence of the method [1]. The singularities in the expansions (which result in the divergence of the numerical scheme) are mandated by the form of the expansions: Only purely imaginary eigenvalues can be produced so that the algorithm *cannot* compute spectrum with a non-zero real part. In the light of this, the author posed a conjecture that not only is the presence of a singularity *necessary* for the onset of (spectral) instability, but also that it is *sufficient*. The purpose of this contribution is to validate or repudiate this conjecture based upon Direct Numerical Simulation of the spectrum via the surface formulation of Zakharov [5] and Craig & Sulem [6].

After careful numerical investigation, we find that the conjecture is largely justified in the case of deep water (as compared to Benjamin & Feir's [7] critical value of the depth $h \approx 1.363$) but underpredicts the onset of instability in shallow water. Interestingly, the author showed [1] that there was also a dichotomy between deep and shallow water when considering the "first collision" of eigenvalues on the imaginary axis (a necessary condition for an eigenvalue to leave the imaginary axis, resulting in spectral instability) and the presence of a singularity in his recursions. In deep water, eigenvalues could be followed beyond the first collision, while in shallow water eigenvalue collision and the presence of a singularity were synonymous. In the light of this discovery one can wonder about the utility of the "singularity detection" method outlined in [1]. First, the algorithm *does* produce largely precise

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results in the deep water regime, while also producing a reliable lower bound on stability for fluids of any depth. Second, this new approach is greatly advantaged in terms of computational complexity when compared to a direct approach mainly due to the fact that its cost is *independent* of the number of waveforms sampled in the study. The algorithm computes the entire Taylor series representation of the spectral data in one step so that particular details (e.g., singularities in the expansion) can be computed *a posteriori* as a post-processing step.

The organization of the paper is as follows: In Section 2 we recall the governing water wave equations. In Section 2.1 we outline the spectral stability framework we have in mind, and in Section 2.2 we review the conclusions of our previous work. In Section 3 we present our new numerical results, including a description of the algorithm in Section 3.1 and new figures in Section 3.2. Concluding remarks are presented in Section 4.

2. Governing equations

We consider the motion of the free interface above an ideal (inviscid, irrotational, incompressible) two-dimensional (one vertical and one horizontal) fluid under the influence of gravity; effects of surface tension can easily be incorporated if desired. If the fluid occupies the domain

$$S_{h,\eta} = \{x \in \mathbf{R} \mid -h < y < \eta(x,t)\}$$

with mean depth h and free surface $\eta = \eta(x,t)$, the equations of motion are known to be [8]:

$$\Delta\varphi = 0 \quad \text{in } S_{h,\eta} \tag{1a}$$

$$\partial_y\varphi(x,-h) = 0 \tag{1b}$$

$$\partial_t\eta + \partial_x\eta\,\partial_x\varphi - \partial_y\varphi = 0 \quad \text{at } y = \eta \tag{1c}$$

$$\partial_t\varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta = 0 \quad \text{at } y = \eta, \tag{1d}$$

where $\varphi = \varphi(x,y,t)$ is the velocity potential ($\vec{v} = \nabla\varphi$) and g is the constant of gravity. We can also consider the case of a fluid of infinite depth by replacing (1b) with

$$\partial_y\varphi \to 0 \quad \text{as } y \to -\infty. \tag{1e}$$

These equations must be supplemented with initial conditions

$$\eta(x,0) = \eta_0(x), \qquad \varphi(x,\eta_0(x),0) = \xi_0(x), \tag{1f}$$

where it suffices (by elliptic theory [9]) to specify φ only at the surface. Boundary conditions are also required to guarantee the existence of a unique solution which, for the study of Stokes waves, are periodicity with respect to some lattice $\Gamma \subset \mathbf{R}$, i.e.

$$\eta(x+\gamma,t) = \eta(x,t), \qquad \varphi(x+\gamma,y,t) = \varphi(x,y,t) \quad \forall\gamma \in \Gamma;$$

this lattice generates the conjugate lattice of wavenumbers [10],

$$\Gamma' := \{k \in \mathbf{R} \mid k\cdot\gamma \in (2\pi)\mathbf{Z}, \forall\gamma \in \Gamma\}.$$

In this paper we consider non-dimensionalized quantities and, as such, choose $\Gamma = 2\pi\mathbf{Z}$ which gives $\Gamma' = \mathbf{Z}$.

It was shown by Zakharov [5] that the system (1) is Hamiltonian in the canonical variables $\eta(x,t)$ and $\xi(x,t) := \varphi(x,\eta(x,t),t)$ with energy

$$H = \frac{1}{2}\int\int_{-h}^{\eta}|\nabla\varphi|^2\,dy\,dx + \frac{1}{2}\int g\eta^2\,dx.$$

With the introduction of the Dirichlet–Neumann operator (DNO), Craig & Sulem [6] rendered this formulation much more explicit. Specifically, if the DNO is given by

$$G(\eta)[\xi] := [\nabla\varphi\cdot N]_{y=\eta} = [\partial_y\varphi - (\partial_x\eta)\partial_x\varphi]_{y=\eta}$$

then the Hamiltonian can be expressed as

$$H = \frac{1}{2}\int \xi G(\eta)[\xi] + g\eta^2 dx.$$

From this (or a direct calculation with the chain rule) the "Zakharov–Craig–Sulem" evolution equations *equivalent* to (1) can be written

$$\partial_t\eta = G(\eta)[\xi], \qquad \partial_t\xi = -g\eta - A(\eta)B(\eta,\xi), \tag{2}$$

where

$$A(\eta) = \frac{1}{2(1+(\partial_x\eta)^2)},$$
$$B(\eta,\xi) = (\partial_x\xi)^2 - (G(\eta)[\xi])^2 - 2\,(\partial_x\eta)(\partial_x\xi)(G(\eta)[\xi]).$$

The purpose of this paper is to study the stability of traveling wave solutions of (2) and thus it is important to work in a frame moving uniformly with velocity c. In such a frame it is not difficult to show that the governing equations become

$$\partial_t\eta + c\partial_x\eta = G(\eta)[\xi], \qquad \partial_t\xi + c\partial_x\xi = -g\eta - A(\eta)B(\eta,\xi). \tag{3}$$

2.1. Spectral stability analysis

The spectral stability analysis that we have in mind is fully described by the author in [11,1], but we briefly outline it here for completeness. To begin, consider a traveling wave solution of (2), i.e. a *steady* solution

$$(\bar\eta,\bar\xi,\bar c) = (\bar\eta(x),\bar\xi(x),\bar c)$$

of (3). With this we seek solutions to the *full* problem (3) with the "spectral stability" form:

$$\eta(x,t) = \bar\eta(x) + \delta e^{\lambda t}\zeta(x), \qquad \xi(x,t) = \bar\xi(x) + \delta e^{\lambda t}\psi(x)$$

where $\delta \ll 1$ measures the magnitude of the small perturbation of the traveling state and λ determines the spectral stability. Inserting this into (3) we find, to order $\mathcal{O}(\delta)$

$$(\lambda + c\partial_x)\zeta = G_\eta(\bar\eta)[\bar\xi]\{\zeta\} + G(\bar\eta)[\psi] \tag{4a}$$

$$\begin{aligned}(\lambda + c\partial_x)\psi \;=\; & -g\zeta - A_\eta(\bar\eta)\{\zeta\}B(\bar\eta,\bar\xi)\\ & - A(\bar\eta)(B_\eta(\bar\eta,\bar\xi)\{\zeta\} + B_\xi(\bar\eta,\bar\xi)\{\psi\})\end{aligned} \tag{4b}$$

where the η and ξ subscripts represent η and ξ variations, respectively.

The final specification we make for our spectral stability problem (4) are the boundary conditions which ζ and ψ must satisfy. For this we require that the "Bloch boundary conditions":

$$\zeta(x+\gamma) = e^{ip\gamma}\zeta(x), \qquad \psi(x+\gamma) = e^{ip\gamma}\psi(x), \quad \forall\gamma \in \Gamma,$$

[11,1]. Notice that if p is a rational number then these functions will be periodic with respect to the lattice Γ. Such functions can be expanded as

$$\zeta(x) = \sum_{k\in\Gamma'}\hat\zeta_k e^{i(k+p)x}, \qquad \psi(x) = \sum_{k\in\Gamma'}\hat\psi_k e^{i(k+p)x}.$$

We also recall that the periodicity of the spectrum with respect to p implies that only values of p in the *conjugate* cell ($P(\Gamma')$) need be considered [11]; for the non-dimensional choice $L = 2\pi$ this cell is $P(\Gamma') = [0,1]$.

2.2. Previous results

The goal of this paper is to assess the utility of a conjecture made by the author in a previous publication [1]. To explain the conjecture and put it into context, let us return to the classical formulation of the water wave problem, (1), which, in a reference frame moving with velocity c, is

$$\Delta\varphi = 0 \quad \text{in } S_{h,\eta} \tag{5a}$$

$$\partial_y\varphi(x,-h) = 0 \tag{5b}$$

$$\partial_t\eta + c\partial_x\eta + \partial_x\eta\,\partial_x\varphi - \partial_y\varphi = 0 \quad \text{at } y=\eta \tag{5c}$$

$$\partial_t\varphi + c\partial_x\varphi + \frac{1}{2}|\nabla\varphi|^2 + g\eta = 0 \quad \text{at } y=\eta. \tag{5d}$$

Rather than pursuing the surface formulation (2), in [1] the author worked directly with the volumetric formulation (5) subsequent to a domain-flattening change of variables, equivalent to

$$x' = x, \qquad y' = h\left(\frac{y-\eta}{h+\eta}\right).$$

This change of variables delivers (upon dropping primes) the system

$$\Delta u = F(x,y;u,\eta) \quad \text{in } S_{h,0} \tag{6a}$$

$$\partial_y u(x,-h) = 0 \tag{6b}$$

$$\partial_t\eta + c\partial_x\eta - \partial_y u = Q(x;u,\eta) \quad \text{at } y=0 \tag{6c}$$

$$\partial_t u + c\partial_x u + g\eta = R(x;u,\eta) \quad \text{at } y=0, \tag{6d}$$

where u is the transformed potential, and the details of F, Q, and R are provided in [1]. The important feature of these right-hand-sides for our purposes is that they are $\mathcal{O}(\varepsilon)^2$ if u and η are $\mathcal{O}(\varepsilon)$.

With the objective of computing traveling wave solutions, one can seek steady solutions of (5) in the form of regular perturbation expansions

$$\begin{aligned}\bar{u}(x,y;\varepsilon) &= \sum_{n=1}^{\infty}\bar{u}_n(x,y)\varepsilon^n,\\ \bar{\eta}(x;\varepsilon) &= \sum_{n=1}^{\infty}\bar{\eta}_n(x)\varepsilon^n, \qquad \bar{c}(\varepsilon) = \bar{c}_0 + \sum_{n=1}^{\infty}\bar{c}_n\varepsilon^n.\end{aligned} \tag{7}$$

These expansions can be shown to be strongly convergent in an appropriate function space [12] and specify a rapid and robust numerical algorithm for the approximation of traveling waveforms [13]. To study the stability of these traveling waves the author considered the spectral stability forms

$$u(x,y,t) = \bar{u}(x,y) + \delta e^{\lambda t}v(x,y), \qquad \eta(x,t) = \bar{\eta}(x) + \delta e^{\lambda t}\zeta(x).$$

Insertion of these into (6) and dropping terms of order $\mathcal{O}(\delta)$ results in

$$\Delta v = \mathcal{F}(x,y;u,\eta,v,\zeta) \quad \text{in } S_{h,0} \tag{8a}$$

$$\partial_y v(x,-h) = 0 \tag{8b}$$

$$(\lambda + c\partial_x)\zeta - \partial_y v = \mathcal{Q}(x;u,\eta,v,\zeta) \quad \text{at } y=0 \tag{8c}$$

$$(\lambda + c\partial_x)v + g\zeta = \mathcal{R}(x;u,\eta,v,\zeta) \quad \text{at } y=0, \tag{8d}$$

and, again, the forms for $\mathcal{F}$, $\mathcal{Q}$, and $\mathcal{R}$ are given in [1]. As with the right-hand-sides (F, Q, R), if v and ζ are $\mathcal{O}(\varepsilon)$ then $(\mathcal{F}, \mathcal{Q}, \mathcal{R})$ are $\mathcal{O}(\varepsilon)^2$ which suggests expansions of the form

$$\begin{aligned}v(x,y;\varepsilon) &= \sum_{n=0}^{\infty}v_n(x,y)\varepsilon^n,\\ \zeta(x;\varepsilon) &= \sum_{n=0}^{\infty}\zeta_n(x)\varepsilon^n, \qquad \lambda(\varepsilon) = \sum_{n=0}^{\infty}\lambda_n\varepsilon^n\end{aligned} \tag{9}$$

for the spectral data. The author showed [11] that these expansions are (generically) strongly convergent in suitable function spaces, while he displayed in [1] the accurate and stable nature of the numerical simulations which these recursions can produce.

For flat water ($\varepsilon = 0$) the eigenvalues $\lambda(0)$ are purely imaginary indicating (weak) spectral stability of small perturbations from the quiescent state. As ε is increased from zero the spectrum will "move" in the complex plane. It is known (see, e.g., [1]) that eigenvalues of a Hamiltonian system such as the water wave problem *cannot* leave the imaginary axis unless they "collide" with another eigenvalue. Furthermore, MacKay & Saffman [14] further specify that such a collision must be between eigenvalues of the opposite Krein signature to leave the imaginary axis. Additionally, the spectrum is symmetric with respect to the imaginary axis so that departure from this axis automatically guarantees instability.

In light of this spectral theory, the author sought [1] to investigate the "first collisions" of eigenvalues for two-dimensional traveling wave solutions of (6) in water of varying depths. In this study several things were discovered:

1. First collision indicates divergence of the numerical scheme for some choices of p.
2. First collision is followed by continued "motion" of the spectrum along the imaginary axis for other values of p.
3. Collision (even of the opposite Krein signature) does *not* always indicate the onset of spectral instability.
4. The formulation of the recursions prevents the study of the spectrum beyond instability as the coefficients λ_n are required to be purely imaginary (in a non-resonant configuration).

With this data in hand, it was conjectured [1] that the onset of instability and singularity in the expansions of the spectral data, (9), occurred at one and the same value of ε. Our purpose in this contribution is to test this conjecture against a Direct Numerical Simulation (DNS) of the spectrum from (4).

3. Numerical results

In this section we display results of our new DNS of the spectral data from (4) versus our previously published computations of "smallest singularity" associated with the perturbative calculation of the spectrum from (9). We find that in deep water (as defined by the applicability of the Benjamin–Feir Instability [7]) the author's conjecture [1] is to a large extent valid: The first singularity in the expansions (9) is predictive of instability. However, for shallow water the first singularity significantly underestimates the onset of instability and thus is an unsatisfactory indicator of dynamical instability.

3.1. Algorithms

As the algorithms for the detection of crossings and singularities has been extensively described in [1], we focus upon an elucidation of our DNS. Clearly, a crucial ingredient for the DNS of the spectrum is a high-fidelity simulation of the underlying traveling wave. This we procure from the algorithm presented in [13] which is simply a numerical approximation of the $\{\bar{u}_n, \bar{\eta}_n, \bar{c}_n\}$ from (7) (note that a surface evaluation of $\bar{u}_n$ gives $\bar{\xi}_n$). The expansions

$$\begin{pmatrix}\bar{\eta}^{N,N_x}(x;\varepsilon)\\ \bar{\xi}^{N,N_x}(x;\varepsilon)\end{pmatrix} := \sum_{n=1}^{N}\sum_{k=-N_x/2}^{N_x/2-1}\begin{pmatrix}d_{k,n}\\ a_{k,n}\end{pmatrix}e^{ikx}\varepsilon^n,$$

$$\bar{c}^N(\varepsilon) := \tilde{c}_0 + \sum_{n=1}^{N}\tilde{c}_n\varepsilon^n,$$

where $\{d_{k,n}, a_{k,n}, \tilde{c}_n\}$ approximate $\{\hat{\bar{\eta}}_{k,n}, \hat{\bar{\xi}}_{k,n}, \bar{c}_n\}$, are inserted into the eigenvalue problem (4) for various values of ε. This eigenvalue

problem is, in turn, approximated by a Fourier collocation method with unknowns

$$\begin{pmatrix} \zeta^{N_x}(x) \\ \psi^{N_x}(x) \end{pmatrix} := \sum_{k=-N_x/2}^{N_x/2-1} \begin{pmatrix} f_k \\ b_k \end{pmatrix} e^{i(k+p)x},$$

resulting in the finite dimensional eigensystem $L\vec{x} = \lambda\vec{x}$, which was simulated via the `eig` command in MATLAB [15].

The only real issue in this numerical approximation is the choice of algorithm for the simulation of the Dirichlet–Neumann operator (DNO), $G(\eta)$, and its first variation, $G_\eta(\eta)$. For this we use the rapid and accurate "Operator Expansions" (OE) method (see, e.g., [6,16]). To summarize this approach, we note that for sufficiently smooth deformations $\eta = \varepsilon f$, the DNO and its first variation depend *analytically* upon the height/slope parameter ε so that expansions of the form

$$G(\varepsilon f) = \sum_{n=0}^{\infty} G_n(f)\varepsilon^n, \qquad G_\eta(\varepsilon f) = \sum_{n=0}^{\infty} G_n^{(1)}(f)\varepsilon^n,$$

converge strongly. The OE approach delivers convenient formulas for the G_n and $G_n^{(1)}$ in terms of Fourier multipliers and convolution products. For instance,

$$G_0[\xi] = |D|\tanh(h|D|)[\xi], \qquad G_1(f)[\xi] = D[fD[\xi]] - G_0[fG_0[\xi]],$$

where $D := (1/i)\partial_x$, and

$$G_0^{(1)}[\xi]\{\zeta\} = D[\zeta D\xi] - G_0[\zeta G_0\xi],$$

for a variation in the ζ direction. If these terms are computed for $0 \le n \le M$ then the DNO and its variation can be approximated by

$$G^M(\varepsilon f) := \sum_{n=0}^{M} G_n(f)\varepsilon^n, \qquad G_\eta^M(\varepsilon f) := \sum_{n=0}^{M} G_n^{(1)}(f)\varepsilon^n.$$

It has been shown [6,17] that these OE recursions deliver highly accurate simulations with extremely low computational cost ($\mathcal{O}(MN_x\log(N_x))$) provided that ε is small and f smooth so that M may be chosen relatively small. However, care must be taken to avoid severe "cancellations" which are inherent to these recursions. In fact, Nicholls & Reitich showed [18] that this method can be quite unstable when used outside its domain of applicability. For the numerical experiments we present below we restrict out choice to $M = 8$ which gave excellent results.

Once the approximate spectral data $\{\zeta^{N_x}, \psi^{N_x}, \lambda^{N_x}\}$ is computed for a particular value of ε it remains to determine if this configuration is "unstable". For this we simply search over the full set of eigenvalues and find the one with the largest (in absolute value) real part. If this largest real part is bigger than some tolerance, say ρ, then we deem the problem unstable. For all of the simulations we present in this paper, we have selected numerical parameters $N_x = 64$, $M = 8$, $N = 30$, $\rho = 10^{-6}$ and physical parameters $g = 1$, $L = 2\pi$, corresponding to non-dimensionalized units. In relation to the well-known non-dimensional quantity for this problem (the Froude number) we note that $L = 2\pi$ implies fundamental wavenumber $k = 1$ which, in turn, delivers (linear) velocity $c = \sqrt{g/k} = 1$, and Froude number $F = \frac{c}{\sqrt{gh}} = \frac{1}{\sqrt{h}}$.

In the algorithm for singularity detection outlined in [1] an important parameter τ was introduced which gives a "cancellation tolerance". More precisely, the truncated Taylor polynomials $\lambda^N(\varepsilon)$, c.f. (9), were approximated by rational functions $A(\varepsilon)/B(\varepsilon)$ using the Padé Approximation algorithm [19]. A natural set of singularities is given by the zeros of the polynomial B, however, "false positives" can be identified if a zero of A occurs for the same value of ε. Therefore, we defined the set of approximate singularities to be $P_\tau := \{\varepsilon \in \mathbf{C} \mid B(\varepsilon) = 0, A(\varepsilon) > \tau\}$.

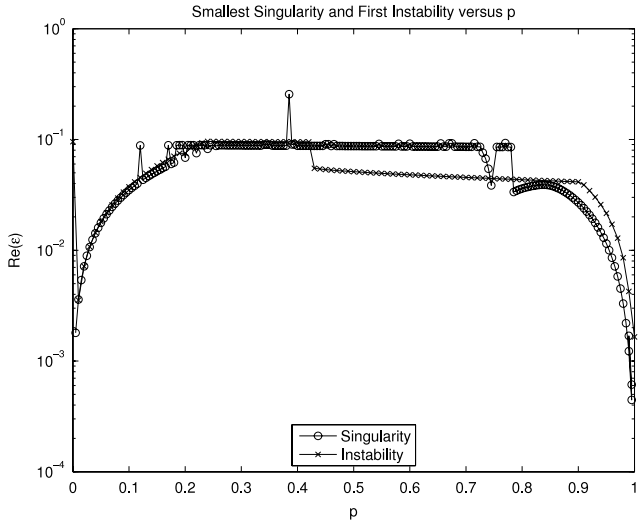

Fig. 1. Plot of ε_s (smallest singularity) and ε_i (first instability) versus p for water of depth $h = \infty$. The underlying Stokes wave is 2π-periodic and $\tau = 10^{-8}$.

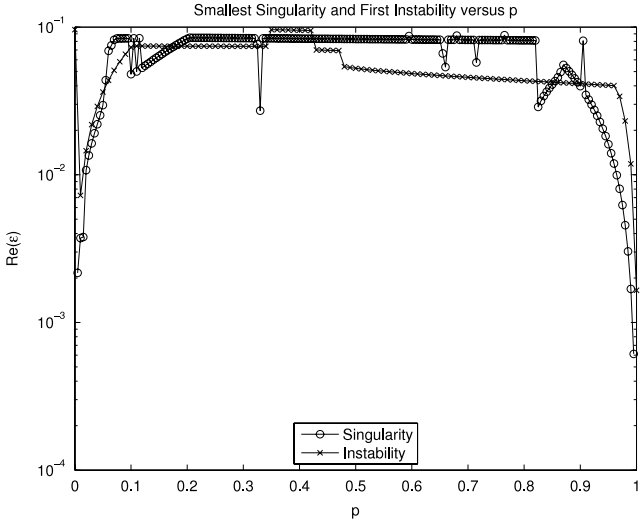

Fig. 2. Plot of ε_s (smallest singularity) and ε_i (first instability) versus p for water of depth $h = 2$. The underlying Stokes wave is 2π-periodic and $\tau = 10^{-8}$.

3.2. Results

In Figs. 1–4 we present, simultaneously, results for the first singularity in our perturbation expansion of the spectral data (ε_s) and the onset of instability (ε_i) based upon our DNS of the spectral data. The figures correspond to depths of $h = \infty$, $h = 2$, $h = 1$, and $h = 1/2$, respectively, and for Bloch parameters $p = p_j = j/100$ for $j = 0, \ldots, 100$.

From these figures we learn several things, particularly relative to Benjamin & Feir's critical depth $h \approx 1.363$ [7]. First of all, in the cases of deep water ($h = 2, \infty$) the smallest singularity in our expansion, ε_i, appears to be predictive of (spectral) instability for these traveling waves. To make this more precise we plot, in Figures 5 and 6 the relative difference in these values

$$\frac{\varepsilon_s - \varepsilon_i}{\varepsilon_i} \tag{10}$$

versus p. For most values of p this difference falls below 30% which lends credence to our claim. However, there are clearly counterexamples for a range of p just above $1/2$ which we attribute to the errors inherent to our two quite different numerical approaches.

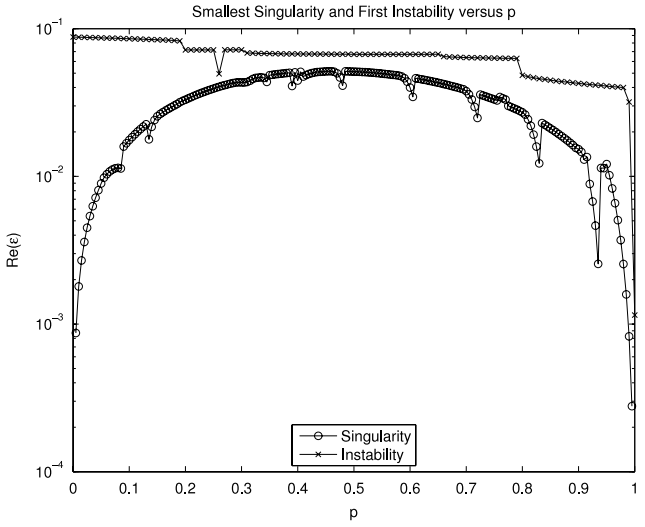

Fig. 3. Plot of ε_s (smallest singularity) and ε_i (first instability) versus p for water of depth $h = 1$. The underlying Stokes wave is 2π-periodic and $\tau = 10^{-8}$.

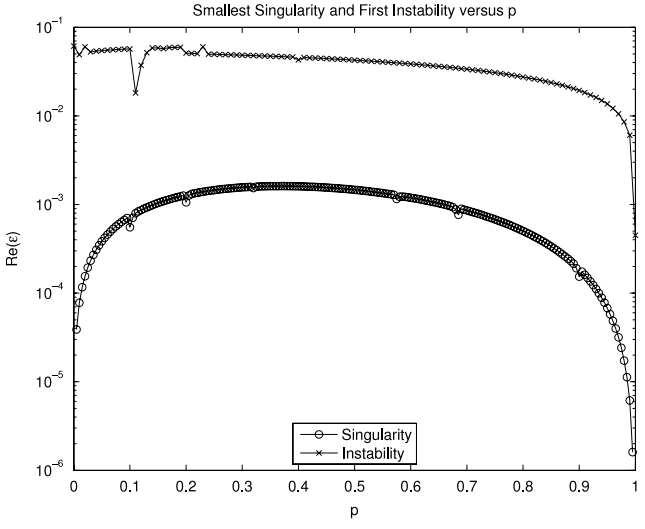

Fig. 4. Plot of ε_s (smallest singularity) and ε_i (first instability) versus p for water of depth $h = 1/2$. The underlying Stokes wave is 2π-periodic and $\tau = 10^{-8}$.

Interestingly, in the cases of shallow water ($h = 1/2, 1$) the smallest singularity is *not* predictive of spectral instability. As before, we make this precise in Figs. 7 and 8 using the measure (10). In fact, the first singularity significantly *underpredicts* the onset of instability: Traveling waves in shallow water are stable for much larger amplitudes than predicted in the results of [1]. Curiously, we recall that it is in precisely this regime of shallow water that the "windows of stability" seemed to disappear. In fact they still exist, however, they are not identified using our perturbation expansions. We also point out that our DNS demonstrates the well-known property that the Benjamin–Feir instability disappears in shallow water: For small p (long wavelength disturbances) there is a non-zero range of ε for which traveling waves are spectrally stable.

At this point one may wonder about the utility of the approach advocated in [1] for classification of spectral stability by detection of singularities. Certainly, for shallow water it does not appear to give the sharpest possible results, however, it always delivers a lower bound which is as accurate as the test of eigenvalue collision. Furthermore, in deep water it *does* appear to give largely accurate results in our framework of spectral stability.

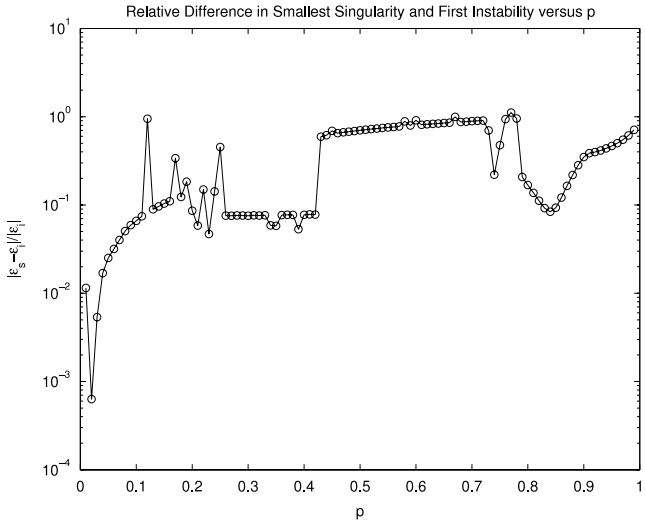

Fig. 5. Plot of relative difference between ε_s (smallest singularity) and ε_i (first instability) versus p for water of depth $h = \infty$. The underlying Stokes wave is 2π-periodic and $\tau = 10^{-8}$.

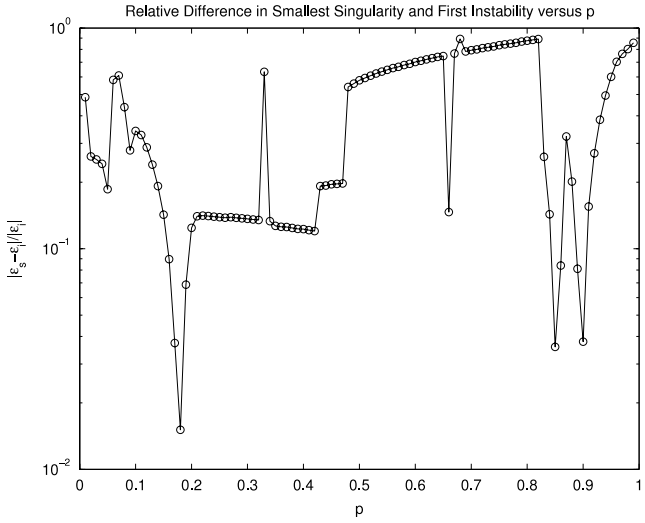

Fig. 6. Plot of relative difference between ε_s (smallest singularity) and ε_i (first instability) versus p for water of depth $h = 2$. The underlying Stokes wave is 2π-periodic and $\tau = 10^{-8}$.

Regardless of the physical configuration of our system is it very important to point out the astronomical cost of the DNS as compared with the singularity detection algorithm. We recall that this latter method requires, for each p, time proportional to

$$\mathcal{O}(NN_x \log(N_x) N_y \log(N_y))$$

to generate the relevant Fourier/Chebyshev/Taylor coefficients and time proportional to $\mathcal{O}(NN_x)$ to find the singularities. By contrast, the bottleneck in our DNS is the computation of the matrix L, for each p, in time $\mathcal{O}(MN_x \log(N_x) N_x)$. However, this must be *recomputed* for *every* waveform along the branch of solutions. In our simulations this amounted to thousands to tens of thousands of computations. As we were able to select the vertical discretization parameter to $N_y = 64$ (please see [1] for full details of this parameter) our DNS simulations were several orders of magnitude more expensive.

4. Conclusions

In this paper we have outlined an algorithm for the Direct Numerical Simulation of the spectrum of the water wave operator

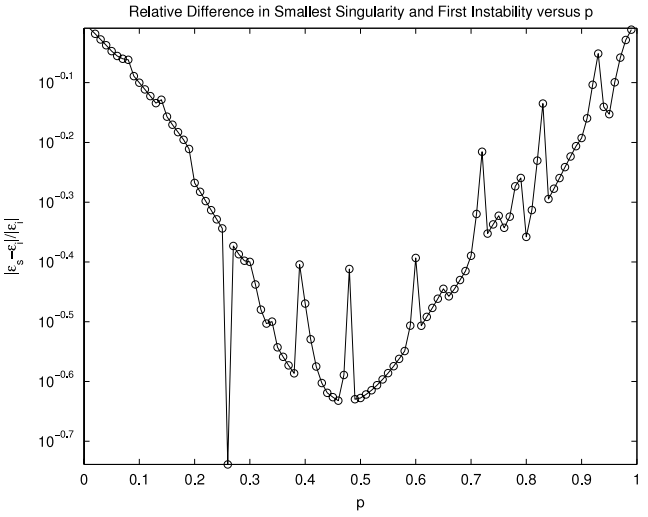

Fig. 7. Plot of relative difference between ε_s (smallest singularity) and ε_i (first instability) versus p for water of depth $h = 1$. The underlying Stokes wave is 2π-periodic and $\tau = 10^{-8}$.

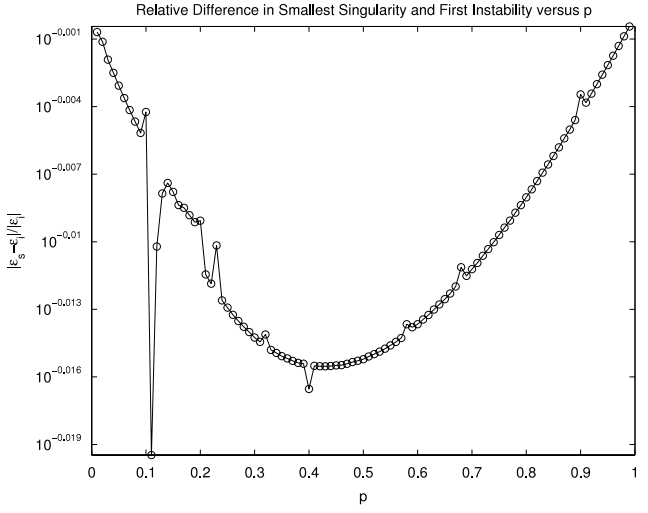

Fig. 8. Plot of relative difference between ε_s (smallest singularity) and ε_i (first instability) versus p for water of depth $h = 1/2$. The underlying Stokes wave is 2π-periodic and $\tau = 10^{-8}$.

linearized about a periodic traveling wave solution. The study of the resulting eigenvalues produces a spectral stability result. This DNS was used to test the conjecture made in [1] regarding the onset of instability and the presence of singularities in recursions advocated by the author. We provide here compelling evidence that the conjecture is largely true in the case of deep (as compared to Benjamin & Feir's critical depth $h \approx 1.363$) water, but false for shallow water. The utility of the singularity identification strategy, while somewhat lessened in the shallow water case, is nonetheless upheld due to its ability to reliably identify a lower bound of stability and its extremely favorable computational complexity.

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