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On analyticity of travelling water waves

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In this paper we establish the existence and analyticity of periodic solutions of a classical free-boundary model of the evolution of three-dimensional, capillary–gravity waves on the surface of an ideal fluid. The result is achieved through the application of bifurcation theory to a boundary perturbation formulation of the problem, and it yields analyticity *jointly* with respect to the perturbation parameter and the spatial variables. The travelling waves we find can be interpreted as resulting from the (nonlinear) interaction of two two-dimensional wavetrains, giving rise to a periodic travelling pattern. Our analyticity theorem extends the most sophisticated results known to date in the absence of resonance; 'short crested waves', which result from the interaction of two wavetrains with unit amplitude ratio are realized as a special case. Our method of proof also sheds light on the convergence and conditioning properties of classical boundary perturbation methods for the numerical approximation of travelling surface waves. Indeed, we demonstrate that the rather unstable numerical behaviour of these approaches can be attributed to the strong but subtle cancellations in the formulas underlying their classical implementations. These observations motivate the derivation and use of an alternative, stable, formulation which, in addition to providing our method of proof, suggests new stabilized implementations of boundary perturbation algorithms.

Keywords: Capillary–gravity water waves; travelling waves; boundary perturbations

1. Introduction

The stable and accurate numerical simulation of free-surface ocean dynamics is one of the central problems in computational fluid mechanics. From shoaling and breaking of waves over nearshore regions to energy, momentum, and scalar transport in the open ocean, the rapid and reliable approximation of the surface of a fluid is a necessary tool in problems of physical relevance. Surface waves that propagate with constant velocity and without change of form (the travelling waves) are a distinguished class of motions which are believed to be a fundamental building-block of surface ocean dynamics.

In this paper we take up the mathematical question of regularity properties of travelling wave solutions of the classical water wave model (see §2), which constitutes an accurate representation for the motion of the free surface of the ocean. In particular, we demonstrate that the water wave problem in d-dimensions admits *surfaces* of solutions, parameterized by $(d-1)$ many

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parameters $\varepsilon \in \mathbf{R}^{d-1}$, which are *jointly* analytic in the parametric and spatial variables. Our method of proof is perturbative in nature and general enough to encompass every case away from resonances (see §4).

The first rigorous existence theorems for travelling wave solutions to the water wave model date to the results in two space dimensions without surface tension by Levi-Civita (1925) (infinite depth) and Struik (1926) (finite depth) who used complex variables techniques. With the advent of the modern computer there was a resurgence of interest in the problem in the 1970s and 1980s as highly nonlinear waveforms could now be simulated (e.g. Schwartz, 1974; Roberts, 1983; Schwartz & Roberts, 1983; Marchant & Roberts 1987). This resurgence was also accompanied by new theoretical developments. For instance, Reeder & Shinbrot (1981*a*,*b*) studied the phenomena of Wilton ripples which arise in two-dimensional travelling capillary–gravity water waves. They showed existence and smoothness of branches of travelling wave solutions which exist in the presence of resonance in the linearized problem. Other important theoretical results in two space dimensions include those of Toland and collaborators, who used various integral formulations of the two-dimensional water wave problem coupled with variational techniques (e.g. minimizers, mountain pass). An important early result of Toland's (1978) established the *global* existence of the bifurcating branch of solutions all the way to the Stokes singularity. Jones & Toland (1986) also looked at surface tension effects in two dimensions, and subharmonic bifurcations in (Jones & Toland 1985). Subharmonic bifurcation was also the object of Buffoni *et al.* (2000).

In three dimensions, on the other hand, the most general results to date are those of Craig & Nicholls (2000) who, in the presence of non-zero surface tension, established existence of travelling capillary–gravity water waves with an arbitrary fundamental period. The theorem of Craig & Nicholls used the surface formulation of Zakharov (1968) and Craig & Sulem (1993), coupled with the Lyapunov–Schmidt procedure from bifurcation theory. Other existence results in three dimensions include that of Sun (1993), who viewed the travelling wave as generated by a surface pressure, and Groves & Mielke (2001) and Groves (2001), who have studied travelling waves using a 'spatial dynamics' approach. In this formulation, the direction of propagation, in the traveling wave equations, is considered the dynamical quantity; the transverse direction is typically considered to be periodic and then periodic (in propagation direction) solutions are sought.

Regarding *spatial* analyticity of solitary travelling water waves, we mention the seminal work of Lewy (1952) who, using complex variables techniques, established that (in the presence of gravity alone) once the surface is known to be C^1, it is automatically analytic. Matei (2002) and Craig & Matei (2003) have extended this result to non-zero capillarity in two and three dimensions, respectively, using a partial hodograph transform. We also mention the broad generalization of these techniques in the recent work of Koch *et al.* (2004). Bona & Li (1997) established both spatial analyticity and decay (at infinity) estimates for travelling waves for a wide class of nonlinear, dispersive wave equations. Using a velocity potential/streamfunction formulation, they extended these theorems to travelling water waves in two dimensions.

The three-dimensional results most closely related to those we present herein are those of Reeder & Shinbrot (1981*c*), who demonstrate the existence and parametric analyticity of 'short-crested' capillary–gravity waves of sufficiently small amplitude; short-crested waves are typically defined (Dias & Kharif 1999) as the waves which result from the (nonlinear) interaction of two periodic wavetrains of equal amplitude, infinite extent and non-zero angle of interaction (as the angle of interaction approaches zero, the waves are typically referred to as 'long-crested'). Akin to the method we adopt in §3, Reeder & Shinbrot also use a 'domain flattening' change of variables. Our results expand on those of (Reeder & Shinbrot 1981*c*) in two important directions: first, our derivations demonstrate that, in fact, the free boundary and velocity potential are *jointly* analytic in space and bifurcation parameter (a fact that, of course, does not follow from separate analytic dependence); and second, our developments allow for the interaction of wavetrains of *arbitrary* amplitude ratio, i.e. not necessarily short-crested waves. To attain the latter, our approach entails the use of *multi-dimensional* perturbation parameters.

In addition to establishing existence and analyticity of hypersurfaces of travelling water waves, our work also sheds light on the convergence and conditioning properties of classical boundary perturbation methods for the numerical simulation of travelling capillary–gravity water waves. In particular, we discuss the method of Stokes (1847) (which we term the method of 'field expansions' (FE)) that was further refined and carried out to high order by Roberts (1983), Schwartz & Roberts (1983), and Marchant & Roberts (1987) for three-dimensional travelling water waves in the absence of surface tension. As we show in §2*d*, this method produces *unstable* results as the perturbation order is increased due to subtle but significant *cancellations* which are present in the underlying recursions. As we explain, a further consequence of this observation is that these FE recursions cannot be used for a direct proof of existence or analyticity. However, as we anticipated above, a direct proof can be realized once a 'domain flattening' change of variables is effected, as this can be shown to implicitly account for all significant cancellations. This latter fact suggests a stabilized approach to numerical simulation, whose thorough investigation we defer to future work (see also Nicholls & Reitich 2001*a*,*b*, 2003, 2004*a*,*b*).

The remainder of the paper is organized as follows: first, in §2, we introduce the equations of motion and the classical FE approach to simulating travelling water waves; in particular, in §2d, we demonstrate how these classical recursions rely heavily on significant cancellations for their convergence. In §3, we introduce a change of variables which substantially ameliorates these cancellations and paves the way for the analyticity proof of §4; some auxiliary results necessary for this latter proof are collected in appendix A.

2. Preliminaries

In this section, we briefly review the equations of motion of the water wave problem (ideal fluid, free-surface flow) and outline the classical FE technique for perturbatively computing solutions. With the aid of a numerical implementation of this algorithm and several simulations, we illustrate how these recursions are inherently unstable at high orders owing to underlying *cancellations*.

(*a*) *Equations of motion*

Consider a d-dimensional ($d=2$, 3) fluid (one vertical dimension specified by the variable y and $(d-1)$ horizontal dimensions specified by x) bounded below by an impermeable bottom at $y=-h$ (h possibly infinite) and above by an undetermined air/fluid interface, $y=\eta(x,\ t)$, which occupies the domain

$$S_{h,\eta} = \{(x,y) \in \mathbf{R}^{d-1} \times \mathbf{R} | -h < y < \eta(x,t)\}.$$

In the case of finite depth, no generality is lost if h is set to one, as this simply amounts to a rescaling of independent variables. Consider also the classical assumption that the fluid be periodic with respect to the lattice $\Gamma \subset \mathbf{R}^{d-1}$, which defines a parallelogram of periodicity $P(\Gamma)$. The equations of motion of an ideal fluid in such a domain under the effects of gravity and capillarity are (Lamb 1993)

$$\Delta\varphi = 0 \quad \text{in} \quad S_{1,\eta}, \tag{2.1a}$$

$$\partial_y \varphi(x,-1) = \int_{P(\Gamma)} \partial_y \varphi(x,-1)\,\mathrm{d}x, \qquad \int_{P(\Gamma)} \varphi(x,-1)\,\mathrm{d}x = 0, \tag{2.1b}$$

$$\partial_t \varphi + \tfrac{1}{2}|\nabla\varphi|^2 + g\eta - \sigma\kappa(\nabla_x \eta) = 0 \quad \text{at } y = \eta, \tag{2.1c}$$

$$-\partial_t \eta - \nabla_x \eta \cdot \nabla_x \varphi + \partial_y \varphi = 0 \quad \text{at } y = \eta, \tag{2.1d}$$

where φ is the velocity potential, g is the constant of gravity, σ is the constant of capillarity and κ is the curvature:

$$\kappa(\nabla_x \eta) = \mathrm{div}_x \left[\frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}} \right].$$

As we stated, we shall be concerned with travelling waves translating uniformly with speed $c \in \mathbf{R}^{d-1}$, which satisfy

$$\Delta\varphi = 0 \quad \text{in } S_{1,\eta}, \tag{2.2a}$$

$$\partial_y \varphi(x,-1) = \int_{P(\Gamma)} \partial_y \varphi(x,-1)\,\mathrm{d}x, \qquad \int_{P(\Gamma)} \varphi(x,-1)\,\mathrm{d}x = 0, \tag{2.2b}$$

$$[c \cdot \nabla_x]\varphi + \tfrac{1}{2}|\nabla\varphi|^2 + g\eta - \sigma\kappa(\nabla_x \eta) = 0 \quad \text{at } y = \eta, \tag{2.2c}$$

$$-[c \cdot \nabla_x]\eta - \nabla_x \eta \cdot \nabla_x \varphi + \partial_y \varphi = 0 \quad \text{at } y = \eta. \tag{2.2d}$$

(*b*) *Bifurcation theory*

Adopting a bifurcation theoretic approach, we seek solutions of equation (2.2) near the quiescent state ($\varphi=\eta=0$ and *any* velocity c) which forms a 'trivial' family of solutions. Bifurcation theory requires the analysis of the linearization of equation (2.2) about these trivial solutions which leads to consideration of the problem

$$\Delta\varphi_1(x,y)=0 \quad \text{in } S_{1,0}, \tag{2.3a}$$

$$\partial_y\varphi_1(x,-1)=\int_{P(\Gamma)}\partial_y\varphi_1(x,-1)\,\mathrm{d}x, \qquad \int_{P(\Gamma)}\varphi_1(x,-1)\,\mathrm{d}x=0, \tag{2.3b}$$

$$[c_0\cdot\nabla_x]\varphi_1(x,0)+[g-\sigma\Delta_x]\eta_1(x)=0, \tag{2.3c}$$

$$-[c_0\cdot\nabla_x]\eta_1(x)+\partial_y\varphi_1(x,0)=0. \tag{2.3d}$$

The periodic boundary conditions (2.3a) and (2.3b) imply that

$$\varphi_1(x,y)=\sum_{k\in\Gamma',k\neq0}a_{1,k}\frac{\cosh(|k|(y+1))}{\cosh(|k|)}\mathrm{e}^{\mathrm{i}k\cdot x}, \quad \eta_1(x)=\sum_{k\in\Gamma',k\neq0}d_{1,k}\,\mathrm{e}^{\mathrm{i}k\cdot x}.$$

Equations (2.3c) and (2.3d) become

$$A(c_0,k)\begin{pmatrix}a_{1,k}\\ d_{1,k}\end{pmatrix}\equiv\begin{pmatrix}c_0\cdot\mathrm{i}k & g+\sigma|k|^2\\ |k|\tanh(|k|) & -c_0\cdot\mathrm{i}k\end{pmatrix}\begin{pmatrix}a_{1,k}\\ d_{1,k}\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}, \tag{2.4}$$

for every $k\in\Gamma'$, $k\neq0$. Thus a non-trivial solution can exist only if the matrix $A(c_0,\,k)$ is singular for some $k\in\Gamma'$, that is, if the determinant

$$\Lambda_\sigma(c_0,k)=(c_0\cdot k)^2-(g+\sigma|k|^2)|k|\tanh(|k|) \tag{2.5}$$

vanishes. In this case, if a pair $(c_0,\,k)$ satisfies $\Lambda_\sigma(c_0,k)=0$, then a non-trivial solution of equation (2.3) is

$$\eta_1(x)=\alpha_k(c_0\cdot k)\mathrm{e}^{\mathrm{i}k\cdot x}+\bar{\alpha}_k(c_0\cdot k)\mathrm{e}^{-\mathrm{i}k\cdot x}, \tag{2.6a}$$

$$\varphi_1(x,y)=\alpha_k\mathrm{i}(g+\sigma|k|^2)\frac{\cosh(|k|(y+1))}{\cosh(|k|)}\mathrm{e}^{\mathrm{i}k\cdot x},$$

$$-\bar{\alpha}_k\mathrm{i}(g+\sigma|k|^2)\frac{\cosh(|k|(y+1))}{\cosh(|k|)}\mathrm{e}^{-\mathrm{i}k\cdot x}, \tag{2.6b}$$

where $\alpha_k\in\mathbf{C}$ is an arbitrary constant.

Our approach to finding non-trivial solutions of equation (2.3) when $d=2$ is to choose a wavenumber $\kappa_1\in\Gamma'(\kappa_1\neq0)$ and solve for the corresponding c_0

such that $\varLambda_\sigma(c_0,\kappa_1)=0$, i.e.

$$c_0=\pm\sqrt{\frac{(g+\sigma|\kappa_1|^2)\tanh(|\kappa_1|)}{|\kappa_1|}}; \tag{2.7}$$

without loss of generality, we can always select the positive root. With c_0 chosen in this way, we can write $\varLambda_\sigma(c_0,k)=k^2(\psi(\kappa_1)-\psi(k))$ where

$$\psi(k)=\frac{(g+\sigma k^2)\tanh(k)}{k},$$

and we have used the fact that $\tanh(k)/k$ is even to drop the absolute value.

Clearly, when $k=0$, $\pm\kappa_1$, $\varLambda_\sigma$ is zero and the null space of the linearized operator is at least three-dimensional. An important question is whether other wavenumbers will produce zeros of $\varLambda_\sigma$ resulting in a higher dimensional null space; this scenario is one of *resonance* and is left outside the scope of our current theory. However, one can easily make some general statements concerning the possibility of resonance. In the case of zero surface tension ($\sigma=0$)

$$\psi(k)=\frac{g\tanh(k)}{k},$$

which (for $k>0$) is strictly decreasing, implying that $\psi(k)=\psi(\kappa_1)$ if and only if $k=\pm\kappa_1$. Thus, in this case, there can be no resonance. However, for $\sigma>0$, the derivative of ψ may vanish at a point if σ/g is sufficiently small. However, even in this case, the existence of an additional *integer* root of $\psi(k)=\psi(\kappa_1)$ will not occur generically.

Our approach to finding non-trivial solutions of equation (2.3) when $d>2$ is to choose $(d-1)$ many wavenumbers $\kappa_1,\kappa_2,\dots,\kappa_{d-1}\in\Gamma'$ $(\kappa_j\neq 0)$ and solve the corresponding set of $(d-1)$ equations $\varLambda_\sigma(c_0,\kappa_j)=0$, i.e.

$$Kc_0=R, \tag{2.8}$$

where $K\in\mathbf{R}^{(d-1)\times(d-1)}$ has rows κ_1, κ_2, …, κ_{d-1}, and $R\in\mathbf{R}^{d-1}$ has jth entry

$$R_j=\pm\sqrt{(g+\sigma|\kappa_j|^2)|\kappa_j|\tanh(|\kappa_j|)}. \tag{2.9}$$

Among the 2^{d-1} choices for the vector R, we will always choose the one such that $R_j>0$. When $d>2$ there is always the possibility, though rare, that additional ‘resonant’ wavenumbers κ_d, …, κ_p may exist such that $\varLambda(c_0,\kappa_j)=0$ for $j=d$, …, p. In fact, when $\sigma=0$ the number p can be infinite; see e.g. (Craig & Nicholls 2000) for a more complete discussion of these issues.

For general $d\geq 2$, in the non-resonant case ($p=d-1$), the first-order solution will be of the form:

$$\eta_1(x)=\sum_{j=1}^{d-1}\eta_{e_j}\varepsilon_j=\sum_{j=1}^{d-1}(\rho_j\varepsilon_j)(c_0\cdot\kappa_j)\cos(\kappa_j\cdot x+\theta_j).$$

Clearly, by varying ε_j, we lose no generality by setting $\rho_j=1$. Furthermore, by fixing the crest of the linear solution at zero we may choose $\theta_j=0$. Therefore our solution surface will be parameterized by $(d-1)$ many parameters, ε_j. We note here that, in particular, our construction in §4 will deliver a unique solution $(\eta(\varepsilon_1,\dots,\varepsilon_{d-1}),\ \varphi(\varepsilon_1,\dots,\varepsilon_{d-1}),\ c(\varepsilon_1,\dots,\varepsilon_{d-1}))$ even in the *degenerate* case where one or more of the ε_j is set to vanish identically. For instance, when $d=3$ if we set $\varepsilon_2=0$, then the linear solution becomes

$$\eta_1(x) = \varepsilon_1 \cos(\kappa_1 \cdot x),$$

and κ_2 plays no role; in fact, the presence of κ_2 is purely artificial. As a consequence of this, our unique solutions comprise, in this situation, only a one-dimensional family within the two-dimensional (Stokes) manifold of solutions corresponding to κ_1 (Craig & Nicholls 2002). Interestingly, however, this manifold can be completely recovered from our solutions by adding appropriate velocity components. More precisely, if our unique solution is $(\eta(\varepsilon_1,0),\ \phi(\varepsilon_1,0),\ c(\varepsilon_1,0))$, then all solutions can be parameterized by $(\varepsilon_1,\ \delta)\to(\eta(\varepsilon_1,0),\ \phi(\varepsilon_1,0),\ c(\varepsilon_1,0)+\delta c^*)$, where $c^*\cdot\kappa_1=0$. Similar remarks apply to the case $\varepsilon_1=0$ and $\varepsilon_1=\varepsilon_2=0$ (trivial branch); in this latter case the parameterization is $(\delta_1,\delta_2)\to(0,0,c(0,0)+(\delta_1,\delta_2))$.

(*c*) *Field expansions*

A classical approach to finding approximate solutions to equation (2.2) was devised by Stokes (1847) in the mid-1800s. It consists of the boundary perturbations philosophy we have termed 'field expansions' (FE) to distinguish it from the alternative 'operator expansions' approach, (e.g. Nicholls & Reitich 2004*a*,*b*) carried out to low (first or second) order. This method was expanded to higher orders by subsequent authors with the most recent attempts being those of Roberts (1983), Roberts & Schwartz (1983), and Marchant & Roberts (1987). For ease of comparison with the results contained in these papers we adopt their notation in the current exposition of the FE approach.

For simplicity, we consider equation (2.2) in the case of two dimensions $(d=2)$, 2π-periodicity, zero capillarity $(\sigma=0)$, and infinite depth $(h=\infty)$. Following Roberts (1983), we define the surface velocities:

$$U(x) = \partial_x\varphi(x,y)|_{y=\eta}, \quad V(x) = \partial_y\varphi(x,y)|_{y=\eta},$$

and expand

$$\eta(x,\varepsilon) = \sum_{n\geq 1}\eta_n(x)\varepsilon^n, \quad \varphi(x,y,\varepsilon) = \sum_{n\geq 1}\varphi_n(x,y)\varepsilon^n, \quad c(\varepsilon) = \sum_{n\geq 0} c_n\varepsilon^n, \tag{2.10a}$$

$$U(x,\varepsilon) = \sum_{n\geq 1} U_n(x)\varepsilon^n, \quad V(x,\varepsilon) = \sum_{n\geq 1} V_n(x)\varepsilon^n. \tag{2.10b}$$

We find that we must solve

$$\Delta\varphi_n = 0, \quad y<0, \tag{2.11a}$$

$$\partial_y\varphi_n \to 0, \quad y\to -\infty, \tag{2.11b}$$

$$c_0 U_n + g\eta_n = Q_n - c_{n-1}U_1 \quad \text{at } y=0, \tag{2.11c}$$

$$-c_0\partial_x\eta_n + V_n = R_n + c_{n-1}\partial_x\eta_1 \quad \text{at } y=0, \tag{2.11d}$$

where

$$Q_n = -\sum_{l=1}^{n-2} c_l U_{n-l} - \frac{1}{2}\sum_{l=1}^{n-1} U_l U_{n-l} - \frac{1}{2}\sum_{l=1}^{n-1} V_l V_{n-l},$$
$$R_n = \sum_{l=1}^{n-2} c_l\partial_x\eta_{n-l} + \sum_{l=1}^{n-1}\partial_x\eta_l U_{n-l}.$$

To solve these equations we note that, on account of equations (2.11a) and (2.11b), and the periodic boundary conditions, η_n and φ_n can be expressed as

$$\eta_n(x) = \sum_{k=-\infty}^{\infty} d_{n,k}\,\mathrm{e}^{\mathrm{i}kx}, \quad \varphi_n(x,y) = \sum_{k=-\infty}^{\infty} a_{n,k}\,\mathrm{e}^{\mathrm{i}kx+|k|y}. \tag{2.12}$$

To find forms for U_n and V_n we first write

$$\mathrm{e}^{\gamma\eta} = \exp\left[\gamma\sum_{n\geq 1}\eta_n\varepsilon^n\right] = \sum_{n\geq 0} E_n(x;\gamma)\varepsilon^n.$$

As can be easily verified, the coefficients E_n are polynomials in γ of degree n which can be recursively found from the relations

$$E_0 = 1, \quad E_n = \sum_{l=1}^{n}\frac{l}{n}\eta_l E_{n-l}(x;\gamma)\gamma. \tag{2.13}$$

Then, we have

$$\begin{aligned} U(x) &= \sum_{n\geq 1}\partial_x\varphi_n(x,y)\bigg|_{y=\eta}\varepsilon^n = \sum_{n\geq 1}\sum_{k=-\infty}^{\infty} a_{n,k}(\mathrm{i}k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{e}^{|k|\eta}\varepsilon^n \\ &= \sum_{n\geq 1}\sum_{k=-\infty}^{\infty}\left(\sum_{m\geq 0} E_m(x;|k|)\varepsilon^m\right) a_{n,k}(\mathrm{i}k)\mathrm{e}^{\mathrm{i}kx}\varepsilon^n \\ &= \sum_{n\geq 1}\varepsilon^n\sum_{k=-\infty}^{\infty}(\mathrm{i}k)a_{n,k}\,\mathrm{e}^{\mathrm{i}kx} + \sum_{n\geq 2}\varepsilon^n\sum_{l=1}^{n-1}\sum_{k=-\infty}^{\infty} E_{n-l}(x;|k|)(\mathrm{i}k)a_{l,k}\,\mathrm{e}^{\mathrm{i}kx}, \end{aligned}$$

so that we can write $U_n = \bar{U}_n + \tilde{U}_n$, where

$$\bar{U}_n = \sum_{k=-\infty}^{\infty}(\mathrm{i}k)a_{n,k}\,\mathrm{e}^{\mathrm{i}kx}, \quad \tilde{U}_n = \sum_{l=1}^{n-1}\sum_{k=-\infty}^{\infty} E_{n-l}(x;|k|)(\mathrm{i}k)a_{l,k}\,\mathrm{e}^{\mathrm{i}kx}. \tag{2.14}$$

Similarly, $V_n = \bar{V}_n + \tilde{V}_n$, where

$$\bar{V}_n = \sum_{k=-\infty}^{\infty}|k|a_{n,k}\,\mathrm{e}^{\mathrm{i}kx}, \quad \tilde{V}_n = \sum_{l=1}^{n-1}\sum_{k=-\infty}^{\infty} E_{n-l}(x;|k|)|k|a_{1,k}\,\mathrm{e}^{\mathrm{i}kx}. \tag{2.15}$$

Finally, we can rewrite equation (2.11) as

$$\Delta\varphi_n = 0 \quad y<0, \tag{2.16a}$$

$$\partial_y\varphi_n \to 0, \quad y \to -\infty, \tag{2.16b}$$

$$c_0\bar{U}_n + g\eta_n = Q_n - c_0\tilde{U}_n - c_{n-1}U_1 \quad \text{at } y=0, \tag{2.16c}$$

$$-c_0\partial_x\eta_n + \bar{V}_n = R_n - \tilde{V}_n + c_{n-1}\partial_x\eta_1 \quad \text{at } y=0, \tag{2.16d}$$

which, using equation (2.12), is equivalent to

$$\begin{pmatrix} ic_0k & g \\ |k| & -ic_0k \end{pmatrix}\begin{pmatrix} a_{n,k} \\ d_{n,k} \end{pmatrix} = \begin{pmatrix} S_{n,k} - (ic_{n-1}k)a_{1,k} \\ T_{n,k} + (ic_{n-1}k)d_{1,k} \end{pmatrix}, \quad -\infty<k<\infty, \tag{2.17}$$

where

$$S_n = \sum_{k=-\infty}^{\infty} S_{n,k}\,\mathrm{e}^{ikx}, \quad T_n = \sum_{k=-\infty}^{\infty} T_{n,k}\,\mathrm{e}^{ikx},$$

and

$$S_n = -c_0\tilde{U}_n - \sum_{l=1}^{n-2} c_l U_{n-l} - \frac{1}{2}\sum_{l=1}^{n-1} U_l U_{n-l} - \frac{1}{2}\sum_{l=1}^{n-1} V_l V_{n-l},$$
$$T_n = -\tilde{V}_n + \sum_{l=1}^{n-2} c_l\partial_x\eta_{n-l} + \sum_{l=1}^{n-1}\partial_x\eta_l U_{n-l}.$$

The FE procedure consists of solving equation (2.17) recursively up to a specified order $n=N$, starting with relations of the form (2.6), where $k=\kappa_1$ and c_0 are chosen to satisfy $\Lambda_0(c_0, \kappa_1)=0$. For example, choosing $\kappa_1=c_0=1$ and normalizing $g=1$ the linear part of the solution (taking α real) is (cf. equation (2.6))

$$\eta_1(x) = 2\alpha\cos(x), \quad \varphi_1(x,y) = -2\alpha\,\mathrm{e}^y\sin(x). \tag{2.18}$$

The procedure can be carried out to an arbitrarily high order N by recursively solving equation (2.17). Of course, by the choice of c_0, the matrix in equation (2.17) is singular at $k=\pm\kappa_1$ and a compatibility condition is required to ensure solvability. This is provided by an appropriate choice of c_{n-1} in equation (2.17) which closes the system of equations.

(*d*) *Cancellations*

It should be noted that this derivation of the FE recursions, equation (2.17), is purely formal in nature. Indeed, for instance, the recurrence entails spatial derivatives of the velocity potential of an increasingly high order (cf. equations (2.14) and (2.15)) whose growth should be controlled if the series in equation (2.10) are to be shown to converge. On the other hand, if such control is to be

Table 1. *Coefficients* $|d_{n,n}|$ *and digits of accuracy*

(Computation of the coefficients $|d_{n,n}|$ (cf. equation (2.12)), in double and quadruple precision—only 16 digits are reported—and the digits of accuracy contained in the double precision calculation.)

n	double precision	quadruple precision	digits of accuracy
5	0.162 760 416 666 666 8	0.162 760 416 666 666 7	15
7	0.182 367 621 527 777 9	0.182 367 621 527 777 8	15
9	0.231 689 889 090 401 3	0.231 689 889 090 401 8	15
11	0.317 278 892 745 832 9	0.317 278 892 745 837 1	14
13	0.456 719 934 443 245 0	0.456 719 934 443 266 4	13
15	0.681 283 829 123 065 3	0.681 283 829 123 126 3	12
17	1.043 768 775 207 166	1.043 768 775 207 084	13
19	1.632 677 013 377 351	1.632 677 013 390 746	11
21	2.596 641 986 288 087	2.596 641 980 321 151	9
23	4.186 275 932 441 967	4.186 277 983 801 273	6
25	6.823 624 125 009 305	6.825 974 125 230 163	3
27	8.836 865 217 279 312	11.237 383 290 065 77	0

based on the relations (2.17), it will demand, for instance, that we bound U_n, V_n recursively from equations (2.14) and (2.15). Here, however, the only obvious bound will, by necessity, use the triangle inequality in the order l (cf. equations (2.14) and (2.15)). As we have shown in related applications of boundary perturbation approaches (Nicholls & Reitich 2001*a*,*b*, 2003, 2004*a*,*b*), these bounds will consistently fail to provide useful growth control on any norm of the solutions as they destroy significant cancellations that are present in the corresponding recurrences.

To substantiate this claim we next present a set of numerical experiments that demonstrate the existence of cancellations in equations (2.14) and (2.15) as well as their implications in attempts at numerically simulating travelling water waves with high-order versions of the FE scheme. We begin by choosing specific parameters that give rise to a bifurcating solution: $d=2$, $\sigma=0$, $h=\infty$, $g=1$, 2π-periodicity, $\kappa_1=1$, $c_0=1$, for which we present results of a suitable FE implementation up to order $N=40$. In particular, all convolution products in S_n and T_n are performed using fast Fourier acceleration in vectors of length $N_x=128$ (which represent wavenumbers $[-N_x/2,\ N_x/2-1]$) greater than $2N=80$ to prevent aliasing.

The first evidence we present concerns the accurate computation of the Fourier coefficients of the wave profile η. It is not difficult to see that, beginning with equation (2.18), $d_{n,n+p}=a_{n,n+p}=0$ for all $p>0$, while $d_{n,n}$ and $a_{n,n}$ will not be equal to zero and represent a 'leading edge' of non-zero Fourier coefficients at order n. In table 1 we report the results of the computation of $|d_{n,n}|$, via the FE recursion (2.17) with $\alpha=1/2$ in equation (2.18), in both double and quadruple precision ($N_x=128$). In the final column we treat the quadruple precision calculations as 'exact solutions' and count the digits of accuracy in the double precision calculation. We point out the precipitous loss of accuracy in the coefficients $|d_{n,n}|$ through all orders of n, which rapidly accelerates beyond $n=17$, resulting in approximations which contain *no accurate information* by $n=27$.

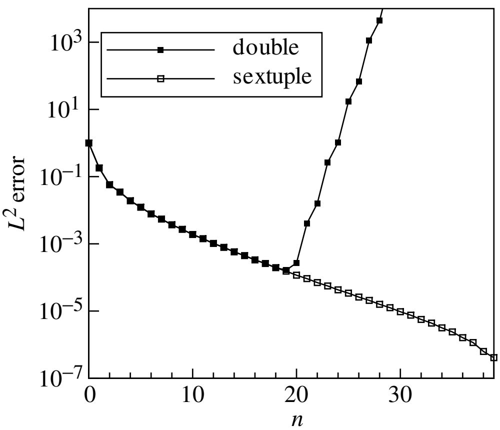

Figure 1. Comparison of double and sextuple precision computations of η_n^{128}, cf. equation (2.19), with a highly resolved solution (sextuple precision calculation with $N_x=128$, $n=40$). Error is measured in the L^2 norm ($N_x=128$, $0\leq n\leq 39$, $\varepsilon=0.3$, $\sigma=0$, $h=\infty$, $g=1$, $\kappa=1$, $c_0=1$).

In the previous calculation, one could argue that for large n an accompanying factor of ε^n (where ε is typically much less than 1) in the approximation of $\eta(x)$ might disguise the inaccurate computation of $d_{n,n}$. However, as the next calculation illustrates, such a hope is unfounded and the accurate computation of high-frequency information is *crucial* for a correct representation. To substantiate this claim, we next approximate a more physically relevant quantity, the L^2-norm of the wave form $\eta(x)$. More specifically, if we denote

$$\eta_n^{N_x}(x;\varepsilon) = \sum_{j=0}^{n} \sum_{k=-N_x/2}^{N_x/2-1} d_{j,k}\, \mathrm{e}^{\mathrm{i}kx} \varepsilon^j, \tag{2.19}$$

which represents the FE approximation to $\eta(x)$, then in figure 1 we present the difference (measured in L^2) between double precision and sextuple precision approximations of $\eta_n^{128}(x;0.3)$, and a highly resolved calculation (sextuple precision with $N_x=128$ and $n=40$); sextuple precision was necessary as we found that quadruple precision calculations were inadequate beyond $n=33$. We note that at $\varepsilon=0.3$ the sextuple precision calculation is fully converged at $n=40$, indicating that $\varepsilon=0.3$ is *within* the disc of convergence of the Taylor series (2.10). We point out in this figure the explosive divergence of the double precision calculation as n is increased. We also note the roughly linear shape of the curve on the log-linear axes indicating the exponential growth of errors.

3. Transformed field expansions

As the calculations of the previous section indicate, the cancellations in equation (2.17) are present for all n and increase in severity with increasing n. As explained above, this has consequences for more than just numerical simulation. Indeed, as we mentioned, the cancellations preclude the use of the most natural approach to estimating the convergence of the series (2.10) based on the derivation of bounds, e.g. of the form $\|\eta_n\|_{H^s} < CB^n$, from the recurrence (2.17).

However, as we explain next and further demonstrate in §4 a direct estimation of the terms in the series (2.10) can be realized upon a change of independent variables in advance of the perturbation expansion, much as in the application of boundary perturbation methods to boundary value problems (Nicholls & Reitich 2004*a*,*b*). Indeed, as in these latter applications, the transformation has the effect of implicitly accounting for all significant cancellations so that the terms in the corresponding recurrence can be inductively estimated. To derive these transformed field expansions (TFE) we begin by considering the transformation

$$x' = x, \quad y' = \frac{y-\eta}{1+\eta}, \tag{3.1}$$

which maps the domain $S_{1,\eta}$ to the strip $S_{1,0}$. The equations (2.2) become, upon dropping primes,

$$\Delta\varphi(x,y) = F(x,y) \quad \text{in } S_{1,0}, \tag{3.2a}$$

$$\partial_y\varphi(x,-1) = \int_{P(\Gamma)} \partial_y\varphi(x,-1)\,\mathrm{d}x, \qquad \int_{P(\Gamma)} \varphi(x,-1)\,\mathrm{d}x = 0, \tag{3.2b}$$

$$[c_0\cdot\nabla_x]\varphi + [g-\sigma\Delta_x]\eta = Q(x) \quad \text{at } y=0, \tag{3.2c}$$

$$-[c_0\cdot\nabla_x]\eta + \partial_y\varphi = R(x) \quad \text{at } y=0, \tag{3.2d}$$

where c_0 will be defined as in equation (2.7), $F(x,y) = \mathrm{div}_x[F^{(1)}(x,y)] + \partial_y F^{(2)}(x,y) + F^{(3)}(x,y)$, and

$$F^{(1)}(x,y) = -\eta^2\nabla_x\varphi - 2\eta\nabla_x\varphi + (1+y)(1+\eta)\nabla_x\eta\partial_y\varphi, \tag{3.3a}$$

$$F^{(2)}(x,y) = (1+y)(1+\eta)\nabla_x\eta\cdot\nabla_x\varphi - (1+y)^2|\nabla_x\eta|^2\partial_y\varphi, \tag{3.3b}$$

$$F^{(3)}(x,y) = (1+\eta)\nabla_x\eta\cdot\nabla_x\varphi - (1+y)|\nabla_x\eta|^2\partial_y\varphi. \tag{3.3c}$$

The functions $Q(x)$ and $R(x)$ are represented by similar formulas.

To solve equations (3.2) in the transformed variables we now propose the following expansions for $\varepsilon\in\mathbf{R}^{d-1}$ and multi-index $n\in\mathbf{N}^{d-1}$

$$\varphi(x,y,\varepsilon) = \sum_{|n|\geq 1}\varphi_n(x,y)\varepsilon^n, \quad \eta(x,\varepsilon) = \sum_{|n|\geq 1}\eta_n(x)\varepsilon^n, \quad c(\varepsilon) = \sum_{|n|\geq 0} c_n\varepsilon^n \tag{3.4}$$

(the 'TFE') and find that we must solve the following problems

$$\Delta\varphi_n(x,y) = (1-\delta_{|n|,0})F_n(x,y) \quad \text{in } S_{1,0}, \tag{3.5a}$$

$$\partial_y\varphi_n(x,-1) = \int_{P(\Gamma)} \partial_y\varphi_n(x,-1)\,\mathrm{d}x, \qquad \int_{P(\Gamma)} \varphi_n(x,-1)\,\mathrm{d}x = 0, \tag{3.5b}$$

$$[c_0\cdot\nabla_x]\varphi_n(x,0)+[g-\sigma\Delta_x]\eta_n(x)+\sum_{\substack{j=1\\ e_j\le n}}^{d-1}[c_{n-e_j}\cdot\nabla_x]\varphi_{e_j}(x,0)=Q_n(x), \qquad (3.5c)$$

$$-[c_0\cdot\nabla_x]\eta_n(x)+\partial_y\varphi_n(x,0)-\sum_{\substack{j=1\\ e_j\le n}}^{d-1}[c_{n-e_j}\cdot\nabla_x]\eta_{e_j}(x,0)=R_n(x). \qquad (3.5d)$$

Here, $\delta_{k,p}$ is the Kronecker delta, $e_j=(0,\dots,0,1,0,\dots,0)$ where e_j is non-zero at index j, and for multi-indices m, $n\in\mathbf{N}^{d-1}$, $m\le n$ if $m_j\le n_j$ for all $j=1,\dots,d-1$. Furthermore, $F_n(x,y)=\mathrm{div}_x[F_n^{(1)}(x,y)]+\partial_y F_n^{(2)}(x,y)+F_n^{(3)}(x,y)$, where

$$\begin{aligned}F_n^{(1)}(x,y)=&\sum_{|m|=2}^{|n|-1}\sum_{|l|=1}^{|m|-1}\eta_l\eta_{m-l}\nabla_x\varphi_{n-m}-2\sum_{|l|=1}^{|n|-1}\eta_l\nabla_x\varphi_{n-l}\\&+(1+y)\sum_{|m|=2}^{|n|-1}\sum_{|l|=1}^{|m|-1}\eta_l\nabla_x\eta_{m-l}\partial_y\varphi_{n-m}+(1+y)\sum_{|l|=1}^{|n|-1}\nabla_x\eta_l\partial_y\varphi_{n-l},\end{aligned} \qquad (3.6a)$$

$$\begin{aligned}F_n^{(2)}(x,y)=&(1+y)\sum_{|m|=2}^{|n|-1}\sum_{|l|=1}^{|m|-1}\eta_l\nabla_x\eta_{m-l}\cdot\nabla_x\varphi_{n-m}+(1+y)\sum_{|l|=1}^{|n|-1}\nabla_x\eta_l\cdot\nabla_x\varphi_{n-l}\\&-(1+y)^2\sum_{|m|=2}^{|n|-1}\sum_{|l|=1}^{|m|-1}\nabla_x\eta_l\cdot\nabla_x\eta_{m-l}\partial_y\varphi_{n-m},\end{aligned} \qquad (3.6b)$$

$$\begin{aligned}F_n^{(3)}(x,y)=&\sum_{|m|=2}^{|n|-1}\sum_{|l|=1}^{|m|-1}\eta_l\nabla_x\eta_{m-l}\cdot\nabla_x\varphi_{n-m}+\sum_{|l|=1}^{|n|-1}\nabla_x\eta_l\cdot\nabla_x\varphi_{n-l}\\&-(1+y)\sum_{|m|=2}^{|n|-1}\sum_{|l|=1}^{|m|-1}\nabla_x\eta_l\cdot\nabla_x\eta_{m-l}\partial_y\varphi_{n-m}.\end{aligned} \qquad (3.6c)$$

The functions Q_n and R_n can be similarly derived. Note that $F_n^{(l)}$ (and Q_n, and R_n) depend only on $\{\eta_j\}_{|j|<|n|}$, $\{\varphi_j\}_{|j|<|n|}$, and $\{c_j\}_{|j|<|n|-1}$, and vanish for $|n|=1$.

4. Analyticity of solutions

Clearly, the nature of equations (3.5) is quite similar to that of equations (2.16); however, there are some important differences. Most notably, in contrast to equations (2.16), the right-hand sides of equation (3.5) contain only derivatives of order one or two which act either on the field, φ_l, or the surface shape, η_l. In what follows, we show that this, in fact, allows for the inductive establishment of bounds:

$$\|\varphi_n\|_X \le CB^{|n|}, \quad \|\eta_n\|_Y \le CB^{|n|}, \quad |c_n| \le CB^{|n|}$$

in appropriate function spaces X and Y, and for some constants C, $B \ge 0$. Furthermore, we will show that *all* spatial derivatives of φ_n and η_n can be similarly bounded, implying that all quantities are *jointly* analytic with respect to all arguments.

To set notation we recall that any L^2 function f periodic on a $(d-1)$-dimensional lattice $\Gamma \subset \mathbf{R}^{d-1}$ can be represented as

$$f(x) = \sum_{k\in\Gamma'} \hat{f}(k) \mathrm{e}^{\mathrm{i}k\cdot x}, \tag{4.1}$$

where Γ' is the conjugate lattice to Γ. Additionally, if an L^2 function $u(x, y)$ is periodic in x with respect to Γ and square integrable in the y variable on $[-1, 0]$ then

$$u(x, y) = \sum_{k\in\Gamma'} \hat{u}(k, y) \mathrm{e}^{\mathrm{i}k\cdot x}. \tag{4.2}$$

Using this representation we can define the L^2 based Sobolev spaces

$$H^s = \{u \in L^2 | \|u\|_{H^s} < \infty\}, \tag{4.3}$$

for $s \in \mathbf{Z}^+$ and where

$$\|u(x, y)\|^2_{H^s} \equiv \sum_{j=0}^{s} \sum_{k\in\Gamma'} \langle k\rangle^{2s-2j} \int_{-1}^{0} |\partial_y^j \hat{u}(k, y)|^2 \mathrm{d}y. \tag{4.4}$$

and $\langle k\rangle = \sqrt{1+k^2}$. Note that if $f = f(x)$ depends on x alone then the space H^s for $s \in \mathbf{R}$ can be defined by the norm

$$\|f(x)\|^2_{H^s} \equiv \sum_{k\in\Gamma'} \langle k\rangle^{2s} |\hat{f}(k)|^2. \tag{4.5}$$

For future reference we note the following algebra property for H^s (Adams, 1975).

Lemma 4.1. *If $s > d/2$ then H^s is an algebra, i.e. for u, $v \in H^s$*

$$\|uv\|_{H^s} \le M \|u\|_{H^s} \|v\|_{H^s} \tag{4.6}$$

for a constant $M = M(d, s)$ depending only on d and s.

In the case of non-zero surface tension ($\sigma > 0$), our main result is as follows.

Theorem 4.2. *Given an integer $s > d/2$, if $|n| \ge 1$ the solutions $\varphi_n(x, y)$, $\eta_n(x)$, and c_{n-e_j} of equation* (3.5) ($\sigma > 0$) *satisfy*

$$\left\| \frac{\partial_x^k \partial_y^l}{(|k|+l)!} \varphi_n \right\|_{H^{s+2}} \le C_1 \frac{B^{|n|-1}}{|n|^p} \frac{D^l}{(l+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}, \tag{4.7a}$$

$$\left\| \frac{\partial_x^k}{|k|!} \eta_n \right\|_{H^{s+5/2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}, \quad |c_{n-e_j}| \leq C_1 \frac{B^{|n|-1}}{|n|^p} \tag{4.7b}$$

for all $j=1,\ldots,\ d-1,\ p>d-1$, *and some constants* $C_1,\ B,\ D,\ A>0$.

We will prove theorem 4.2 by induction on l; thus our first objective is to establish the result for $l=0$. This is done in the following lemma, for all k, with an induction on the order $|n|$.

Lemma 4.3. *Given an integer* $s>d/2$, *if* $|n|\geq 1$ *the solutions* $\varphi_n(x,\ y)$, $\eta_n(x)$, *and* c_{n-e_j} *of equation (3.5)* $(\sigma>0)$ *satisfy*

$$\left\| \frac{\partial_x^k}{|k|!} \varphi_n \right\|_{H^{s+2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2},$$

$$\left\| \frac{\partial_x^k}{|k|!} \eta_n \right\|_{H^{s+5/2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}, \quad |c_{n-e_j}| \leq C_1 \frac{B^{|n|-1}}{|n|^p}$$

for all $j=1,\ \ldots,\ d-1,\ p>d-1$, *and some constants* $C_1,\ B,\ A>0$.

To prove Lemma 4.3 we need two lemmas: the following which estimates the right hand sides of equation (3.5) of the inhomogeneous problems, and the sequel which provides estimates on solutions of these problems.

Lemma 4.4. *Given an integer* $s>d/2$, *suppose that*

$$\left\| \frac{\partial_x^k}{|k|!} \varphi_n \right\|_{H^{s+2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \quad \forall k,$$

$$\left\| \frac{\partial_x^k}{|k|!} \eta_n \right\|_{H^{s+5/2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \quad \forall k, \quad |c_{n-e_j}| \leq C_1 \frac{B^{|n|-1}}{|n|^p},$$

for all $1\leq n<N$ *(i.e.* $n_j<N_j$*)*, $p>d-1$, *and for some constants* $C_1,\ B,\ A>0$. *Then there exists a constant* C_2 *such that the functions* $F_N^{(j)}$, $Q_N^{(j)}$, *and* R_N *in equation (3.5) satisfy*

$$\left\| \frac{\partial_x^k}{|k|!} F_N^{(j)} \right\|_{H^s} \leq C_1 C_2 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \quad \forall k, \tag{4.8a}$$

$$\left\| \frac{\partial_x^k}{|k|!} Q_N^{(j)} \right\|_{H^{s+1/2}} \leq C_1 C_2 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \quad \forall k, \tag{4.8b}$$

$$\left\| \frac{\partial_x^k}{|k|!} R_N \right\|_{H^{s+1/2}} \leq C_1 C_2 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \quad \forall k. \tag{4.8c}$$

Proof of lemma 4.4. For the sake of brevity consider the first term of $F_N^{(3)}$:

$$Z_1 = \sum_{|m|=2}^{|n|-1} \sum_{|l|=1}^{|m|-1} \eta_l \eta_{m-l} \nabla_x \varphi_{n-m};$$

every other term in F_N, Q_N, and R_N can be similarly estimated. We begin

$$\left\| \frac{\partial_x^k}{|k|!} Z_1 \right\|_{H^s} \leq \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} M^2 \sum_{\sigma \leq \tau} \sum_{\tau \leq k} \frac{k!|\sigma|!|\tau-\sigma|!|k-\tau|!}{|k|!\sigma!(\tau-\sigma)!(k-\tau)!} \left\| \frac{\partial_x^\sigma}{|\sigma|!} \eta_l \right\|_{H^s} \times \left\| \frac{\partial_x^{\tau-\sigma}}{|\tau-\sigma|!} \nabla_x \eta_{m-l} \right\|_{H^s} \left\| \frac{\partial_x^{k-\tau}}{|k-\tau|!} \nabla_x \varphi_{N-m} \right\|_{H^s}.$$

Since

$$\frac{k!|\sigma|!|\tau-\sigma|!|k-\tau|!}{|k|!\sigma!(\tau-\sigma)!(k-\tau)!} \leq 1,$$

we can deduce that

$$\left\| \frac{\partial_x^k}{|k|!} Z_1 \right\|_{H^s} \leq \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} M^2 \sum_{\sigma \leq \tau} \sum_{\tau \leq k} C_1 \frac{B^{|l|-1}}{|l|^p} \prod_{q=1}^{d-1} \frac{A^{\sigma_q}}{(\sigma_q+1)^2} \times C_1 \frac{B^{|m-l|-1}}{|m-l|^p} \prod_{q=1}^{d-1} \frac{A^{\tau_q-\sigma_q}}{(\tau_q-\sigma_q+1)^2} C_1 \frac{B^{|N-m|-1}}{|N-m|^p} \prod_{q=1}^{d-1} \frac{A^{k_q-\tau_q}}{(k_q-\tau_q+1)^2}.$$

Continuing,

$$\left\| \frac{\partial_x^k}{|k|!} Z_1 \right\|_{H^s} \leq M^2 C_1^3 \frac{B^{|N|-3}}{|N|^p} S^{2(d-1)} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} \frac{|N|^p}{|l|^p |m-l|^p |N-m|^p},$$

and

$$\left\| \frac{\partial_x^k}{|k|!} Z_1 \right\|_{H^s} \leq \left[M^2 C_1^3 \frac{S^{2(d-1)} \Sigma_{d-1}^2}{B} \right] \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2},$$

where

$$S = \max_k \sum_{\tau=0}^{k} \frac{k^2}{(\tau+1)^2(k-\tau-1)^2}, \quad \Sigma_{d-1} \equiv \max_m \sum_{|l|=1}^{|m|-1} \frac{|m|^p}{|l|^p|m-l|^p},$$

which is bounded uniformly in $|m|$ for $p > d-1$. The proof is complete provided $B > MC_1 S^{d-1} \Sigma_{d-1}$. ■

The second lemma necessary to prove lemma 4.3 is now presented; the proof, based on classical elliptic estimates, is given in appendix A.

Lemma 4.5. *Consider any integer $s \geq 0$. Given linearly independent wavenumbers $\kappa_1, \ldots, \kappa_{d-1} \in \Gamma' \subset \mathbf{R}^{d-1}$ there exists a unique speed $c = (c_1, \ldots, c_{d-1}) \in \mathbf{R}^{d-1}$ satisfying equation (2.8) such that $R_j > 0$. Given this c, if for all multi-indices $|n| = \bar{n}$, $p_n \in H^s$, $q_n \in H^{s+1/2}$, and $r_n \in H^{s+1/2}$ then there exist for all $|n| = \bar{n}$ real solutions $w_n \in H^{s+2}$, $v_n \in H^{s+5/2}$, and μ_{n-e_j} $(j = 1, \ldots, d)$ of*

$$\Delta w_n(x, y) = p_n(x, y) \quad \text{in } S_{1,0}, \tag{4.9a}$$

$$\partial_y w_n(x, -1) = \int\limits_{P(\Gamma)} \partial_y w_n(x, -1) \, \mathrm{d}x, \qquad \int\limits_{P(\Gamma)} w_n(x, -1) \, \mathrm{d}x = 0, \tag{4.9b}$$

$$(g - \sigma \Delta_x) v_n(x) + [c \cdot \nabla_x] w_n(x, 0) + \sum_{\substack{j=1 \\ e_j \leq n}}^{d-1} [\mu_{n-e_j} \cdot \nabla_x] b_{e_j}(x) = q_n(x), \tag{4.9c}$$

$$-[c \cdot \nabla_x] v_n(x) + \partial_y w_n(x, 0) - \sum_{\substack{j=1 \\ e_j \leq n}}^{d-1} [\mu_{n-e_j} \cdot \nabla_x] f_{e_j}(x) = r_n(x), \tag{4.9d}$$

where

$$b_{e_j}(x) = \alpha_j \mathrm{i}(g + \sigma |\kappa_j|^2) \mathrm{e}^{\mathrm{i}\kappa_j \cdot x} - \bar{\alpha}_j \mathrm{i}(g + \sigma |\kappa_j|^2) \mathrm{e}^{-\mathrm{i}\kappa_j \cdot x}, \tag{4.10a}$$

$$f_{e_j}(x) = \alpha_j (c \cdot \kappa_j) \mathrm{e}^{\mathrm{i}\kappa_j \cdot x} + \bar{\alpha}_j (c \cdot \kappa_j) \mathrm{e}^{-\mathrm{i}\kappa_j \cdot x}. \tag{4.10b}$$

If, in addition, we require that

$$\int\limits_{P(\Gamma)} v_n(x) \mathrm{e}^{\pm \mathrm{i}\kappa_j \cdot x} \, \mathrm{d}x = 0, \tag{4.11}$$

then this solution is unique. Furthermore there exists a constant C_e such that the solutions satisfy

$$\| w_n \|_{H^{s+2}} \leq C_e [\| p_n \|_{H^s} + \| q_n \|_{H^{s-1/2}} + \| r_n \|_{H^{s+1/2}}], \tag{4.12a}$$

$$\| v_n \|_{H^{s+5/2}} \leq C_e [\| p_n \|_{H^s} + \| q_n \|_{H^{s+1/2}} + \| r_n \|_{H^{s+1/2}}], \tag{4.12b}$$

$$|\mu_{n-e_j}| \leq C_e [\| p_n \|_{H^s} + \| q_n \|_{H^{s+1/2}} + \| r_n \|_{H^{s+1/2}}]. \tag{4.12c}$$

We can now complete the proof of lemma 4.3.

Proof of lemma 4.3. The proof proceeds via induction on $|n|$; since φ_n, η_n, c_n satisfy equation (3.5), $\partial_x^k / |k|! \varphi_n$, $\partial_x^k / |k|! \eta_n$, and c_{n-1} satisfy

$$\Delta \frac{\partial_x^k}{|k|!} \varphi_n = (1 - \delta_{n,1}) \frac{\partial_x^k}{|k|!} F_n \quad \text{in } S_{1,0}, \tag{4.13a}$$

$$\partial_y \frac{\partial_x^k}{|k|!}\varphi_n(x,-1) = \int_{P(\Gamma)} \partial_y \frac{\partial_x^k}{|k|!}\varphi_n(x,-1)\,\mathrm{d}x, \qquad \int_{P(\Gamma)} \frac{\partial_x^k}{|k|!}\varphi_n(x,-1)\,\mathrm{d}x = 0, \tag{4.13b}$$

$$\begin{aligned}&[c_0\cdot\nabla_x]\frac{\partial_x^k}{|k|!}\varphi_n(x) + [g-\sigma\Delta_x]\frac{\partial_x^k}{|k|!}\eta_n(x) + (1-\delta_{n,1})[c_{n-1}\cdot\nabla_x]\frac{\partial_x^k}{|k|!}\varphi_1(x)\\ &\quad = (1-\delta_{n,1})\frac{\partial_x^k}{|k|!}Q_n(x),\end{aligned} \tag{4.13c}$$

$$\begin{aligned}&-[c_0\cdot\nabla_x]\frac{\partial_x^k}{|k|!}\eta_n(x,0) + \partial_y\frac{\partial_x^k}{|k|!}\varphi_n(x,0) - (1-\delta_{n,1})[c_{n-1}\cdot\nabla_x]\frac{\partial_x^k}{|k|!}\eta_1(x,0)\\ &\quad = (1-\delta_{n,1})\frac{\partial_x^k}{|k|!}R_n(x).\end{aligned} \tag{4.13d}$$

In the case $|n|=1$ we have the *explicit* formulas

$$\begin{aligned}\varphi_{e_j}(x,y) &= \alpha_{k_j}\mathrm{i}(g+\sigma|k_j|^2)\frac{\cosh(|k_j|(y+1))}{\cosh(|k_j|)}\mathrm{e}^{\mathrm{i}k_j\cdot x}\\ &\qquad - \bar{\alpha}_{k_j}\mathrm{i}(g+\sigma|k_j|^2)\frac{\cosh(|k_j|(y+1))}{\cosh(|k_j|)}\mathrm{e}^{-\mathrm{i}k_j\cdot x},\end{aligned} \tag{4.14a}$$

$$\eta_{e_j}(x) = \alpha_{k_j}(c\cdot k_j)\mathrm{e}^{\mathrm{i}k_j\cdot x} + \bar{\alpha}_{k_j}(c\cdot k_j)\mathrm{e}^{-\mathrm{i}k_j\cdot x}, \tag{4.14b}$$

and, for $c_0\in\mathbf{R}^{d-1}$, the set of $(d-1)$ equations $\Lambda_\sigma(c_0,k_j)=0$. From these formulas the estimates for $|n|=1$ follow easily. Next, consider order $|N|$ and note that $N=\nu+e_j$ for some multi-index ν. Suppose that

$$\left\|\frac{\partial_x^k}{|k|!}\varphi_n\right\|_{H^{s+2}} \le C_1\frac{B^{|n|-1}}{|n|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2},$$

$$\left\|\frac{\partial_x^k}{|k|!}\eta_n\right\|_{H^{s+5/2}} \le C_1\frac{B^{|n|-1}}{|n|^p}\prod_{q=1}^{d-1}\frac{A^{k_q}}{(k_q+1)^2}, \quad |c_{n-e_j}| \le C_1\frac{B^{|n|-1}}{|n|^p}$$

for all k and for all $1\le|n|<|N|$, $n<N$. Using lemma 4.5 we have

$$\left\|\frac{\partial_x^k}{|k|!}\varphi_{\nu+e_j}\right\|_{H^{s+2}} \le C_e\left[\left\|\frac{\partial_x^k}{|k|!}F_{\nu+e_j}\right\|_{H^s} + \left\|\frac{\partial_x^k}{|k|!}Q_{\nu+e_j}\right\|_{H^{s-1/2}} + \left\|\frac{\partial_x^k}{|k|!}R_{\nu+e_j}\right\|_{H^{s+1/2}}\right],$$

$$\left\|\frac{\partial_x^k}{|k|!}\eta_{\nu+e_j}\right\|_{H^{s+5/2}} \le C_e\left[\left\|\frac{\partial_x^k}{|k|!}F_{\nu+e_j}\right\|_{H^s} + \left\|\frac{\partial_x^k}{|k|!}Q_{\nu+e_j}\right\|_{H^{s+1/2}} + \left\|\frac{\partial_x^k}{|k|!}R_{\nu+e_j}\right\|_{H^{s+1/2}}\right],$$

$$|c_\nu| \le C_e\left[\left\|\frac{\partial_x^k}{|k|!}F_{\nu+e_j}\right\|_{H^s} + \left\|\frac{\partial_x^k}{|k|!}Q_{\nu+e_j}\right\|_{H^{s+1/2}} + \left\|\frac{\partial_x^k}{|k|!}R_{\nu+e_j}\right\|_{H^{s+1/2}}\right].$$

Finally, using lemma 4.4 we obtain

$$\left\| \frac{\partial_x^k}{|k|!} \varphi_{\nu+e_j} \right\|_{H^{s+2}} \leq C_e 6 C_1 C_2 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2},$$

$$\left\| \frac{\partial_x^k}{|k|!} \eta_{\nu+e_j} \right\|_{H^{s+5/2}} \leq C_e 6 C_1 C_2 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2},$$

$$|c_\nu| \leq C_e 6 C_1 C_2 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}.$$

The proof is complete, provided that $B>6C_eC_2$ and $p>d-1$. ■

Lemma 4.3 proves equation (4.7) for $l=0$. In order to complete the induction for $l>0$ we need the following two results.

Lemma 4.6. *Given an integer $s>0$ the following estimate holds*

$$\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_{e_j} \right\|_{H^{s+2}} \leq C_1 \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \qquad \forall k, L \tag{4.15}$$

for some constants C_1, D, $A>0$.

Proof of lemma 4.6. The proof comes immediately from the explicit formula (4.14) for φ_{e_j}. ■

Lemma 4.7. *Given an integer $s>d/2$, suppose that*

$$\left\| \frac{\partial_x^k \partial_y^l}{(|k|+l)!} \varphi_n \right\|_{H^{s+2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \frac{D^l}{(l+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}, \tag{4.16a}$$

$$\left\| \frac{\partial_x^k}{|k|!} \eta_n \right\|_{H^{s+5/2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}, \quad |c_{n-e_j}| \leq C_1 \frac{B^{|n|-1}}{|n|^p} \tag{4.16b}$$

for all indices $k\geq 0$, $|n|\geq 1$, when $l<L$; for all $k\geq 0$, $|n|<|N|$ when $l=L$; and $p>d-1$. Then there exists a constant C_3 such that F_N, Q_N, and *R_N in equations (3.6) satisfy*

$$\left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} F_N \right\|_{H^{s+1}} \leq C_1 C_3 \frac{B^{|N|-2}}{|N|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \qquad \forall k, \tag{4.17a}$$

$$\left\| \frac{\partial_x^k}{|k|!} Q_N \right\|_{H^{s+1/2}} \leq C_1 C_3 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \qquad \forall k, \tag{4.17b}$$

$$\left\| \frac{\partial_x^k}{|k|!} R_N \right\|_{H^{s+1/2}} \leq C_1 C_3 \frac{B^{|N|-2}}{|N|^p} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \quad \forall k. \tag{4.17c}$$

Proof of lemma 4.7. Estimates (4.17b) and (4.17c) are true by lemma 4.4. For brevity, in regard to equation (4.17a) let us consider only the third portion of $F_N^{(2)}$, namely

$$Z_3(x,y) \equiv (1+y)^2 \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} \nabla_x \eta_l \cdot \nabla_x \eta_{m-l} \partial_y \varphi_{N-m}.$$

Then

$$\begin{aligned}\left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \partial_y Z_3 \right\|_{H^{s+1}} \leq & \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} M^2 \sum_{\sigma \leq \tau} \sum_{\tau \leq k} \frac{k!|\sigma|!|\tau-\sigma|!(|k-\tau|+L)!}{\sigma!(\tau-\sigma)!(k-\tau)!(|k|+L)!} \\ & \times \left\| \frac{\partial_x^\sigma}{|\sigma|!} \nabla_x \eta_l \right\|_{H^{s+1}} \left\| \frac{\partial_x^{\tau-\sigma}}{|\tau-\sigma|!} \nabla_x \eta_{m-l} \right\|_{H^{s+1}} \\ & \times \left\| \frac{\partial_x^{k-\tau} \partial_y^L}{(|k-\tau|+L)!} [(1+y)^2 \partial_y \varphi_{N-m}] \right\|_{H^{s+1}}.\end{aligned}$$

Since

$$\frac{k!|\sigma|!|\tau-\sigma|!(|k-\tau|+L)!}{\sigma!(\tau-\sigma)!(k-\tau)!(|k|+L)!} \leq 1,$$

and

$$\begin{aligned}\partial_y^L[(1+y)^2 \partial_y \varphi_{N-m}] = & (1+y)^2 \partial_y^{L+1} \varphi_{N-m} + 2L(1+y) \partial_y^L \varphi_{N-m} \\ & + L(L-1) \partial_y^{L-1} \varphi_{N-m},\end{aligned}$$

we can continue

$$\begin{aligned}\left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \partial_y Z_3 \right\|_{H^{s+1}} \leq & \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} M^2 \sum_{\sigma \leq \tau} \sum_{\tau \leq k} C_1 \frac{B^{|l|-1}}{|l|^p} \prod_{q=1}^{d-1} \frac{A^{\sigma_q}}{(\sigma_q+1)^2} \\ & \times C_1 \frac{B^{|m-l|-1}}{|m-l|^p} \prod_{q=1}^{d-1} \frac{A^{\tau_q - \sigma q}}{(\tau_q - \sigma_q + 1)^2} \\ & \times \left\{ Y^2 M^2 \left\| \frac{\partial_x^{k-\tau} \partial_y^L}{(|k-\tau|+L)!} \varphi_{N-m} \right\|_{H^{s+2}} \right. \\ & + 2LYM \left\| \frac{\partial_x^{k-\tau} \partial_y^{L-1}}{(|k-\tau|+L)!} \varphi_{N-m} \right\|_{H^{s+2}} \\ & \left. + L(L-1) \left\| \frac{\partial_x^{k-\tau} \partial_y^{L-2}}{(|k-\tau|+L)!} \varphi_{N-m} \right\|_{H^{s+2}} \right\}.\end{aligned}$$

Using equation (4.16) we obtain

$$\left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \partial_y Z_3 \right\|_{H^s} \le C_1^2 M^2 \frac{B^{|N|-3}}{|N|^p} \sum_{|m|=2}^{|N|-1} \sum_{|l|=1}^{|m|-1} \frac{|N|^p}{|l|^p |m-l|^p |N-m|^p}$$

$$\times \prod_{j=1}^{d-1} \left\{ \frac{A^{k_j}}{(k_j+1)^2} \sum_{\sigma \le \tau} \sum_{\tau \le k} \frac{(k_j+1)^2}{(\sigma_j+1)^2 (\tau_j - \sigma_j + 1)^2 (k_j - \tau_j + 1)^2} \right\}$$

$$\times \left\{ Y^2 M^2 \frac{D^L}{(L+1)^2} + \frac{2LYM}{(|k-\tau|+L)} \frac{D^{L-1}}{L^2} + \frac{L(L-1)}{(|k-\tau|+L)(|k-\tau|+L-1)} \frac{D^{L-2}}{(L-1)^2} \right\}$$

and

$$\left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \partial_y Z_3 \right\|_{H^s} \le C_1 \frac{C_1 M^2 S^{2(d-1)}}{B} \left\{ Y^2 M^2 + \frac{2LYM(L+1)^2}{(|k-\tau|+L) L^2 D} + \frac{L(L-1)(L+1)^2}{(|k-\tau|+L)(|k-\tau|+L-1)(L-1)^2 D^2} \right\}$$

$$\times \frac{B^{|N|-2}}{|N|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}$$

$$\le C_1 C_3 \frac{B^{|N|-2}}{|N|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2},$$

provided that C_3 is chosen appropriately. ■

We are now in a position to prove theorem 4.2.

Proof of theorem 4.2. We begin by noting that lemma 4.3 has already established equation (4.7b) so we need only focus on equation (4.7a). For this estimate we work by induction on l. For $l=0$ and any $k \ge 0$, $|n| \ge 1$ we use lemma 4.3. Now we assume

$$\left\| \frac{\partial_x^k \partial_y^l}{(|k|+l)!} \varphi_n \right\|_{H^{s+2}} \le C_1 \frac{B^{|n|-1}}{|n|^p} \frac{D_l}{(l+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}$$

for all $l < L$ and any $k \ge 0$ and $|n| \ge 1$, and seek to prove

$$\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_n \right\|_{H^{s+2}} \le C_1 \frac{B^{|n|-1}}{|n|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}$$

for any $k \ge 0$ and $|n| \ge 1$. We accomplish this via a second induction on $|n|$. Lemma 4.6 establishes equation (4.7a) in the case $|n|=1$. We now assume that

$$\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_n \right\|_{H^{s+2}} \leq C_1 \frac{B^{|n|-1}}{|n|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}$$

for all $|n| < |N|$ and seek to prove

$$\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_N \right\|_{H^{s+2}} \leq C_1 \frac{B^{|N|-1}}{|N|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2}.$$

We make the estimate

$$\begin{aligned}
&\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_N \right\|_{H^{s+2}} \\
&\quad \leq \left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_N \right\|_{H^{s+1}} + \left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \nabla_x \varphi_N \right\|_{H^{s+1}} + \left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \partial_y \varphi_N \right\|_{H^{s+1}} \\
&\quad \leq \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \varphi_N \right\|_{H^{s+2}} + \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \nabla_x \varphi_N \right\|_{H^{s+2}} + \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \Delta_x \varphi_N \right\|_{H^{s+1}} \\
&\qquad + \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} F_N \right\|_{H^{s+1}} \\
&\quad \leq \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \varphi_N \right\|_{H^{s+2}} + \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \nabla_x \varphi_N \right\|_{H^{s+2}} + C(d) \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} \nabla_x \varphi_N \right\|_{H^{s+2}} \\
&\qquad + \left\| \frac{\partial_x^k \partial_y^{L-1}}{(|k|+L)!} F_N \right\|_{H^{s+1}},
\end{aligned}$$

where C is a generic function of dimension alone and we have used the fact that φ_N solves equation (3.5). Finally, using lemma 4.7 we have

$$\begin{aligned}
\left\| \frac{\partial_x^k \partial_y^L}{(|k|+L)!} \varphi_N \right\|_{H^{s+2}} \leq & C_1 [1 + C(d) A] \frac{B^{|N|-1}}{|N|^p} \frac{D^{L-1}}{L^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2} \\
& + C_1 C_3 \frac{B^{|N|-2}}{|N|^p} \frac{D^L}{(L+1)^2} \prod_{q=1}^{d-1} \frac{A^{k_q}}{(k_q+1)^2},
\end{aligned}$$

which completes the proof provided that $D > (1 + C(d)A)$ and $B > C_3$. ■

We remark here that the same existence and analyticity proofs given above for the case of capillary gravity waves ($\sigma > 0$) can also be given for pure gravity waves ($\sigma = 0$) provided that $d = 2$. The key differences lie in the different nature of Λ_σ when $\sigma = 0$, and the elliptic estimate, lemma 4.5.

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Appendix A. Elliptic estimate

First, we state two lemmas which are needed in the proof of lemma 4.5; the proofs are straightforward and therefore omitted.

Lemma A.1. *If the unknowns* $\mu_m \in \mathbf{R}^{d-1}$, $m \in \mathbf{N}^{d-1}$, $|m| = \bar{n} - 1 \geq 0$ *are constrained by the linear equations*

$$E_{n,j} : \kappa_j \cdot \mu_{n-e_j} = R_{n,j}, \quad \forall |n| = \bar{n}, \tag{A 1}$$

and all $j = 1, \ldots, d-1$ *such that* $n \geq e_j$, $\kappa_1, \ldots, \kappa_{d-1}$ *are linearly independent, then a unique solution of equation (A 1) exists.*

Lemma A.2. *The convolution integrals*

$$T_1(F)(y) = \int_{-1}^{y} \mathrm{e}^{|k|(s-y)} F(s)\, \mathrm{d}s, \quad T_2(F)(y) = \int_{y}^{0} \mathrm{e}^{|k|(y-s)} F(s)\, \mathrm{d}s,$$

$(k \neq 0)$ satisfy the estimates

$$\| T_1(F) \|_{L^2} \leq \frac{1}{|k|} \| F \|_{L^2}, \quad \| T_2(F) \|_{L^2} \leq \frac{1}{|k|} \| F \|_{L^2},$$

$$| T_1(F)(0) | \leq \frac{1}{\sqrt{2|k|}} \| F \|_{L^2}, \quad | T_2(F)(-1) | \leq \frac{1}{\sqrt{2|k|}} \| F \|_{L^2}.$$

We will now establish the elliptic estimate lemma 4.5.

Proof of lemma 4.5. The periodicity of solutions of equation (4.9) permit the Fourier series expansions

$$w_n(x, y) = \sum_{k \in \Gamma'} \hat{w}_n(k, y) \mathrm{e}^{\mathrm{i}k \cdot x}, \quad v_n(x, y) = \sum_{k \in \Gamma'} \hat{v}_n(k) \mathrm{e}^{\mathrm{i}k \cdot x}.$$

If $k \neq 0$ variation of parameters gives a solution of equation (4.9a)

$$\hat{w}_n(k, y) = A_n(k) \mathrm{e}^{|k|y} - \frac{1}{2|k|} T_2(y) + B_n(k) \mathrm{e}^{-|k|y} - \frac{1}{2|k|} T_1(y), \tag{A 2}$$

where

$$T_1(y) \equiv \int_{-1}^{y} \mathrm{e}^{|k|(s-y)} \hat{p}_n(k,s)\,\mathrm{d}s, \qquad T_2(y) \equiv \int_{y}^{0} \mathrm{e}^{|k|(y-s)} \hat{p}_n(k,s)\,\mathrm{d}s.$$

Then equations (4.9*b*) to (4.9*d*) can be used to solve for $A_n(k)$, $B_n(k)$, and $\hat{v}_n(k)$. Provided that $k\neq 0$ and $k\neq \pm\kappa_j$ the solution of this problem is

$$\begin{aligned}
\hat{w}_n(k,y) = {} & \frac{1}{\varLambda_\sigma(c,k)\cosh(|k|)}\Bigg\{ -(\mathrm{i}c\cdot k)\hat{q}_n(k)\cosh(|k|(y+1)) \\
& -(g+\sigma|k|^2)\hat{r}_n(k)\cosh(|k|(y+1)) + \frac{(c\cdot k)^2\,T_1(0)}{2|k|}\cosh(|k|(y+1)) \\
& + \frac{(g+\sigma|k|^2)|k|\,T_1(0)}{2|k|}\cosh(|k|(y+1)) + \frac{(c\cdot k)^2\,T_2(-1)}{2|k|}\sinh(|k|y) \\
& + \frac{(g+\sigma|k|^2)|k|\,T_2(-1)}{2|k|}\cosh(|k|y)\Bigg\} - \frac{1}{2|k|}\,T_2(y) - \frac{1}{2|k|}\,T_1(y), \\
\hat{v}_n(k) = {} & \frac{-(\mathrm{i}c\cdot k)}{2\varLambda_\sigma(c,k)\cosh(|k|)}\{\mathrm{e}^{|k|}\,T_1(0) + T_2(-1)\} - \frac{|k|\tanh(|k|)}{\varLambda_\sigma(c,k)}\,\hat{q}_n(k) \\
& + \frac{(\mathrm{i}c\cdot k)}{\varLambda_\sigma(c,k)}\,\hat{r}_n(k).
\end{aligned}$$

In the case $k=0$, variation of parameters and equation (4.9) require that

$$\hat{w}_n(0,y) = (y+1)\hat{r}_n(0) - y\int_y^0 \hat{p}_n(0,s)\,\mathrm{d}s - \int_{-1}^{y} s\hat{p}_n(0,s)\,ds - \int_{-1}^{0}\hat{p}_n(0,s)\,\mathrm{d}s,$$
$$\hat{v}_n(0) = \frac{\hat{q}_n(0)}{g},$$

where we have used equation (4.9*b*) to uniquely specify $\hat{w}_n(0,y)$.

For $k=\kappa_j$ (and similarly for $k=-\kappa_j$) we again use the variation of parameters formula (A 2) and seek $A_n(\kappa_j)$, $B_n(\kappa_j)$, and $\hat{v}_n(\kappa_j)$. Combining equations (4.9*c*) and (4.9*d*) it can be shown that

$$\begin{aligned}
& -(c\cdot\kappa_j)^2\hat{w}_j(\kappa_j,0) + (g+\sigma|\kappa_j|^2)\partial_y\hat{w}_j(\kappa_j,0) \\
& \quad = 2\mathrm{i}(c\cdot\kappa_j)(\mu_n\cdot\kappa_j)(g+\sigma|\kappa_j|^2)\alpha_j + (\mathrm{i}c\cdot\kappa_j)\hat{q}_j(\kappa_j) + (g+\sigma|\kappa_j|^2)\hat{r}_j(\kappa_j)
\end{aligned} \tag{A 3}$$

must hold. Equations (4.9*b*) and (A 3) result in a linear system of equations for $A_n(\kappa_j)$ and $B_n(\kappa_j)$

$$\begin{pmatrix} |\kappa_j|\mathrm{e}^{-|\kappa_j|} & -|\kappa_j|\mathrm{e}^{|\kappa_j|} \\ -(c\cdot\kappa_j)^2 + |\kappa_j|(g+\sigma|\kappa_j|^2) & -(c\cdot\kappa_j)^2 - |\kappa_j|(g+\sigma|\kappa_j|^2) \end{pmatrix} \begin{pmatrix} A_n(\kappa_j) \\ B_n(\kappa_j) \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2}\,T_2(-1) \\ \varPhi(\kappa_j) \end{pmatrix}, \tag{A 4}$$

where

$$\Phi(\kappa_j) = 2\mathrm{i}(c\cdot\kappa_j)(\mu_{n-e_j}\cdot\kappa_j)(g+\sigma|\kappa_j|^2)\alpha_j + (\mathrm{i}c\cdot\kappa_j)\hat{q}_j(\kappa_j) + (g+\sigma|\kappa_j|^2)\hat{r}_j(\kappa_j)$$
$$-\frac{(c\cdot\kappa_j)^2}{2|\kappa_j|}\,T_1(0) - \frac{(g+\sigma|\kappa_j|^2)}{2}\,T_1(0).$$

Since $\Lambda_\sigma(c,\, k_j)=0$, the matrix on the left-hand side of equation (A 4) is singular, and thus there is a compatibility condition required for solvability. This condition reads

$$\Phi(\kappa_j) + \frac{(c\cdot\kappa_j)^2 - |\kappa_j|(g+\sigma|\kappa_j|^2)}{2|\kappa_j|\mathrm{e}^{-|\kappa_j|}}\,T_2(-1) = 0,$$

and can be written generically as $\kappa_j\cdot\mu_{n-e_j} = R_{n,j}$. From lemma A.1 we conclude that there exist unique μ_{n-e_j} to satisfy all of these compatibility conditions at order $|n|=\bar{n}$. We note that while w_n and v_n can be completely determined from equation (4.9) at multi index n, the μ_{n-e_j} require equation (4.9) at all multi-indices $|n|=\bar{n}$ for their unique resolution. Once we have solvability we need to specify an orthogonality condition to ensure uniqueness. We choose equation (4.11) which requires that $\hat{v}_n(\pm\kappa_j)=0$. From equation (4.9*c*),

$$\hat{v}_n(\kappa_j) = \frac{1}{g+\sigma|\kappa_j|^2}\,[\hat{q}_n(\kappa_j) - (\mathrm{i}c\cdot\kappa_j)\hat{w}_n(\kappa_j,0) - (\mu_n\cdot\kappa_j)(g+\sigma|\kappa_j|^2)\alpha_j],$$

and from equation (A 2),

$$\hat{w}_n(\kappa_j,0) = A_n(\kappa_j) + B_n(\kappa_j) - \frac{1}{2|\kappa_j|}\,T_1(0),$$

so we now have a second equation to specify $A_n(\kappa_j)$ and $B_n(\kappa_j)$ (in addition to the one from equation (A 4)) uniquely; the result is

$$A_n(\kappa_j) = \frac{1}{2|\kappa_j|\cosh(|\kappa_j|)} \times\left[\frac{|\kappa_j|\mathrm{e}^{|\kappa_j|}}{\mathrm{i}c\cdot\kappa_j}\,(\hat{q}_n(\kappa_j) - (\mu\cdot\kappa_j)(g+\sigma|\kappa_j|^2)) + \frac{\mathrm{e}^{|\kappa_j|}}{2}\,T_1(0) + \frac{1}{2}\,T_2(-1)\right],$$
$$B_n(\kappa_j) = \frac{1}{2|\kappa_j|\cosh(|\kappa_j|)} \times\left[\frac{|\kappa_j|\,e^{-|\kappa_j|}}{ic\cdot\kappa_j}\,(\hat{q}_n(\kappa_j) - (\mu_n\cdot\kappa_j)(g+\sigma|\kappa_j|^2)) + \frac{e^{-|\kappa_j|}}{2}\,T_1(0) - \frac{1}{2}\,T_2(-1)\right].$$

Now, using the estimates given in lemma A.2 it can be shown that for all $k\in\Gamma'$

$$\|\hat{w}_n(k,y)\|^2_{L^2(\mathrm{d}y)} \le C[\langle k\rangle^{-5}|\hat{q}_n(k)|^2 + \langle k\rangle^{-3}|\hat{r}_n(k)|^2 + \langle k\rangle^{-4}\|\hat{p}_n(k,y)\|^2_{L^2(\mathrm{d}y)}],$$
$$\|\partial_y\hat{w}_n(k,y)\|^2_{L^2(\mathrm{d}y)} \le C[\langle k\rangle^{-3}|\hat{q}_n(k)|^2 + \langle k\rangle^{-1}|\hat{r}_n(k)|^2 + \langle k\rangle^{-2}\|\hat{p}_n(k,y)\|^2_{L^2(\mathrm{d}y)}],$$

$$|\hat{v}_n(k)|^2 \le C[\langle k\rangle^{-4}|\hat{q}_n(k)|^2 + \langle k\rangle^{-4}|\hat{r}_n(k)|^2 + \langle k\rangle^{-5}\|\hat{p}_n(k,y)\|^2_{L^2(\mathrm{d}y)}],$$

which give the desired estimates on w_n and v_n. ■

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