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Well-posedness and analyticity of solutions to a water wave problem with viscosity

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Abstract

The water wave problem models the free-surface evolution of an ideal fluid under the influence of gravity and surface tension. The governing equations are a central model in the study of open ocean wave propagation, but they possess a surprisingly difficult and subtle well-posedness theory. In this paper we establish the existence and uniqueness of spatially periodic solutions to the water wave equations augmented with physically inspired viscosity suggested in the recent work of Dias et al. (2008) [16]. As we show, this viscosity (which can be arbitrarily weak) not only delivers an enormously simplified well-posedness theory for the governing equations, but also justifies a greatly stabilized numerical scheme for use in studying solutions of the water wave problem.

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1. Introduction

One of the central problems in fluid mechanics is the accurate modeling of the free-surface motion of a large body of water (e.g., a lake or an ocean) [25,37,2]. It is not only a problem of

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classical interest [35,4,36,9,22], but also one of present importance due to its role in a number of applications from the formation and movement of sandbars, to the forces generated by waves on open–ocean oil rigs, to the propagation of tsunamis. The "water wave equations" are the most faithful and successful model for this problem, but they have a surprisingly difficult and subtle well-posedness theory [38,39,6,5,26]. We refer the interested reader to these papers, their extensive bibliographies, and the recent collection [8] for the state of the art in the field (in particular, see the chapters by Ambrose [7] and Wu [40]).

Due to the extremely important role of this model, we were inspired to find a new proof of well-posedness which did not rely on the sophisticated technology required in the papers mentioned above. While this has proven elusive, we now demonstrate that if a physically motivated viscosity is added, then a straightforward existence and uniqueness result can be established. For this we imitate the second author's previous program [1] where such a philosophy was pursued for a weakly nonlinear approximation of the full water wave problem.

For the incorporation of viscosity, as in [1] we follow the lead of Dias, Dyachenko, and Zakharov [16] who advocated for, essentially, (2.1) below. These equations describe potential flow with dissipation featuring a dispersion relation which, in the small viscosity limit, corresponds to that presented in the classic book of Lamb [25]. They also admit the fundamental problem with "viscous potential flow," that the modeling assumptions inherent in potential flow are incompatible with the presence of viscosity. However, the model has proven useful and we believe that this is a natural way to add viscosity to the water wave problem with the goals we have in mind. We also note the work of one of the authors and Kakleas [24] who used this approach to build a stabilized numerical scheme to model surface wave propagation in the weakly nonlinear regime. It is our intent to implement such a scheme for the full water wave equations, but delay our description for a future publication.

Before proceeding, we point out that our method of proof is rather different from the standard techniques, e.g., described in [28,1]. Rather than seeking a fixed point of a contraction mapping, we follow the approach of Friedman and Reitich to free boundary problems, more specifically in the contexts of the classical Stefan problem [19] and the capillary drop problem [20]. Friedman and Reitich's method is perturbative in nature, expanding the solution in a Taylor series in a parameter which characterizes the deformation of the free interface from a simple, separable geometry. Their proof uses, very strongly, the unique solvability of the governing equations on a *fixed*, *trivial* domain (using separation of variables) to show that higher order corrections satisfy appropriate bounds which demonstrate the strong convergence of the Taylor series for the solution. The difficulties here are certain algebra properties of the relevant function spaces and trace lemmas; these are different in the current context, but we show that their demonstrations can be extended.

Additionally, we point out that due to the nature of the function spaces we introduce, the conclusion of our theorem is not only the well-posedness of our model of viscous water waves, but also the very strong stability of our solutions. Our function spaces demand *exponential* decay in time with the rate determined by the value of the viscosity. Thus, not only do unique solutions exist, they persist globally in time and decay exponentially fast to zero. We can state all of this rather informally in the following result.

Theorem 1.1. *There exists a unique solution to the water wave problem with viscosity provided that the initial data resides in a Sobolev space akin to* $L_t^2(e^{2\alpha t})H_x^s H_y^s$, $s \geq 4$. *The solutions are analytic with respect to a parameter which measures the deformation of the fluid interface* *from its rest value, and the nature of the function classes demands that solutions decay at an exponential rate in time* $0 < \alpha < 2\mu$*, where* μ *is the constant of (surface) viscosity.*

The rest of the paper is organized as follows. In § 2 we recall the governing equations of Dias, Dyachenko, and Zakharov [16] which we modify slightly to suit our purposes. In § 3 we follow the lead of our previous work on free boundary problems [32,33] by applying a domain flattening change of variables which maps the moving problem domain to a fixed one. In § 4 we introduce the function spaces we require to establish our well-posedness result, together with crucial lemmas on algebra properties of functions in these spaces, and trace estimates on the same. In § 5 and 6 we state and prove fundamental estimates on the elliptic and parabolic problems which arise in the linearization of our governing equations about the trivial configuration which we analyze in § 7. In § 8 we state and prove an inductive lemma which enables the proof of our central well-posedness result which is established in § 9. We make concluding remarks in § 10. We collect the proofs of the trace lemma in § A and a lemma on products of analytic functions in § B.

2. Governing equations

The well-known [25,37,2] equations governing the motion of two dimensional laterally periodic gravity-capillary water waves on a fluid of depth h are

$$\begin{aligned}
&\Delta\varphi = 0, && -h < y < \eta,\\
&\partial_y\varphi = 0, && y = -h,\\
&\partial_t\eta = \partial_y\varphi - (\partial_x\eta)\partial_x\varphi, && y = \eta,\\
&\partial_t\varphi = -g\eta + \sigma\partial_x^2\eta + \sigma\partial_x\left[(\partial_x\eta)H(\partial_x\eta)\right] - \frac{1}{2}(\partial_x\varphi)^2 - \frac{1}{2}(\partial_y\varphi)^2, && y = \eta,
\end{aligned}$$

where φ is the velocity potential ($\mathbf{u} = \nabla\varphi$), $y = \eta$ is the free air-fluid interface,

$$H(\partial_x\eta) := \frac{1}{\sqrt{1 + (\partial_x\eta)^2}} - 1,$$

and $g > 0$ and $\sigma > 0$ are the constants of gravity and surface tension, respectively. These are supplemented with the boundary conditions

$$\varphi(x + 2\pi, y, t) = \varphi(x, y, t), \quad \eta(x + 2\pi) = \eta(x),$$

and initial conditions

$$\eta(x, 0) = \eta^{(0)}(x), \quad \varphi(x, \eta^{(0)}, 0) = \xi^{(0)}(x),$$

where standard elliptic theory [17] reveals that specifying the initial velocity potential at the *surface* is sufficient. We supplement this with viscous terms first introduced by Dias, Dyachenko, and Zakharov [16] resulting in the “water wave equations with viscosity”

$$\Delta\varphi = 0, \qquad -h < y < \eta, \tag{2.1a}$$

$$\partial_y \varphi = 0, \qquad y = -h, \tag{2.1b}$$

$$\partial_t \eta = \partial_y \varphi + 2\mu \partial_x^2 \eta - (\partial_x \eta)\partial_x \varphi, \qquad y = \eta, \tag{2.1c}$$

$$\partial_t \varphi = -g\eta + \sigma \partial_x^2 \eta - 2\mu \partial_y^2 \varphi + \sigma \partial_x \left[(\partial_x \eta) H(\partial_x \eta) \right] - \frac{1}{2}(\partial_x \varphi)^2 - \frac{1}{2}(\partial_y \varphi)^2, \qquad y = \eta, \tag{2.1d}$$

$$\varphi(x, \eta, 0) = \xi^{(0)}(x), \tag{2.1e}$$

$$\eta(x, 0) = \eta^{(0)}(x), \tag{2.1f}$$

for a surface viscosity parameter $\mu > 0$. In these we have slightly modified Dias, Dyachenko, and Zakharov's equations by dropping the bottom viscosity terms included in [15]. We will show that this problem is well-posed with analytic solutions.

Remark 2.1. We make two important observations: First, the mass

$$M = \int_0^{2\pi} \eta(x, t)\, dx,$$

is conserved by the flow. Indeed, it is well-known in the inviscid case (see, e.g., [12]) that

$$\partial_t [M]_{\mu=0} = \partial_t \int_0^{2\pi} \eta(x, t)\, dx = \int_0^{2\pi} \partial_t \eta(x, t)\, dx = \int_0^{2\pi} \left[\partial_y \varphi - (\partial_x \eta)\partial_x \varphi \right]_{y=\eta} = 0.$$

The addition of viscosity introduces only the term $\int_0^{2\pi} \mu \partial_x^2 \eta \, dx$ to this computation, which is zero as it features an exact derivative and η is periodic.

Second, one can arrange for the mean of the *surface* velocity

$$\xi(x, t) := \varphi(x, \eta(x, t), t),$$

to remain zero if it is initially set that way. For this one must remember that the velocity potential is only meaningfully defined up to a time-dependent constant [25,37,2],

$$\varphi(x, y, t) = \tilde{\varphi}(x, y, t) + C(t).$$

With this we can consider the average surface velocity potentials

$$\Xi(t) = \int_0^{2\pi} \varphi(x, \eta, t)\, dx, \qquad \tilde{\Xi}(t) = \int_0^{2\pi} \tilde{\varphi}(x, \eta, t)\, dx = \Xi(t) - 2\pi C(t).$$

So, if we choose $C(t) = \Xi(t)/(2\pi)$ and drop the tildes we are done. Thus we restrict our function spaces by requiring $M = \Xi = 0$.

3. Reformulation on a fixed domain

It was shown in [32,33,21] how a simple change of variables could be used to demonstrate the analyticity of Dirichlet–Neumann Operators (DNOs) with respect to sufficiently small and regular surface deformations $\eta = \varepsilon f$. The problem of computing DNOs for Laplace's equation is closely related to the water wave problem and, in fact, our equations could be equivalently restated at the fluid surface in terms of these operators (see, e.g., [13,14,29]), though we do not pursue it here.

To imitate this success for DNOs in the context of the water wave problem with viscosity, we follow the lead of [32] and perform the domain–flattening change of variables (known as σ-coordinates [34] and the C-Method [11])

$$x' = x, \quad y' = h\left(\frac{y-\eta}{h+\eta}\right), \quad t' = t,$$

from which we define

$$u(x', y', t') := \varphi\left(x', \left(\frac{h+\eta}{h}\right) y' + \eta, t'\right).$$

It is not difficult to show [30] that derivatives change as

$$\begin{aligned} M(x,t)\partial_x &= M(x',t')\partial_{x'} + N(x',y',t')\partial_{y'}, \\ M(x,t)\partial_t &= M(x',t')\partial_{t'} + P(x',y',t')\partial_{y'}, \\ M(x,t)\partial_y &= h\partial_{y'}, \end{aligned}$$

where

$$\begin{aligned} M(x,t) &= h + \eta(x,t), \\ N(x,y,t) &= -(\partial_x \eta(x,t))(y+h), \\ P(x,y,t) &= -(\partial_t \eta(x,t))(y+h). \end{aligned}$$

The governing equations (2.1) transform, upon dropping the primes, to

$$\Delta u = F, \qquad -h < y < 0, \tag{3.1a}$$

$$\partial_y u = 0, \qquad y = -h, \tag{3.1b}$$

$$\partial_t \eta = \partial_y u + 2\mu \partial_x^2 \eta + Q, \qquad y = 0, \tag{3.1c}$$

$$\partial_t u = -g\eta + \sigma \partial_x^2 \eta - 2\mu \partial_y^2 u + R, \qquad y = 0, \tag{3.1d}$$

$$u(x,0,0) = \xi^{(0)}(x), \tag{3.1e}$$

$$\eta(x,0) = \eta^{(0)}(x). \tag{3.1f}$$

The form for F can be shown [30] to be

$$h^2 F = -\text{div}\left[A^{(1)}(\eta)\nabla u\right] - \text{div}\left[A^{(2)}(\eta)\nabla u\right] + (\partial_x \eta) B^{(0)} \cdot \nabla u + (\partial_x \eta) B^{(1)}(\eta) \cdot \nabla u,$$

where

$$A^{(1)}(\eta) = \begin{pmatrix} A^{(1),xx} & A^{(1),xy} \\ A^{(1),yx} & A^{(1),yy} \end{pmatrix} := \begin{pmatrix} 2h\eta & -h(y+h)\partial_x \eta \\ -h(y+h)\partial_x \eta & 0 \end{pmatrix},$$
$$A^{(2)}(\eta) = \begin{pmatrix} A^{(2),xx} & A^{(2),xy} \\ A^{(2),yx} & A^{(2),yy} \end{pmatrix} := \begin{pmatrix} \eta^2 & -(y+h)\eta(\partial_x \eta) \\ -(y+h)\eta(\partial_x \eta) & (y+h)^2(\partial_x \eta)^2 \end{pmatrix},$$
$$B^{(0)} = \begin{pmatrix} B^{(0),x} \\ B^{(0),y} \end{pmatrix} := \begin{pmatrix} h \\ 0 \end{pmatrix}, \quad B^{(1)}(\eta) = \begin{pmatrix} B^{(1),x} \\ B^{(1),y} \end{pmatrix} := \begin{pmatrix} \eta \\ -(y+h)\partial_x \eta \end{pmatrix}.$$

Therefore,

$$\begin{aligned} h^2 F = & -\partial_x \left[2h\eta \partial_x u\right] + \partial_x \left[h(y+h)(\partial_x \eta)\partial_y u\right] + \partial_y \left[h(y+h)(\partial_x \eta)\partial_x u\right] \\ & - \partial_x \left[\eta^2 \partial_x u\right] + \partial_x \left[(y+h)\eta(\partial_x \eta)\partial_y u\right] + \partial_y \left[(y+h)\eta(\partial_x \eta)\partial_x u\right] \\ & - \partial_y \left[(y+h)^2(\partial_x \eta)^2 \partial_y u\right] + h(\partial_x \eta)\partial_x u + \eta(\partial_x \eta)\partial_x u - (y+h)(\partial_x \eta)^2 \partial_y u. \end{aligned}$$

Furthermore, multiplying (2.1c) by M and evaluating at $y = 0$, we find

$$hQ = -\eta(\partial_t \eta) + 2\mu\eta(\partial_x^2 \eta) - h(\partial_x \eta)\partial_x u - \eta(\partial_x \eta)\partial_x u + h(\partial_x \eta)^2 \partial_y u,$$

where we have used the fact that, since η is independent of y, we have

$$\partial_t \eta = \partial_{t'} \eta, \quad \partial_x \eta = \partial_{x'} \eta.$$

Finally, multiplying (2.1d) by M^2 and evaluating at $y = 0$, we discover

$$\begin{aligned} h^2 R = & -2h\eta \partial_t u - \eta^2 \partial_t u + h^2(\partial_t \eta)\partial_y u + h\eta(\partial_t \eta)\partial_y u - 2gh\eta^2 - g\eta^3 \\ & + 2\sigma h\eta(\partial_x^2 \eta) + \sigma\eta^2(\partial_x^2 \eta) \\ & + \sigma h^2 \partial_x \left[(\partial_x \eta) H(\partial_x \eta)\right] + 2\sigma h\eta \partial_x \left[(\partial_x \eta) H(\partial_x \eta)\right] + \sigma\eta^2 \partial_x \left[(\partial_x \eta) H(\partial_x \eta)\right] \\ & - \frac{1}{2}\left\{h^2(\partial_x u)^2 + 2h\eta(\partial_x u)^2 + \eta^2(\partial_x u)^2 - 2h^2(\partial_x \eta)(\partial_x u)\partial_y u \right. \\ & \left. -2h\eta(\partial_x \eta)(\partial_x u)\partial_y u + h^2(\partial_x \eta)^2(\partial_y u)^2\right\} - \frac{1}{2}h^2(\partial_y u)^2. \end{aligned}$$

Our procedure for establishing well-posedness is to seek solutions of the form

$$\eta = \eta(x,t;\varepsilon) = \sum_{n=1}^{\infty} \eta_n(x,t)\varepsilon^n, \quad u = u(x,y,t;\varepsilon) = \sum_{n=1}^{\infty} u_n(x,y,t)\varepsilon^n, \tag{3.2}$$

given initial data $\{\eta^{(0)}(x), \xi^{(0)}(x)\}$. We will show that if this data lies in appropriate Sobolev spaces, then the $\{\eta_n, \varphi_n\}$ also exist in (different) Sobolev classes satisfying estimates which justify the strong convergence of the series (3.2). Upon insertion of these into (3.1) we find that, at each perturbation order,

$$\Delta u_n = F_n, \qquad -h < y < 0, \tag{3.3a}$$

$$\partial_y u_n = 0, \qquad y = -h, \tag{3.3b}$$

$$\partial_t \eta_n = \partial_y u_n + 2\mu \partial_x^2 \eta_n + Q_n, \qquad y = 0, \tag{3.3c}$$

$$\partial_t u_n = -g\eta_n + \sigma \partial_x^2 \eta_n - 2\mu \partial_y^2 u_n + R_n, \qquad y = 0. \tag{3.3d}$$

$$u_n(x,0,0) = \delta_{n,0}\xi^{(0)}(x), \tag{3.3e}$$

$$\eta_n(x,0) = \delta_{n,0}\eta^{(0)}(x), \tag{3.3f}$$

where $\delta_{n,k}$ is the Kronecker delta function. In these F_n, Q_n, and R_n can be shown, using the notation $[\![\cdot]\!]_n$ from Appendix B which connotes the n-th coefficient in the formal Cauchy product expansion, to be

$$\begin{aligned} h^2 F_n = & -\partial_x \left[2h [\![\eta \partial_x u]\!]_n\right] + \partial_x \left[h(y+h) [\![(\partial_x \eta)\partial_y u]\!]_n\right] + \partial_y \left[h(y+h) [\![(\partial_x \eta)\partial_x u]\!]_n\right] \\ & - \partial_x \left[\left[\!\!\left[\eta^2 \partial_x u\right]\!\!\right]_n\right] + \partial_x \left[(y+h) [\![\eta(\partial_x \eta)\partial_y u]\!]_n\right] + \partial_y \left[(y+h) [\![\eta(\partial_x \eta)\partial_x u]\!]_n\right] \\ & - \partial_y \left[(y+h)^2 \left[\!\!\left[(\partial_x \eta)^2 \partial_y u\right]\!\!\right]_n\right] + h [\![(\partial_x \eta)\partial_x u]\!]_n \\ & + [\![\eta(\partial_x \eta)\partial_x u]\!]_n - (y+h) \left[\!\!\left[(\partial_x \eta)^2 \partial_y u\right]\!\!\right]_n, \end{aligned} \tag{3.4a}$$

$$hQ_n = -[\![\eta(\partial_t \eta)]\!]_n + 2\mu \left[\!\!\left[\eta(\partial_x^2 \eta)\right]\!\!\right]_n - h [\![(\partial_x \eta)\partial_x u]\!]_n - [\![\eta(\partial_x \eta)\partial_x u]\!]_n + h \left[\!\!\left[(\partial_x \eta)^2 \partial_y u\right]\!\!\right]_n, \tag{3.4b}$$

and

$$\begin{aligned} h^2 R_n = & -2h [\![\eta \partial_t u]\!]_n - \left[\!\!\left[\eta^2 \partial_t u\right]\!\!\right]_n + h^2 [\![(\partial_t \eta)\partial_y u]\!]_n + h [\![\eta(\partial_t \eta)\partial_y u]\!]_n \\ & - 2gh \left[\!\!\left[\eta^2\right]\!\!\right]_n - g \left[\!\!\left[\eta^3\right]\!\!\right]_n + 2\sigma h \left[\!\!\left[\eta(\partial_x^2 \eta)\right]\!\!\right]_n + \sigma \left[\!\!\left[\eta^2(\partial_x^2 \eta)\right]\!\!\right]_n \\ & + \sigma h^2 [\![\partial_x [(\partial_x \eta) H(\partial_x \eta)]]\!]_n + 2\sigma h [\![\eta \partial_x [(\partial_x \eta) H(\partial_x \eta)]]\!]_n + \sigma \left[\!\!\left[\eta^2 \partial_x [(\partial_x \eta) H(\partial_x \eta)]\right]\!\!\right]_n \\ & - \frac{1}{2} \left\{ h^2 \left[\!\!\left[(\partial_x u)^2\right]\!\!\right]_n + 2h \left[\!\!\left[\eta(\partial_x u)^2\right]\!\!\right]_n + \left[\!\!\left[\eta^2(\partial_x u)^2\right]\!\!\right]_n - 2h^2 [\![(\partial_x \eta)(\partial_x u)\partial_y u]\!]_n \right. \\ & \left. -2h [\![\eta(\partial_x \eta)(\partial_x u)\partial_y u]\!]_n + h^2 \left[\!\!\left[(\partial_x \eta)^2(\partial_y u)^2\right]\!\!\right]_n \right\} - \frac{1}{2} h^2 \left[\!\!\left[(\partial_y u)^2\right]\!\!\right]_n. \end{aligned} \tag{3.4c}$$

Remark 3.1. For the expansion of H we recall that

$$H(\psi) = \frac{1}{\sqrt{1+\psi^2}} - 1,$$

which, upon squaring, can be written as

$$(H^2 + 2H + 1)(1 + \psi^2) = 1,$$

or

$$H = -\frac{1}{2}\left\{\psi^2 + H^2 + 2\psi^2 H + \psi^2 H^2\right\}. \tag{3.5}$$

If we make the expansion, cf. (3.2),

$$\psi = \psi(x,t;\varepsilon) = \sum_{n=1}^{\infty} \psi_n(x,t)\varepsilon^n,$$

then it is easy to see that

$$H = H(x,t;\varepsilon) = \sum_{n=2}^{\infty} H_n \varepsilon^n.$$

In fact, from (3.5) we have

$$H_n = -\frac{1}{2}\left\{ \left[\!\left[\psi^2\right]\!\right]_n + \left[\!\left[H^2\right]\!\right]_n + 2\left[\!\left[\psi^2 H\right]\!\right]_n + \left[\!\left[\psi^2 H^2\right]\!\right]_n \right\}, \tag{3.6}$$

and it is clear that H_n depends only on $\{\partial_x \eta_1, \ldots, \partial_x \eta_{n-1}\}$.

4. Function spaces

Following Friedman and Reitich [19,20] we define, for the function $g = g(t)$, the norm

$$[g]_t^2 := \int_0^t e^{2\alpha u} |g(u)|^2 \, du,$$

for some $\alpha > 0$, and recall, for the function $f = f(x)$, the classical Sobolev norm [27,3,17], for real $s \geq 0$,

$$\|f\|^2_{H^s([0,2\pi])} := \sum_{p=-\infty}^{\infty} \langle p \rangle^{2s} \left|\hat{f}_p\right|^2, \quad \langle p \rangle^2 := 1 + |p|^2, \quad \hat{f}_p := \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-ipx} \, dx.$$

With these, for the function $U = U(x,t)$, we define, for real $s \geq 4$,

$$\|U\|^2_{X^s} := \left[\|U\|^2_{H^s([0,2\pi])}\right]^2_\infty + \left[\|\partial_t U\|^2_{H^{s-2}([0,2\pi])}\right]^2_\infty + \left[\left\|\partial_t^2 U\right\|^2_{H^{s-4}([0,2\pi])}\right]^2_\infty$$

$$= \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2s} \left| \hat{U}_p(u) \right|^2 du + \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2(s-2)} \left| \partial_t \hat{U}_p(u) \right|^2 du$$
$$+ \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2(s-4)} \left| \partial_t^2 \hat{U}_p(u) \right|^2 du,$$

where the second time derivative is required for the algebra property (Lemma 4.1 below) to be valid, cf. [19,20].

In addition, in the next section we require volumetric norms. For the function $v = v(x, y)$, the classical Sobolev norm is [27,3,17], for integer $s \geq 0$,

$$\|v\|^2_{H^s([0,2\pi]\times[-h,0])} := \sum_{\ell=0}^{s} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2(s-\ell)} \int_{-h}^{0} \left| \partial_y^\ell \hat{v}_p(y) \right|^2 dy.$$

Finally, for the function $w = w(x, y, t)$, we define the following norm for integer $s \geq 4$,

$$|||w|||^2_{V^s} := \left[\|w\|_{H^s([0,2\pi]\times[-h,0])} \right]^2_\infty + \left[\|\partial_t w\|_{H^{s-2}([0,2\pi]\times[-h,0])} \right]^2_\infty$$
$$+ \left[\left\| \partial_t^2 w \right\|_{H^{s-4}([0,2\pi]\times[-h,0])} \right]^2_\infty$$
$$= \int_0^\infty e^{2\alpha u} \sum_{\ell=0}^{s} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2(s-\ell)} \int_{-h}^{0} \left| \partial_y^\ell \hat{w}_p(y, u) \right|^2 dy\, du$$
$$+ \int_0^\infty e^{2\alpha u} \sum_{\ell=0}^{s-2} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2(s-2-\ell)} \int_{-h}^{0} \left| \partial_y^\ell \partial_t \hat{w}_p(y, u) \right|^2 dy\, du$$
$$+ \int_0^\infty e^{2\alpha u} \sum_{\ell=0}^{s-4} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2(s-4-\ell)} \int_{-h}^{0} \left| \partial_y^\ell \partial_t^2 \hat{w}_p(y, u) \right|^2 dy\, du.$$

With these norms we define the function spaces, for real $s \geq 0$,

$$H^s([0, 2\pi]) := \left\{ f(x) \in L^2([0, 2\pi]) \;\middle|\; \int_0^{2\pi} f(x)\, dx = 0, \|f\|_{H^s([0,2\pi])} < \infty \right\},$$

and, for real $s \geq 4$,

$$X^s([0, 2\pi] \times [0, \infty)) := \left\{ U(x, t) \in L^2([0, 2\pi] \times [0, \infty)) \;\middle|\; \int_0^{2\pi} U(x, t)\, dx = 0, \|U\|_{X^s} < \infty \right\},$$

and, for integer $s \geq 0$,

$$H^s([0,2\pi] \times [-h,0]) := \left\{ v(x,y) \in L^2([0,2\pi] \times [-h,0]) \;\middle|\; \|v\|_{H^s([0,2\pi]\times[-h,0])} < \infty \right\},$$

and, for integer $s \geq 4$,

$$V^s([0,2\pi] \times [-h,0] \times [0,\infty)) := \left\{ w(x,y,t) \in L^2([0,2\pi] \times [-h,0] \times [0,\infty)) \;\middle|\; |||w|||_{V^s} < \infty \right\}.$$

From this point we suppress the domain dependence unless there is danger of confusion.

Of fundamental importance to our proof are the following algebra properties which are straightforward generalizations of Friedman and Reitich's Theorem A.4 in [19].

Lemma 4.1. *If $s \geq 4$; $f, g \in X^s$; $v, w \in V^s$; then there is a constant $M > 0$ such that*

$$\|fg\|_{X^s} \leq M \|f\|_{X^s} \|g\|_{X^s}, \tag{4.1a}$$

$$|||fv|||_{V^s} \leq M \|f\|_{X^s} |||v|||_{V^s}, \tag{4.1b}$$

$$|||vw|||_{V^s} \leq M |||v|||_{V^s} |||w|||_{V^s}. \tag{4.1c}$$

Remark 4.2. While the lemma above is true for any real $s \geq 4$ by interpolation [19], we will only utilize (4.1b) and (4.1c) for integer s, and (4.1a) for integer s or $s = m + 1/2$ for m integer.

In addition, we require a temporal trace theorem due to Friedman and Reitich [19] (see (4.53) on page 360 and the following discussion) suitably modified to our space X^s.

Lemma 4.3. *If $s \geq 4$ and $\sigma(x,t) \in X^s$ then there exists a constant $C_t > 0$ such that*

$$\max\left\{ \|\sigma(x,0)\|_{H^{s-1}}, \|\partial_t \sigma(x,0)\|_{H^{s-3}} \right\} \leq C_t \|\sigma\|_{X^s}. \tag{4.2}$$

Finally, we recall two auxiliary lemmas from [33].

Lemma 4.4. *If $s \geq 4$ and $w \in V^s$ then there exists a constant $Y = Y(s)$ such that*

$$|||(y+h)w|||_{V^s} < Y |||w|||_{V^s}.$$

Lemma 4.5. *There exists a universal constant $\Sigma > 0$ such that*

$$\max\left\{ \sum_{m=0}^{N} \frac{(N+1)^2}{(N-m+1)^2(m+1)^2}, \right.$$

$$\sum_{m=0}^{N} \sum_{\ell=0}^{m} \frac{(N+1)^2}{(N-m+1)^2(m-\ell+1)^2(\ell+1)^2},$$

$$\sum_{m=0}^{N}\sum_{\ell=0}^{m}\sum_{q=0}^{\ell}\frac{(N+1)^2}{(N-m+1)^2(m-\ell+1)^2(\ell-q+1)^2(q+1)^2}\Bigg\} < \Sigma.$$

5. A fundamental lemma for the elliptic problem

To prove our well-posedness result, Theorem 9.1, we must establish the following elliptic estimate which generalizes the results found in [32] to the spaces V^s and X^s.

Theorem 5.1. *Given an integer $s \geq 4$, if $F \in V^{s+1}$ and $\psi \in X^{s+5/2}$, then there exists a unique solution of*

$$\Delta w = F, \qquad -h < y < 0, \tag{5.1a}$$

$$w(x,0,t) = \psi(x,t), \tag{5.1b}$$

$$\partial_y w(x,-h,t) = 0, \tag{5.1c}$$

in V^{s+3} satisfying

$$\max\left\{\|w(x,0,t)\|_{X^{s+5/2}}, |||w|||_{V^{s+3}}\right\} \leq K_e\left\{|||F|||_{V^{s+1}} + \|\psi\|_{X^{s+5/2}}\right\}, \tag{5.2}$$

where $K_e > 0$ is a universal constant.

Proof. In a recent publication [31] one of the authors proved a similar result for the Helmholtz equation and we follow those developments here. To begin, we use the lateral periodicity of the solution to express

$$\{w(x,y,t), F(x,y,t), \psi(x,t)\} = \sum_{p=-\infty}^{\infty}\left\{\hat{w}_p(y,t), \hat{F}_p(y,t), \hat{\psi}_p(t)\right\} e^{ipx},$$

and note that (5.1) then demands that

$$\partial_y^2 \hat{w}_p - |p|^2 \hat{w}_p = \hat{F}_p, \qquad -h < y < 0, \tag{5.3a}$$

$$\hat{w}_p(0,t) = \hat{\psi}_p(t), \tag{5.3b}$$

$$\partial_y \hat{w}_p(-h,t) = 0. \tag{5.3c}$$

Existence and Uniqueness: To show the existence and uniqueness of a solution we appeal to the classical results of Keller [23], later extended in the "Integrated Solution Method" of Zhang [41,42] (see also [10]). Using the notation of [10] we consider, after a trivial change of variables $y \to y + h$, the problem

$$\mathbf{u}'(y) + \mathbf{M}(y)\mathbf{u}(y) = \mathbf{f}(y), \qquad 0 < y < h,$$

$$\mathbf{A}_0\mathbf{u} = \mathbf{r}_0, \qquad y = 0,$$

$$\mathbf{B}_1\mathbf{u} = \mathbf{s}_1, \qquad y = h,$$

where

$$\mathbf{f}(y) \in \mathbf{C}^m, \quad \mathbf{r}_0 \in \mathbf{C}^{m_1}, \quad \mathbf{s}_1 \in \mathbf{C}^{m_2},$$

are vector fields ($m = m_1 + m_2$). Further,

$$\mathbf{M}(y) \in \mathbf{C}^{m \times m}, \quad \mathbf{A}_0 \in \mathbf{C}^{m_1 \times m}, \quad \mathbf{B}_1 \in \mathbf{C}^{m_2 \times m},$$

are full rank matrices. Let $\mathbf{\Phi}(y)$ be the fundamental matrix solution of the system

$$\mathbf{\Phi}'(y) + \mathbf{M}(y)\mathbf{\Phi}(y) = 0, \quad \mathbf{\Phi}(0) = I_m,$$

where I_m is the $m \times m$ identity matrix. Keller shows [23] that the two-point value problem above has a unique solution if and only if

$$\det \begin{pmatrix} \mathbf{A}_0 \\ \mathbf{B}_1 \mathbf{\Phi}(h) \end{pmatrix} \neq 0.$$

In this instance we have $m = 2$, $m_1 = m_2 = 1$, and

$$\mathbf{u} = \begin{pmatrix} \hat{w}_p \\ \partial_y \hat{w}_p \end{pmatrix}, \quad \mathbf{M}(y) = \begin{pmatrix} 0 & -1 \\ -|p|^2 & 0 \end{pmatrix}, \quad \mathbf{f}(y) = \begin{pmatrix} 0 \\ \hat{F}_p \end{pmatrix},$$
$$\mathbf{A}_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \mathbf{r}_0 = 0, \quad \mathbf{s}_1 = \hat{\psi}_p.$$

There are two cases of p to consider.

1. The case $p = 0$: Here we may show that

$$\mathbf{\Phi}(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix},$$

and

$$\det \begin{pmatrix} \mathbf{A}_0 \\ \mathbf{B}_1 \mathbf{\Phi}(h) \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ 1 & h \end{pmatrix} = -1 \neq 0.$$

Thus, a unique solution exists in this case.

2. The case $p \neq 0$: Here one can see that

$$\mathbf{\Phi}(y) = \begin{pmatrix} \cosh(|p|\, y) & \sinh(|p|\, y)/\,|p| \\ |p| \sinh(|p|\, y) & \cosh(|p|\, y) \end{pmatrix},$$

and

$$\det \begin{pmatrix} \mathbf{A}_0 \\ \mathbf{B}_1 \mathbf{\Phi}(h) \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ \cosh(|p|\, h) & \sinh(|p|\, h)/\,|p| \end{pmatrix} = -\cosh(|p|\, h) \neq 0.$$

Again, a unique solution exists in this case.

We note that existence and uniqueness of solutions can also be verified by simply (but less elegantly) writing down the exact solution as in [32].

Estimates: In order to accommodate Sobolev spaces with very low smoothness, we consider the slightly generalized form of (5.1)

$$
\begin{aligned}
&\Delta w = \partial_x F^{(1)} + \partial_y F^{(2)} + F^{(3)}, && -h < y < 0,\\
&w(x,0,t) = \psi(x,t),\\
&\partial_y w(x,-h,t) = 0,
\end{aligned}
$$

which, upon Fourier transform, becomes

$$
\begin{aligned}
&\partial_y^2 \hat{w}_p - |p|^2 \hat{w}_p = (ip)\hat{F}_p^{(1)} + \partial_y \hat{F}_p^{(2)} + \hat{F}_p^{(3)}, && -h < y < 0,\\
&\hat{w}_p(0,t) = \hat{\psi}_p(t),\\
&\partial_y \hat{w}_p(-h,t) = 0.
\end{aligned}
$$

Following the developments of [32], we set

$$
\hat{w}_p(y,t) = \hat{w}_p^{(0)}(y,t) + \hat{w}_p^{(1)}(y,t) + \hat{w}_p^{(2)}(y,t) + \hat{w}_p^{(3)}(y,t),
$$

where, for $j = 0, 1, 2, 3$,

$$
\partial_y^2 \hat{w}_p^{(j)} - |p|^2 \hat{w}_p^{(j)} = \delta_{j,1}(ip)\hat{F}_p^{(1)} + \delta_{j,2}\partial_y \hat{F}_p^{(2)} + \delta_{j,3}\hat{F}_p^{(3)}, \qquad -h < y < 0, \tag{5.4a}
$$

$$
\hat{w}_p^{(j)}(0,t) = \delta_{j,0}\hat{\psi}_p(t), \tag{5.4b}
$$

$$
\partial_y \hat{w}_p^{(j)}(-h,t) = 0. \tag{5.4c}
$$

It can be shown (see Lemma A.2 of [31]) from the solution formula for $\hat{w}_p(y,t)$, that the following volumetric estimates hold for the unique solution when $\ell = 0, 1$,

$$
\left\| \partial_y^\ell \hat{w}_p^{(0)}(y,t) \right\|_{L^2(dy)}^2 \le K_e \langle p \rangle^{-1+2\ell} \left| \hat{\psi}_p(t) \right|^2, \tag{5.5a}
$$

$$
\left\| \partial_y^\ell \hat{w}_p^{(1)}(y,t) \right\|_{L^2(dy)}^2 \le K_e \langle p \rangle^{-2+2\ell} \left\| \hat{F}_p^{(1)}(y,t) \right\|_{L^2(dy)}^2, \tag{5.5b}
$$

$$
\left\| \partial_y^\ell \hat{w}_p^{(2)}(y,t) \right\|_{L^2(dy)}^2 \le K_e \langle p \rangle^{-2+2\ell} \left\| \hat{F}_p^{(2)}(y,t) \right\|_{L^2(dy)}^2, \tag{5.5c}
$$

$$
\left\| \partial_y^\ell \hat{w}_p^{(3)}(y,t) \right\|_{L^2(dy)}^2 \le K_e \langle p \rangle^{-4+2\ell} \left\| \hat{F}_p^{(3)}(y,t) \right\|_{L^2(dy)}^2, \tag{5.5d}
$$

for some $K_e > 0$. In addition, the subsequent boundary estimates hold for $\ell = 0, 1$

$$\left|\partial_y^\ell \hat{w}_p^{(0)}(0,t)\right|^2 \le K_e \langle p\rangle^{2\ell} \left|\hat{\psi}_p(t)\right|^2, \tag{5.6a}$$

$$\left|\partial_y^\ell \hat{w}_p^{(1)}(0,t)\right|^2 \le K_e \langle p\rangle^{-1+2\ell} \left\|\hat{F}_p^{(1)}(y,t)\right\|_{L^2(dy)}^2, \tag{5.6b}$$

$$\left|\partial_y^\ell \hat{w}_p^{(2)}(0,t)\right|^2 \le K_e \langle p\rangle^{-1+2\ell} \left\|\hat{F}_p^{(2)}(y,t)\right\|_{L^2(dy)}^2, \tag{5.6c}$$

$$\left|\partial_y^\ell \hat{w}_p^{(3)}(0,t)\right|^2 \le K_e \langle p\rangle^{-3+2\ell} \left\|\hat{F}_p^{(3)}(y,t)\right\|_{L^2(dy)}^2, \tag{5.6d}$$

with $K_e > 0$ sufficiently large. Furthermore, as the governing equations (5.4) treat time as a parameter, by simply applying time derivatives one can achieve the same estimates with $\{\hat{w}_p^{(j)}, \hat{F}_p^{(j)}, \hat{\psi}_p\}$ replaced by $\{\partial_t \hat{w}_p^{(j)}, \partial_t \hat{F}_p^{(j)}, \partial_t \hat{\psi}_p\}$, and $\{\partial_t^2 \hat{w}_p^{(j)}, \partial_t^2 \hat{F}_p^{(j)}, \partial_t^2 \hat{\psi}_p\}$.

Consider the H^1-type norm of w

$$\left[\|w\|_{H^1}^2\right]_\infty^2 = \int_0^\infty e^{2\alpha u} \sum_{\ell=0}^{1} \sum_{p=-\infty}^{\infty} \langle p\rangle^{2(1-\ell)} \left\|\partial_y^\ell \hat{w}_p(y,u)\right\|_{L^2(dy)}^2 \, du,$$

and the H^{-1} analogue for F

$$\left[\|F\|_{H^{-1}}^2\right]_\infty^2 = \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \sum_{j=1}^{3} \left\|\hat{F}_p^{(j)}(y,u)\right\|_{L^2(dy)}^2 \, du, \quad F = \partial_x F^{(1)} + \partial_y F^{(2)} + F^{(3)},$$

for some $F^{(j)} \in L^2([-h,0])$ (see Chapter 6 of Evans [17]). Using the estimates above (and being a little wasteful in our estimate of $F^{(3)}$) we have

$$\begin{aligned}\left[\|w\|_{H^1}^2\right]_\infty^2 &\le \sum_{j=0}^{3} \left[\left\|w^{(j)}\right\|_{H^1}^2\right]_\infty^2 \\ &\le \int_0^\infty e^{2\alpha u} \sum_{\ell=0}^{1} \sum_{p=-\infty}^{\infty} \langle p\rangle^{2(1-\ell)} K_e \left\{ \langle p\rangle^{-1+2\ell} \left|\hat{\psi}_p(u)\right|^2 \right. \\ &\qquad \left. + \sum_{j=1}^{3} \langle p\rangle^{-2+2\ell} \left\|\hat{F}_p^{(j)}(y,u)\right\|_{L^2(dy)}^2 \right\} du.\end{aligned}$$

Rearranging this we find

$$\begin{aligned}\left[\|w\|_{H^1}^2\right]_\infty^2 &\le K_e \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \langle p\rangle^1 \left|\hat{\psi}_p(u)\right|^2 du \\ &\quad + K_e \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \sum_{j=1}^{3} \left\|\hat{F}_p^{(j)}(y,u)\right\|_{L^2(dy)}^2 du\end{aligned}$$

$$\leq K_e \left\{ \|\psi\|_{H^{1/2}}^2 + \|F\|_{H^{-1}}^2 \right\}.$$

Either by conducting the tedious manipulations to produce the higher-order analogues of (5.5) and (5.6), or by proceeding as in Chapter 6 of Evans [17], we can deduce

$$\left[\|w\|_{H^{s+3}}^2 \right]_\infty^2 \leq C K_e \left\{ \left[\|F\|_{H^{s+1}}^2 \right]_\infty^2 + \left[\|\psi\|_{H^{s+5/2}}^2 \right]_\infty^2 \right\}.$$

Applying this estimate to $\partial_t w$ and $\partial_t^2 w$, and recalling that $s \geq 4$, we discover

$$\left[\|\partial_t w\|_{H^{s+1}}^2 \right]_\infty^2 \leq C K_e \left\{ \left[\|\partial_t F\|_{H^{s-1}}^2 \right]_\infty^2 + \left[\|\partial_t \psi\|_{H^{s+1/2}}^2 \right]_\infty^2 \right\},$$

and

$$\left[\left\|\partial_t^2 w\right\|_{H^{s-1}}^2 \right]_\infty^2 \leq C K_e \left\{ \left[\left\|\partial_t^2 F\right\|_{H^{s-3}}^2 \right]_\infty^2 + \left[\left\|\partial_t^2 \psi\right\|_{H^{s-1/2}}^2 \right]_\infty^2 \right\},$$

which, upon summation, delivers the conclusion of the theorem. □

6. A fundamental lemma for the parabolic problem

To state our next result we recall the definition of an order-k Fourier multiplier.

Definition 6.1. Suppose that $\psi \in L^2([0, 2\pi])$ then the equation

$$m(D)\psi(x) := \sum_{p=-\infty}^{\infty} m(p)\hat{\psi}_p e^{ipx},$$

defines the Fourier multiplier $m(D)$. If, for some $k \in \mathbf{R}$, we have for any s real

$$\|m(D)\psi\|_{H^s} \leq C \|\psi\|_{H^{s+k}},$$

then we say that $m(D)$ is order-k.

Remark 6.2. The classical derivative, ∂_x, is clearly an order-one Fourier multiplier with symbol (iD). Of relevance to the current contribution are the order-one multiplier

$$G_0 := |D| \tanh(h |D|),$$

which is the flat-interface DNO for Laplace's equation on a strip [13,14,29], the order-three-halves $(i\omega_D)$ operator

$$(i\omega_D)\psi(x) = \sum_{p=-\infty}^{\infty} (i\omega_p)\hat{\psi}_p e^{i\alpha_p x} := \sum_{p=-\infty}^{\infty} i\sqrt{(g + \sigma |p|^2) \left|\hat{G}_{0,p}\right|}\hat{\psi}_p e^{i\alpha_p x},$$

which comes from the dispersion relation for water waves [25,37,2], and the order-two operator $(-|D|^2) = \partial_x^2$.

We require the following parabolic estimate for our inductive proof to proceed.

Theorem 6.3. *Given a real number $s \geq 4$, if $Q \in X^{s+1}$, $R \in X^{s+1/2}$, $\eta^{(0)} \in H^{s+2}$, and $\xi^{(0)} \in H^{s+3/2}$ then there exists a unique solution of*

$$\partial_t \eta = G_0[\xi] + 2\mu \partial_x^2 \eta + Q, \qquad \eta(x,0) = \eta^{(0)}(x), \tag{6.1a}$$

$$\partial_t \xi = -g\eta + \sigma \partial_x^2 \eta - 2\mu |D|^2 \xi + R, \qquad \xi(x,0) = \xi^{(0)}(x), \tag{6.1b}$$

satisfying

$$\max\left\{ \|\eta\|_{X^{s+3}}, \|\partial_t \eta\|_{X^{s+1}}, \|\xi\|_{X^{s+5/2}}, \|\partial_t \xi\|_{X^{s+1/2}} \right\} \leq K_p \left\{ \|Q\|_{X^{s+1}} + \|R\|_{X^{s+1/2}} + \left\|\eta^{(0)}\right\|_{H^{s+2}} + \left\|\xi^{(0)}\right\|_{H^{s+3/2}} \right\}, \tag{6.2}$$

where $K_p > 0$ is a universal constant.

To establish this we require the following result from Friedman and Reitich [20] (Lemma 7.1).

Lemma 6.4. *Consider the initial value problem*

$$\dot{B}(t) + (K + iM)B(t) = F(t), \qquad t > 0,$$
$$B(0) = B_0,$$

where $K, M \in \mathbf{R}$; $K > 0$; and $F \in L^2(0,T)$ for any $T > 0$. If $0 < \alpha < K$ then the following inequalities hold

$$[B]_t^2 \leq \frac{2}{(K-\alpha)^2} [F]_t^2 + \frac{|B_0|^2}{K-\alpha}, \tag{6.3a}$$

$$\left[\dot{B}\right]_t^2 \leq 2\left(\frac{2K^2}{(K-\alpha)^2} + 1\right)[F]_t^2 + \frac{2K^2}{K-\alpha}|B_0|^2. \tag{6.3b}$$

To prove Theorem 6.3 we establish a similar result for a decoupled version of (6.1).

Theorem 6.5. *Given a real number $s \geq 4$, if $W \in X^s$, $f \in H^{s+1}$ then there exists a unique solution of*

$$\partial_t U = \left[-2\mu |D|^2 \pm (i\omega_D)\right] U + W, \quad U(x,0) = f(x), \tag{6.4}$$

satisfying

$$\max\left\{\|U\|_{X^{s+2}}, \|\partial_t U\|_{X^s}\right\} \le \tilde{K}_p \left\{\|W\|_{X^s} + \|f\|_{H^{s+1}}\right\}, \tag{6.5}$$

where $\tilde{K}_p > 0$ is a universal constant.

Proof. We focus on the case of (6.4) with the minus sign in front of $(i\omega_D)$; the other case is handled similarly. We expand W and f in Fourier series

$$W(x,t) = \sum_{p=-\infty}^{\infty} \hat{W}_p(t)e^{ipx}, \quad f(x) = \sum_{p=-\infty}^{\infty} \hat{f}_p e^{ipx},$$

where $\hat{W}_0(t) \equiv 0$ and $\hat{f}_p = 0$ by the definitions of X^s and H^s, respectively, and seek a solution

$$U(x,t) = \sum_{p=-\infty}^{\infty} \hat{U}_p(t)e^{ipx}.$$

Inserting these into (6.4) we find

$$\partial_t \hat{U}_p(t) = -\Omega(p)\hat{U}_p(t) + \hat{W}_p(t), \tag{6.6a}$$

$$\hat{U}_p(0) = \hat{f}_p, \tag{6.6b}$$

where

$$\Omega(p) := \left(2\mu\,|p|^2 + i\omega_p\right) = \mathcal{O}(p^2), \quad p \to \infty.$$

It is clear that $\hat{U}_0(t) \equiv 0$ is the unique solution in the case $p = 0$ so we now concentrate on $p \neq 0$. Using Lemma 6.4 we find

$$\left[\hat{U}_p\right]_\infty^2 \le C_{0,W}(p)\left[\hat{W}_p\right]_\infty^2 + C_{0,f}(p)\left|\hat{f}_p\right|^2, \tag{6.7a}$$

$$\left[\partial_t \hat{U}_p\right]_\infty^2 \le C_{1,W}(p)\left[\hat{W}_p\right]_\infty^2 + C_{1,f}(p)\left|\hat{f}_p\right|^2, \tag{6.7b}$$

with, since $p \neq 0$,

$$K = K_p := 2\mu\,|p|^2 > 0, \quad 0 < \alpha < 2\mu\,|1|^2 \le \min_{p\neq 0} K_p,$$

for some α. In these

$$C_{0,W}(p) := \frac{2}{(2\mu\,|p|^2 - \alpha)^2} = \mathcal{O}(p^{-4}), \qquad p \to \infty$$

$$C_{0,f}(p) := \frac{1}{(2\mu\,|p|^2 - \alpha)} = \mathcal{O}(p^{-2}), \qquad p \to \infty$$

$$C_{1,W}(p) := 2\left[\frac{2(2\mu\,|p|^2)^2}{(2\mu\,|p|^2 - \alpha)^2} + 1\right] = \mathcal{O}(1), \qquad p \to \infty$$

$$C_{1,f}(p) := \frac{2(2\mu|p|^2)^2}{(2\mu|p|^2-\alpha)} = \mathcal{O}(p^2), \qquad p \to \infty.$$

Differentiating (6.6a) once with respect to t yields

$$\partial_t(\partial_t \hat{U}_p)(t) = -\Omega(p)(\partial_t \hat{U}_p)(t) + \partial_t \hat{W}_p(t), \tag{6.8a}$$

$$\partial_t \hat{U}_p(0) = -\Omega(p)\hat{f}_p + \hat{W}_p(0), \tag{6.8b}$$

where we have used (6.6a) and (6.6b) for the initial condition. Again, appealing to Lemma 6.4 we find

$$\left[\partial_t \hat{U}_p\right]_\infty^2 \le C_{0,W}(p)\left[\partial_t \hat{W}_p\right]_\infty^2 + C_{0,f}(p)\,|\Omega(p)|^2\left|\hat{f}_p\right|^2 + C_{0,f}(p)\left|\hat{W}_p(0)\right|^2, \tag{6.9a}$$

$$\left[\partial_t^2 \hat{U}_p\right]_\infty^2 \le C_{1,W}(p)\left[\partial_t \hat{W}_p\right]_\infty^2 + C_{1,f}(p)\,|\Omega(p)|^2\left|\hat{f}_p\right|^2 + C_{1,f}(p)\left|\hat{W}_p(0)\right|^2. \tag{6.9b}$$

Differentiating (6.8a) once with respect to t yields

$$\partial_t(\partial_t^2 \hat{U}_p)(t) = -\Omega(p)(\partial_t^2 \hat{U}_p)(t) + \partial_t^2 \hat{W}_p, \tag{6.10a}$$

$$\partial_t^2 \hat{U}_p(0) = \Omega(p)^2 \hat{f}_p - \Omega(p)\hat{W}_p(0) + \partial_t \hat{W}_p(0), \tag{6.10b}$$

where we have used (6.8a) and (6.8b) for the initial condition. Appealing to Lemma 6.4 as before we find

$$\begin{aligned}\left[\partial_t^2 \hat{U}_p\right]_\infty^2 \le{}& C_{0,W}(p)\left[\partial_t^2 \hat{W}_p\right]_\infty^2 + C_{0,f}(p)\,|\Omega(p)|^4\left|\hat{f}_p\right|^2 \\ &+ C_{0,f}(p)\,|\Omega(p)|^2\left|\hat{W}_p(0)\right|^2 + C_{0,f}(p)\left|\partial_t \hat{W}_p(0)\right|^2,\end{aligned} \tag{6.11a}$$

$$\begin{aligned}\left[\partial_t^3 \hat{U}_p\right]_\infty^2 \le{}& C_{1,W}(p)\left[\partial_t^2 \hat{W}_p\right]_\infty^2 + C_{1,f}(p)\,|\Omega(p)|^4\left|\hat{f}_p\right|^2 \\ &+ C_{1,f}(p)\,|\Omega(p)|^2\left|\hat{W}_p(0)\right|^2 + C_{1,f}(p)\left|\partial_t \hat{W}_p(0)\right|^2.\end{aligned} \tag{6.11b}$$

If we multiply (6.7a) by $\langle p\rangle^{2s+4}$, (6.9a) by $\langle p\rangle^{2s}$, and (6.11a) by $\langle p\rangle^{2s-4}$ and sum over p we find

$$\begin{aligned}\|U\|_{X^{s+2}} ={}& \sum_{p=-\infty}^{\infty}\left[\langle p\rangle^{2s+4}\left[\hat{U}_p\right]_\infty^2 + \langle p\rangle^{2s}\left[\partial_t \hat{U}_p\right]_\infty^2 + \langle p\rangle^{2s-4}\left[\partial_t^2 \hat{U}_p\right]_\infty^2\right] \\ \le{}& \sum_{p=-\infty}^{\infty}\left[C_{0,W}(p)\left\{\langle p\rangle^{2s+4}\left[\hat{W}_p\right]_\infty^2 + \langle p\rangle^{2s}\left[\partial_t \hat{W}_p\right]_\infty^2 + \langle p\rangle^{2s-4}\left[\partial_t^2 \hat{W}_p\right]_\infty^2\right\}\right. \\ &+ C_{0,f}(p)\left\{\langle p\rangle^{2s+4}\left|\hat{f}_p\right|^2 + \langle p\rangle^{2s}\,|\Omega(p)|^2\left|\hat{f}_p\right|^2 + \langle p\rangle^{2s}\left|\hat{W}_p(0)\right|^2\right. \\ &\left.\left.+\langle p\rangle^{2s-4}\,|\Omega(p)|^4\left|\hat{f}_p\right|^2 + \langle p\rangle^{2s-4}\,|\Omega(p)|^2\left|\hat{W}_p(0)\right|^2 + \langle p\rangle^{2s-4}\left|\partial_t \hat{W}_p(0)\right|^2\right\}\right].\end{aligned}$$

From this we easily find that

$$\|U\|_{X^{s+2}} \leq K_0 \left[\|W\|_{X^s} + \|f\|_{H^{s+1}} + \|W(\cdot,0)\|_{H^{s-1}} + \|\partial_t W(\cdot,0)\|_{H^{s-3}} \right]. \tag{6.12}$$

In a similar way, if we multiply (6.7b) by $\langle p\rangle^{2s}$, (6.9b) by $\langle p\rangle^{2s-4}$, and (6.11b) by $\langle p\rangle^{2s-8}$ and sum over p we find

$$\|\partial_t U\|_{X^s} \leq K_0 \left[\|W\|_{X^s} + \|f\|_{H^{s+1}} + \|W(\cdot,0)\|_{H^{s-1}} + \|\partial_t W(\cdot,0)\|_{H^{s-3}} \right]. \tag{6.13}$$

We now appeal to Lemma 4.3 (which requires $s \geq 4$) and use this and estimate (6.12) to deliver

$$\|U\|_{X^{s+2}} \leq \tilde{K}_p \left[\|W\|_{X^s} + \|f\|_{H^{s+1}} \right],$$

and (6.13) to give

$$\|\partial_t U\|_{X^s} \leq \tilde{K}_p \left[\|W\|_{X^s} + \|f\|_{H^{s+1}} \right],$$

for some $\tilde{K}_p > 0$ and we are done. □

Proof of Theorem 6.3. To establish this result we express our initial value problem, (6.1), on the Fourier side by using

$$\eta(x,t) = \sum_{p=-\infty}^{\infty} \hat{\eta}_p(t) e^{ipx}, \quad \xi(x,t) = \sum_{p=-\infty}^{\infty} \hat{\xi}_p(t) e^{ipx},$$

which, upon insertion into (6.1), delivers

$$\partial_t \hat{\eta}_p = \hat{G}_{0,p} \hat{\xi}_p - 2\mu |p|^2 \hat{\eta}_p + \hat{Q}_p, \qquad \hat{\eta}_p(0) = \widehat{\eta^{(0)}}_p, \tag{6.14a}$$

$$\partial_t \hat{\xi}_p = -(g + \sigma |p|^2)\hat{\eta}_p - 2\mu |p|^2 \hat{\xi}_p + \hat{R}_p, \qquad \hat{\xi}_p(0) = \widehat{\xi^{(0)}}_p, \tag{6.14b}$$

or

$$\partial_t \begin{pmatrix} \hat{\eta}_p \\ \hat{\xi}_p \end{pmatrix} = \begin{pmatrix} -2\mu |p|^2 & \hat{G}_{0,p} \\ -(g+\sigma |p|^2) & -2\mu |p|^2 \end{pmatrix} \begin{pmatrix} \hat{\eta}_p \\ \hat{\xi}_p \end{pmatrix} + \begin{pmatrix} \hat{Q}_p \\ \hat{R}_p \end{pmatrix}.$$

If we use $\omega_p = \sqrt{(g+\sigma |p|^2)\hat{G}_{0,p}}$, define

$$\hat{P}_p := \begin{pmatrix} -i\omega_p & i\omega_p \\ g+\sigma |p|^2 & g+\sigma |p|^2 \end{pmatrix}, \quad \hat{P}_p^{-1} = \frac{1}{2} \begin{pmatrix} -1/(i\omega_p) & 1/(g+\sigma |p|^2) \\ 1/(i\omega_p) & 1/(g+\sigma |p|^2) \end{pmatrix},$$

and make the change of variables

$$\begin{pmatrix} \hat{\zeta}_p \\ \hat{\chi}_p \end{pmatrix} := \hat{P}_p^{-1} \begin{pmatrix} \hat{\eta}_p \\ \hat{\xi}_p \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\hat{\eta}_p/(i\omega_p) + \hat{\xi}_p/(g+\sigma |p|^2) \\ \hat{\eta}_p/(i\omega_p) + \hat{\xi}_p/(g+\sigma |p|^2) \end{pmatrix},$$

we find

$$\partial_t \hat{\zeta}_p = \left(-2\mu\,|p|^2 + i\omega_p\right)\hat{\zeta}_p + \hat{V}_p$$
$$\partial_t \hat{\chi}_p = \left(-2\mu\,|p|^2 - i\omega_p\right)\hat{\chi}_p + \hat{W}_p,$$

where

$$\begin{pmatrix} \hat{V}_p \\ \hat{W}_p \end{pmatrix} := \hat{P}_p^{-1} \begin{pmatrix} \hat{Q}_p \\ \hat{R}_p \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\hat{Q}_p/(i\omega_p) + \hat{R}_p/(g+\sigma\,|p|^2) \\ \hat{Q}_p/(i\omega_p) + \hat{R}_p/(g+\sigma\,|p|^2) \end{pmatrix}.$$

We note that

$$\eta = -(i\omega_D)\zeta + (i\omega_D)\chi,$$
$$\xi = (g - \sigma\partial_x^2)\zeta + (g - \sigma\partial_x^2)\chi,$$

so

$$\max\left\{\|\eta\|_{X^{s+3}}, \|\partial_t\eta\|_{X^{s+1}}, \|\xi\|_{X^{s+5/2}}, \|\partial_t\xi\|_{X^{s+1/2}}, \right\} \le C\left\{\|\zeta\|_{X^{s+9/2}} + \|\partial_t\zeta\|_{X^{s+5/2}} + \|\chi\|_{X^{s+9/2}} + \|\partial_t\chi\|_{X^{s+5/2}}\right\}.$$

From Theorem 6.5 we have

$$\max\left\{\|\eta\|_{X^{s+3}}, \|\partial_t\eta\|_{X^{s+1}}, \|\xi\|_{X^{s+5/2}}, \|\partial_t\xi\|_{X^{s+1/2}}\right\} \le C\tilde{K}_P\left\{\|V\|_{X^{s+5/2}} + \left\|\zeta^{(0)}\right\|_{H^{s+7/2}} + \|W\|_{X^{s+5/2}} + \left\|\chi^{(0)}\right\|_{H^{s+7/2}}\right\}.$$

Now, since

$$V = \frac{1}{2}\left\{(i\omega_D)^{-1}Q + (g-\sigma\partial_x^2)^{-1}R\right\},$$
$$W = \frac{1}{2}\left\{(i\omega_D)^{-1}Q + (g-\sigma\partial_x^2)^{-1}R\right\},$$
$$\zeta^{(0)} = \frac{1}{2}\left\{(i\omega_D)^{-1}\eta^{(0)} + (g-\sigma\partial_x^2)^{-1}\xi^{(0)}\right\},$$
$$\chi^{(0)} = \frac{1}{2}\left\{(i\omega_D)^{-1}\eta^{(0)} + (g-\sigma\partial_x^2)^{-1}\xi^{(0)}\right\},$$

we have

$$\max\left\{\|\eta\|_{X^{s+3}}, \|\partial_t\eta\|_{X^{s+1}}, \|\xi\|_{X^{s+5/2}}, \|\xi\|_{X^{s+1/2}}\right\} \le C\tilde{K}_P\tilde{C}\left\{\|Q\|_{X^{s+1}} + \|R\|_{X^{s+1/2}} + \left\|\eta^{(0)}\right\|_{H^{s+2}} + \left\|\xi^{(0)}\right\|_{H^{s+3/2}}\right\},$$

and we are done if we choose $K_P = C\tilde{K}_P\tilde{C}$. □

7. A fundamental lemma for the linearized water wave problem with viscosity

We require the following estimate of the linearization of water wave problem (3.1) in order to proceed.

Lemma 7.1. *Given an integer $s \geq 4$, if $F \in V^{s+1}$, $Q \in X^{s+1}$, $R \in X^{s+1/2}$, $\eta^{(0)} \in H^{s+2}$, and $\xi^{(0)} \in H^{s+3/2}$ then there exists a unique solution of*

$$\Delta u = F, \qquad -h < y < 0, \tag{7.1a}$$

$$\partial_y u = 0, \qquad y = -h, \tag{7.1b}$$

$$\partial_t \eta = \partial_y u + 2\mu \partial_x^2 \eta + Q, \qquad y = 0, \tag{7.1c}$$

$$\partial_t u = -g\eta + \sigma \partial_x^2 \eta - 2\mu \partial_y^2 u + R, \qquad y = 0, \tag{7.1d}$$

$$u(x,0,0) = \xi^{(0)}(x), \tag{7.1e}$$

$$\eta(x,0) = \eta^{(0)}(x), \tag{7.1f}$$

satisfying

$$\begin{aligned} &\max \left\{ \|\eta\|_{X^{s+3}}, \|\partial_t \eta\|_{X^{s+1}}, \|u(x,0,t)\|_{X^{s+5/2}}, \|\partial_t u(x,0,t)\|_{X^{s+1/2}}, |||u|||_{V^{s+3}} \right\} \\ &\qquad \leq K \left\{ |||F|||_{V^{s+1}} + \|Q\|_{X^{s+1}} + \|R\|_{X^{s+1/2}} + \left\|\eta^{(0)}\right\|_{H^{s+2}} + \left\|\xi^{(0)}\right\|_{H^{s+3/2}} \right\}, \end{aligned} \tag{7.2}$$

for a universal constant $K > 0$.

Proof. Using the periodicity of solutions we write

$$\{u, F\} = \{u, F\}(x, y, t) = \sum_{p=-\infty}^{\infty} \{\hat{u}_p, \hat{F}_p\}(y, t) e^{ipx},$$

$$\{\eta, Q, R\} = \{\eta, Q, R\}(x, t) = \sum_{p=-\infty}^{\infty} \{\hat{\eta}_p, \hat{Q}_p, \hat{R}_p\}(t) e^{ipx},$$

$$\{\eta^{(0)}, \xi^{(0)}\} = \{\eta^{(0)}, \xi^{(0)}\}(x) = \sum_{p=-\infty}^{\infty} \{\widehat{\eta^{(0)}}_p, \widehat{\xi^{(0)}}_p\} e^{ipx},$$

which transforms (7.1) into

$$\partial_y^2 \hat{u}_p - |p|^2 \hat{u}_p = \hat{F}_p, \qquad -h < y < 0, \tag{7.3a}$$

$$\partial_y \hat{u}_p = 0, \qquad y = -h, \tag{7.3b}$$

$$\partial_t \hat{\eta}_p = \partial_y \hat{u}_p - 2\mu |p|^2 \hat{\eta}_p + \hat{Q}_p, \qquad y = 0, \tag{7.3c}$$

$$\partial_t \hat{u}_p = -g\hat{\eta}_p - \sigma |p|^2 \hat{\eta}_p - 2\mu \partial_y^2 \hat{u}_p + \hat{R}_p, \qquad y = 0, \tag{7.3d}$$

$$\hat{u}_p(0,0) = \widehat{\xi^{(0)}}_p, \tag{7.3e}$$

$$\hat{\eta}_p(0) = \widehat{\eta^{(0)}}_p. \tag{7.3f}$$

We now decompose the solution into two parts

$$\{\hat{u}_p, \hat{\eta}_p\} = \{U^P, H^P\} + \{U^E, H^E\},$$

which essentially solve the parabolic (§ 6) and elliptic (§ 5) problems respectively, and we have suppressed the p subscript for clarity. More specifically, $\{U^P, H^P\}$ solves (7.3) in the case $\hat{F}_p \equiv 0$,

$$\partial_y^2 U^P - |p|^2 U^P = 0, \qquad -h < y < 0, \tag{7.4a}$$

$$\partial_y U^P = 0, \qquad y = -h, \tag{7.4b}$$

$$\partial_t H^P = \partial_y U^P - 2\mu |p|^2 H^P + \hat{Q}_p, \qquad y = 0, \tag{7.4c}$$

$$\partial_t U^P = -g H^P - \sigma |p|^2 H^P - 2\mu \partial_y^2 U^P + \hat{R}_p, \qquad y = 0, \tag{7.4d}$$

$$U^P(0,0) = \widehat{\xi^{(0)}}_p, \tag{7.4e}$$

$$H^P(0) = \widehat{\eta^{(0)}}_p, \tag{7.4f}$$

while $\{U^E, H^E\}$ solves (7.3) where $\hat{Q}_p \equiv \hat{R}_p \equiv \widehat{\xi^{(0)}}_p \equiv \widehat{\eta^{(0)}}_p \equiv 0$,

$$\partial_y^2 U^E - |p|^2 U^E = \hat{F}_p, \qquad -h < y < 0, \tag{7.5a}$$

$$\partial_y U^E = 0, \qquad y = -h, \tag{7.5b}$$

$$\partial_t H^E = \partial_y U^E - 2\mu |p|^2 H^E, \qquad y = 0, \tag{7.5c}$$

$$\partial_t U^E = -g H^E - \sigma |p|^2 H^E - 2\mu \partial_y^2 U^E, \qquad y = 0, \tag{7.5d}$$

$$U^E(0,0) = 0, \tag{7.5e}$$

$$H^E(0) = 0. \tag{7.5f}$$

It is not difficult to show that the solution of (7.4a) and (7.4b) is

$$U^P(y,t) = U^P(0,t) \frac{\cosh(|p|(y+h))}{\cosh(|p| h)}.$$

Upon insertion of this form into (7.4c)–(7.4f) we find

$$\partial_t H^P = |p| \tanh(h|p|) U^P - 2\mu |p|^2 H^P + \hat{Q}_p, \qquad y = 0,$$

$$\partial_t U^P = -(g + \sigma |p|^2) H^P - 2\mu |p|^2 U^P + \hat{R}_p, \qquad y = 0,$$

$$U^P(0,0) = \widehat{\xi^{(0)}}_p,$$

$$H^P(0) = \widehat{\eta^{(0)}}_p.$$

Upon inverse Fourier transform we find that this equation is identical to that appearing in Theorem 6.3 with $\hat{\xi}_p(t) = U^P(0,t)$. From this we learn that

$$\max\left\{\left\|H^P\right\|_{X^{s+3}}, \left\|\partial_t H^P\right\|_{X^{s+1}}, \left\|U^P(x,0,t)\right\|_{X^{s+5/2}}, \left\|\partial_t U^P(x,0,t)\right\|_{X^{s+1/2}}\right\} \\ \leq K_p\left\{\|Q\|_{X^{s+1}} + \|R\|_{X^{s+1/2}} + \left\|\eta^{(0)}\right\|_{H^{s+2}} + \left\|\xi^{(0)}\right\|_{H^{s+3/2}}\right\}. \quad (7.6)$$

Turning to (7.5) it is easy to see that (7.5c)–(7.5f) demand that

$$U^E(0,t) \equiv H^E(t) \equiv 0,$$

so that we are left to solve

$$\begin{aligned} &\partial_y^2 U^E - |p|^2 U^E = \hat{F}_p, && -h < y < 0, \\ &\partial_y U^E = 0, && y = -h, \\ &U^E = 0, && y = 0. \end{aligned}$$

However, upon inverse Fourier transform, we realize that this is simply the system of equations in Theorem 5.1 with $\psi \equiv 0$. Thus, we conclude that

$$\max\left\{\left\|U^E(x,0,t)\right\|_{X^{s+5/2}}, \left|\left|\left|U^E\right|\right|\right|_{V^{s+3}}\right\} \leq K_e \left|\left|\left|F\right|\right|\right|_{V^{s+1}}. \quad (7.7)$$

Combining (7.6) and (7.7) to estimate $\hat{\eta}_p = H^P + H^E$ and $\hat{u}_p = U^P + U^E$ we realize (7.2) for some $K > 0$. □

8. An inductive lemma

To complete the proof of our theorem we require the following recursive estimates.

Lemma 8.1. *For an integer $s \geq 4$, suppose for some $C, B > 0$ we have*

$$\max\left\{\|\eta_n\|_{X^{s+3}}, \|\partial_t \eta_n\|_{X^{s+1}}, \|u_n(x,0,t)\|_{X^{s+5/2}}, \|\partial_t u_n(x,0,t)\|_{X^{s+1/2}}, |||u_n|||_{V^{s+3}}\right\} \\ \leq C\frac{B^{n-1}}{(n+1)^2}, \quad \forall n < N,$$

then the functions F_N, Q_N and R_N satisfy

$$\max\left\{|||F_N|||_{V^{s+1}}, \|Q_N\|_{X^{s+1}}, \|R_N\|_{X^{s+1/2}}\right\} \leq C_i C\left\{\frac{B^{N-2}}{(N+1)^2} + \frac{B^{N-3}}{(N+1)^2} + \frac{B^{N-4}}{(N+1)^2}\right\},$$

for a universal constant $C_i > 0$.

Proof. To begin, we consider (3.4a) and estimate

$$\begin{aligned} h^2 \left\|\left|F_N\right|\right\|_{V^{s+1}} \leq & \, 2h \left\|\left|[\![\eta\partial_x u]\!]_N\right|\right\|_{V^{s+2}} + hY \left\|\left|[\![(\partial_x\eta)\partial_y u]\!]_N\right|\right\|_{V^{s+2}} \\ & + hY \left\|\left|[\![(\partial_x\eta)\partial_x u]\!]_N\right|\right\|_{V^{s+2}} + \left\|\left|\left[\!\left[\eta^2\partial_x u\right]\!\right]_N\right|\right\|_{V^{s+2}} \\ & + Y \left\|\left|[\![\eta(\partial_x\eta)\partial_y u]\!]_N\right|\right\|_{V^{s+2}} + Y \left\|\left|[\![\eta(\partial_x\eta)\partial_x u]\!]_N\right|\right\|_{V^{s+2}} \\ & + Y^2 \left\|\left|\left[\!\left[(\partial_x\eta)^2\partial_y u\right]\!\right]_N\right|\right\|_{V^{s+2}} + h \left\|\left|[\![(\partial_x\eta)\partial_x u]\!]_N\right|\right\|_{V^{s+1}} \\ & + \left\|\left|[\![\eta(\partial_x\eta)\partial_x u]\!]_N\right|\right\|_{V^{s+1}} + Y \left\|\left|\left[\!\left[(\partial_x\eta)^2\partial_y u\right]\!\right]_N\right|\right\|_{V^{s+1}}. \end{aligned}$$

From Theorem B.1 we find, since $s+2, s+1 \geq 4$,

$$\begin{aligned} h^2 \left\|\left|F_N\right|\right\|_{V^{s+1}} \leq & \left\{2hC[\eta, \partial_x u] + hYC[\partial_x\eta, \partial_y u] \right. \\ & \left. + hYC[\partial_x\eta, \partial_x u] + hC[\partial_x\eta, \partial_x u]\right\} \frac{B^{N-2}}{(N+1)^2} \\ & + \left\{C[\eta, \eta, \partial_x u] + YC[\eta, \partial_x\eta, \partial_y u] + YC[\eta, \partial_x\eta, \partial_x u] \right. \\ & \left. + Y^2 C[\partial_x\eta, \partial_x\eta, \partial_y u] + C[\eta, \partial_x\eta, \partial_x u] + YC[\partial_x\eta, \partial_x\eta, \partial_y u]\right\} \frac{B^{N-3}}{(N+1)^2}, \end{aligned}$$

where we have used $\eta \in X^{s+3}$ and $u \in V^{s+3}$. Since we have chosen the same constant C for the estimates above we find

$$\begin{aligned} h^2 \left\|\left|F_N\right|\right\|_{V^{s+1}} \leq & \, (3h + 2hY)C^2 M\Sigma \frac{B^{N-2}}{(N+1)^2} \\ & + (2 + 3Y + Y^2)C^3 M^2 \Sigma \frac{B^{N-3}}{(N+1)^2}, \end{aligned}$$

and we are done provided

$$C_i > \frac{1}{h^2} \max\{(3h + 2hY)CM\Sigma, (2 + 3Y + Y^2)C^2 M^2 \Sigma\}.$$

We continue by considering (3.4b) and estimate

$$\begin{aligned} h \left\|Q_N\right\|_{X^{s+1}} \leq & \left\|[\![\eta(\partial_t\eta)]\!]_N\right\|_{X^{s+1}} + 2\mu \left\|\left[\!\left[\eta(\partial_x^2\eta)\right]\!\right]_N\right\|_{X^{s+1}} + h \left\|[\![(\partial_x\eta)\partial_x u]\!]_N\right\|_{X^{s+1}} \\ & + \left\|[\![\eta(\partial_x\eta)\partial_x u]\!]_N\right\|_{X^{s+1}} + h \left\|\left[\!\left[(\partial_x\eta)^2\partial_y u\right]\!\right]_N\right\|_{X^{s+1}}. \end{aligned}$$

From Theorem B.1 we find, since $s+1 \geq 4$,

$$h \|Q_N\|_{X^{s+1}} \leq \left\{ C[\eta, \partial_t \eta] + 2\mu C[\eta, \partial_x^2 \eta] + hC[\partial_x \eta, \partial_x u] \right\} \frac{B^{N-2}}{(N+1)^2} + \left\{ C[\eta, \partial_x \eta, \partial_x u] + hC[\partial_x \eta, \partial_x \eta, \partial_y u] \right\} \frac{B^{N-3}}{(N+1)^2},$$

where we have used $\eta_n \in X^{s+3}$, $\partial_t \eta_n \in X^{s+1}$, and $u_n(x, 0, t) \in X^{s+5/2}$. Again, as we have chosen the same constant C for the estimates above, we find

$$h \|Q_N\|_{X^{s+1}} \leq (1 + 2\mu + h) C^2 M \Sigma \frac{B^{N-2}}{(N+1)^2} + (1+h) C^3 M^2 \Sigma \frac{B^{N-3}}{(N+1)^2},$$

and we are done provided

$$C_i > \frac{1}{h} \max\{(1 + 2\mu + h) C M \Sigma, (1+h) C^2 M^2 \Sigma\}.$$

Finally, we consider R_N and for this we require the following estimate on H_N from (3.6). If

$$\|\eta_n\|_{X^{s+3}} \leq C \frac{B^{n-1}}{(n+1)^2}, \quad \forall n < N,$$

then

$$\|H_N\|_{X^{s+2}} \leq C_i C \frac{B^{N-2}}{(N+1)^2}.$$

This can be established either by an argument analogous to the one given here for $\{F_N, Q_N, R_N\}$, or by simply appealing to the fact that the composition of two analytic functions is analytic. With this fact we return to (3.4c) and estimate

$$\begin{aligned} h^2 \|R_N\|_{X^{s+1/2}} \leq\, & 2h \left\| [\![\eta \partial_t u]\!]_N \right\|_{X^{s+1/2}} + \left\| [\![\eta^2 \partial_t u]\!]_N \right\|_{X^{s+1/2}} + h^2 \left\| [\![(\partial_t \eta) \partial_y u]\!]_N \right\|_{X^{s+1/2}} \\ & + h \left\| [\![\eta (\partial_t \eta) \partial_y u]\!]_N \right\|_{X^{s+1/2}} + 2gh \left\| [\![\eta^2]\!]_N \right\|_{X^{s+1/2}} + g \left\| [\![\eta^3]\!]_N \right\|_{X^{s+1/2}} \\ & + 2\sigma h \left\| [\![\eta (\partial_x^2 \eta)]\!]_N \right\|_{X^{s+1/2}} + \sigma \left\| [\![\eta^2 (\partial_x^2 \eta)]\!]_N \right\|_{X^{s+1/2}} \\ & + \sigma h^2 \left\| [\![\partial_x [(\partial_x \eta) H(\partial_x \eta)]]\!]_N \right\|_{X^{s+1/2}} + 2\sigma h \left\| [\![\eta \partial_x [(\partial_x \eta) H(\partial_x \eta)]]\!]_N \right\|_{X^{s+1/2}} \\ & + \sigma \left\| [\![\eta^2 \partial_x [(\partial_x \eta) H(\partial_x \eta)]]\!]_N \right\|_{X^{s+1/2}} \\ & + \frac{1}{2} \left\{ h^2 \left\| [\![(\partial_x u)^2]\!]_N \right\|_{X^{s+1/2}} + 2h \left\| [\![\eta (\partial_x u)^2]\!]_N \right\|_{X^{s+1/2}} \right. \\ & + \left\| [\![\eta^2 (\partial_x u)^2]\!]_N \right\|_{X^{s+1/2}} + 2h \left\| [\![(\partial_x \eta)(\partial_x u) \partial_y u]\!]_N \right\|_{X^{s+1/2}} \end{aligned}$$

$$+2h \left\| [\![\eta(\partial_x \eta)(\partial_x u)\partial_y u]\!]_N \right\|_{X^{s+1/2}} + h^2 \left\| \left[\!\!\left[(\partial_x \eta)^2 (\partial_y u)^2\right]\!\!\right]_N \right\|_{X^{s+1/2}} \Big\}$$
$$+ \frac{1}{2} h^2 \left\| \left[\!\!\left[(\partial_y u)^2\right]\!\!\right]_N \right\|_{X^{s+1/2}}.$$

Once again using Theorem B.1 we find, since $s + 1/2 \geq 4$,

$$\begin{aligned} h^2 \left\| R_N \right\|_{X^{s+1/2}} \leq &\Big\{ 2hC[\eta, \partial_t u] + h^2 C[\partial_t \eta, \partial_y u] + 2ghC[\eta, \eta] + 2\sigma h C[\eta, \partial_x^2 \eta] \\ &+ \sigma h^2 C[\partial_x \eta, H] + \frac{h^2}{2} C[\partial_x u, \partial_x u] + \frac{h^2}{2} C[\partial_y u, \partial_y u] \Big\} \frac{B^{N-2}}{(N+1)^2} \\ &+ \Big\{ C[\eta, \eta, \partial_t u] + hC[\eta, \partial_t \eta, \partial_y u] + gC[\eta, \eta, \eta] + \sigma C[\eta, \eta, \partial_x^2 \eta] \\ &+ 2\sigma h C[\eta, \partial_x \eta, H] + hC[\eta, \partial_x u, \partial_x u] + hC[\partial_x \eta, \partial_x u, \partial_y u] \Big\} \frac{B^{N-3}}{(N+1)^2} \\ &+ \Big\{ \sigma C[\eta, \eta, \partial_x \eta, H] + \frac{1}{2} C[\eta, \eta, \partial_x u, \partial_x u] + hC[\eta, \partial_x \eta, \partial_x u, \partial_y u] \\ &+ \frac{h^2}{2} C[\partial_x \eta, \partial_x \eta, \partial_y u, \partial_y u] \Big\} \frac{B^{N-4}}{(N+1)^2}, \end{aligned}$$

where we have used $\eta_n \in X^{s+3}$, $\partial_t \eta_n \in X^{s+1}$, $u_n(x, 0, t) \in X^{s+5/2}$, and $\partial_t u_n(x, 0, t) \in X^{s+1/2}$. Finally, as above, since we have chosen the same constant C for the estimates above we find

$$\begin{aligned} h^2 \left\| R_N \right\|_{X^{s+1/2}} \leq &\left(2(1 + g + \sigma)h + (2 + \sigma)h^2 \right) \frac{B^{N-2}}{(N+1)^2} + (1 + 3h + \sigma + g + 2\sigma h) \frac{B^{N-3}}{(N+1)^2} \\ &+ \left(\sigma + \frac{1}{2} + h + \frac{h^2}{2} \right) \frac{B^{N-4}}{(N+1)^2}, \end{aligned}$$

and we are done provided

$$\begin{aligned} C_i > \frac{1}{h^2} \max \Big\{ &\left(2(1 + g + \sigma)h + (2 + \sigma)h^2 \right) CM\Sigma, (1 + 3h + \sigma + g + 2\sigma h) C^2 M^2 \Sigma, \\ &\left(\sigma + \frac{1}{2} + h + \frac{h^2}{2} \right) C^3 M^3 \Sigma \Big\}. \quad \square \end{aligned}$$

9. Well-posedness proof

At last we are in a position to establish our main result.

Theorem 9.1. *Given an integer $s \geq 4$, if $\eta^{(0)} \in H^{s+2}$ and $\xi^{(0)} \in H^{s+3/2}$ then there exists a unique solution, of (3.1) of the form (3.2) satisfying*

$$\max\left\{\|\eta_n\|_{X^{s+3}}, \|\partial_t\eta_n\|_{X^{s+1}}, \|u_n(x,0,t)\|_{X^{s+5/2}}, \|\partial_t u_n(x,0,t)\|_{X^{s+1/2}}, |||u_n|||_{V^{s+3}}\right\} \leq C\frac{B^{n-1}}{(n+1)^2}, \quad \forall n>0, \quad (9.1)$$

for universal constants $C, B > 0$.

Proof. We work by induction on n and begin at order $n = 1$ where (3.3) gives us

$$\begin{aligned}
&\Delta u_1 = 0, && -h < y < 0,\\
&\partial_y u_1 = 0, && y = -h,\\
&\partial_t \eta_1 = \partial_y u_1 + 2\mu\partial_x^2\eta_1, && y = 0,\\
&\partial_t u_1 = -g\eta_1 + \sigma\partial_x^2\eta_1 - 2\mu\partial_y^2 u_1, && y = 0,\\
&u_1(x,0,0) = \xi^{(0)}(x),\\
&\eta_1(x,0) = \eta^{(0)}(x).
\end{aligned}$$

This can be solved explicitly and we set

$$C := \max\left\{\|\eta_1\|_{X^{s+3}}, \|\partial_t\eta_1\|_{X^{s+1}}, \|u_1(x,0,t)\|_{X^{s+5/2}}, \|\partial_t u_1(x,0,t)\|_{X^{s+1/2}}, |||u_1|||_{V^{s+3}}\right\},$$

which, of course, depends upon $\left\|\eta^{(0)}\right\|_{H^{s+2}}$ and $\left\|\xi^{(0)}\right\|_{H^{s+3/2}}$. We now assume estimate (9.1) for all $n < N$, and apply Lemma 7.1 to (3.3) at order N to realize

$$\max\left\{\|\eta_N\|_{X^{s+3}}, \|\partial_t\eta_N\|_{X^{s+1}}, \|u_N(x,0,t)\|_{X^{s+5/2}}, \|\partial_t u_N(x,0,t)\|_{X^{s+1/2}}, |||u_N|||_{V^{s+3}}\right\} \leq K\left\{\|Q_N\|_{X^{s+1}} + \|R_N\|_{X^{s+1/2}} + |||F_N|||_{V^{s+1}}\right\},$$

where we have used that $\eta^{(0)} \equiv \xi^{(0)} \equiv 0$ for $n > 1$. From Lemma 8.1 we have

$$\max\left\{\|\eta_N\|_{X^{s+3}}, \|\partial_t\eta_N\|_{X^{s+1}}, \|u_N(x,0,t)\|_{X^{s+5/2}}, \|\partial_t u_N(x,0,t)\|_{X^{s+1/2}}, |||u_N|||_{V^{s+3}}\right\} \leq KC_iC\left\{\frac{B^{N-2}}{(N+1)^2} + \frac{B^{N-3}}{(N+1)^2} + \frac{B^{N-4}}{(N+1)^2}\right\},$$

and we are done if we choose $B > \max\{KC_i, 1\}/3$. □

Remark 9.2. Before closing, we remark on a limitation of our method of proof. There is clearly a very specific choice of function spaces for the unknowns: $\eta_n \in X^{s+3}$, $\xi_n \in X^{s+5/2}$, $u_n \in V^{s+3}$. One can wonder if these can be changed. However, we believe that these choices are fixed for the following reasons:

1. Since u_n represents the field in the solution of an elliptic equation and ξ_n is its trace, it must be the case that

$$u_n \in V^t \quad \Longleftrightarrow \quad \xi_n \in X^{t-1/2}.$$

2. Our change of variables induces a relationship between the field and the surface deformation, namely, since $\Delta u_n = F_n$ and F_n involves the second derivative of η_n, cf. (3.4a), it appears that

$$u_n \in V^t \quad \Longleftrightarrow \quad \eta_n \in X^t.$$

3. Finally, the parabolic estimate *with* capillarity features the balance

$$\eta_n \in X^t \quad \Longleftrightarrow \quad \xi_n \in X^{t-1/2}.$$

So, if we select $t = s + 3$ we can satisfy all three demands. However, if we drop the capillarity term the final balance in Point 3 becomes

$$\eta_n \in X^t \quad \Longleftrightarrow \quad \xi_n \in X^{t+1/2},$$

and our argument falls apart. However, it is quite possible that a different change of variables or a more subtle analysis could "improve" the relationship in Point 2 and allow us to consider waves without capillarity.

Remark 9.3. We mention that another motivation for this work is the goal of modeling the Faraday wave experiment [18] which considers the motion of the free air-fluid interface of a container of fluid which is being periodically shaken from below. We believe that the viscous water waves problem presented here will be a reasonable model of this physical problem provided that a periodically modulated gravity is introduced, e.g., g replaced by $g + g\phi(t)$, $\phi(t + \Omega) = \phi(t)$. If the forcing is small, e.g. $\phi = \mathcal{O}(\varepsilon)$, then our new theorem can be used to establish existence and uniqueness of solutions; a couple of additional terms appear in the definition of R_n which are readily estimated. However, this is not completely satisfying as our results conclude that solutions decay exponentially as time evolves which is *not* the interesting regime of the Faraday wave experiment. To address the situation where $\phi = \mathcal{O}(1)$ requires an analysis of a new linearized parabolic problem which has the character of a Mathieu equation. We save such considerations for a future publication.

10. Conclusions

In this contribution we have established the existence and uniqueness of solutions to the capillary–gravity water wave problem supplemented with physically motivated viscosity. Our method of proof follows that of Friedman and Reitich in the contexts of the classical Stefan problem [19] and the capillary drop problem [20] which produces somewhat different results than those which can be attained by more standard techniques. It should be noted that due to the nature of the function spaces, the conclusion of the theorem is not only the well-posedness of our model, but also the stability of our solutions. More specifically, we discover *exponential* decay in time with the rate determined by the value of the viscosity. Thus, not only do unique solutions exist, they persist globally in time and decay exponentially fast to zero.

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Appendix A. Proof of the trace inequality

The goal in this appendix is the proof of Lemma 4.3.

Proof of Lemma 4.3. We begin by showing that

$$\|\sigma(x,0)\|_{H^{s-1}} \le C_t \|\sigma\|_{X^s} . \tag{A.1}$$

Following [19] we specify a function $\rho(x,t)$, defined for $0 \le x \le 2\pi$ and $-\infty < t < \infty$, which agrees with σ for $0 \le t \le 1$ and vanishes for $t \le -1$ and $t \ge 2$, such that

$$\begin{aligned}
M^2 &:= \int_{-1}^{2} \sum_{p=-\infty}^{\infty} \left[\langle p\rangle^{2s} \left|\hat{\rho}_p(u)\right|^2 + \langle p\rangle^{2s-4} \left|\partial_t \hat{\rho}_p(u)\right|^2 + \langle p\rangle^{2s-8} \left|\partial_t^2 \hat{\rho}_p(u)\right|^2 \right] du \\
&= \int_{-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left[\langle p\rangle^{2s} \left|\hat{\rho}_p(u)\right|^2 + \langle p\rangle^{2s-4} \left|\partial_t \hat{\rho}_p(u)\right|^2 + \langle p\rangle^{2s-8} \left|\partial_t^2 \hat{\rho}_p(u)\right|^2 \right] du \\
&= \int_{-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left\{ \langle p\rangle^{2s} + \langle \tau\rangle^2 \langle p\rangle^{2s-4} + \langle \tau\rangle^4 \langle p\rangle^{2s-8} \right\} \left|\tilde{\rho}_p(\tau)\right|^2 \, d\tau \\
&\le C \|\sigma\|_{X^s}^2 ,
\end{aligned}$$

where $\tilde{\rho}_p(\tau)$ is the space–time Fourier transform of ρ, and the penultimate equality comes from Parseval's relation. Since $\|\sigma(x,0)\|_{H^{s-1}} = \|\rho(x,0)\|_{H^{s-1}}$, to prove (A.1) it suffices to show that

$$\|\rho(x,0)\|_{H^{s-1}} \le C_t M.$$

Now, by interpolation [3], we have

$$\|\rho(x,0)\|_{H^{s-1}}^2 \le C \left\{ \|\rho(x,0)\|_{L^2}^2 + \left\|\partial_x^{s-1} \rho(x,0)\right\|_{L^2}^2 \right\},$$

and, from the classical trace theorem [3], we can bound the right hand side to deliver

$$\begin{aligned}
\|\rho(x,0)\|_{H^{s-1}}^2 &\le C \left\{ \|\rho\|_{H^{1/2}(dx,dt)}^2 + \left\|\partial_x^{s-1} \rho\right\|_{H^{1/2}(dx,dt)}^2 \right\} \\
&\le C \int_{-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left(1 + |p|^2 + |\tau|^2\right)^{1/2} \left(1 + |p|^{2(s-1)}\right) \left|\tilde{\rho}_p(\tau)\right|^2 \, d\tau.
\end{aligned} \tag{A.2}$$

Next, since $\sqrt{1+x^2} \leq 1+x$ for any $x>0$, we have

$$\left[1+|p|^2+|\tau|^2\right]^{1/2} \leq 1+\left[|p|^2+|\tau|^2\right]^{1/2} = 1+|p|\left[1+\left(\frac{|\tau|}{|p|}\right)^2\right]^{1/2}$$
$$\leq 1+|p|\left[1+\frac{|\tau|}{|p|}\right] = 1+|p|+|\tau|\,.$$

Thus we can conclude that

$$\|\rho(x,0)\|^2_{H^{s-1}} \leq C\int_{-\infty}^{\infty}\sum_{p=-\infty}^{\infty}(1+|p|+|\tau|)\left(1+|p|^{2(s-1)}\right)\left|\tilde{\rho}_p(\tau)\right|^2\,d\tau$$
$$= C\int_{-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\left(1+|p|^{2s-2}+|p|+|p|^{2s-1}+|\tau|+|\tau|\,|p|^{2s-2}\right)\left|\tilde{\rho}_p(\tau)\right|^2\,d\tau.$$

Now, all of the terms on the right hand side will be bounded by M^2 provided, for $|p|\,,|\tau|>1$,

$$\begin{aligned}
&1 \leq C\,|p|^{2s} && \text{requires } s \geq 0\\
&|p|^{2s-2} \leq C\,|p|^{2s} && \text{requires } s \geq 0\\
&|p| \leq C\,|p|^{2s} && \text{requires } s \geq 1/2\\
&|p|^{2s-1} \leq C\,|p|^{2s} && \text{requires } s \geq 0\\
&|\tau| \leq C\,|\tau|^2\,|p|^{2s-4} && \text{requires } s \geq 2,
\end{aligned}$$

and

$$|\tau|\,|p|^{2s-2} \leq C\,|\tau|^2\,|p|^{2s-4}\,,$$

which requires more analysis. We note that

$$|\tau|\,|p|^{2s-2} = |\tau|\,|p|^a\,|p|^b \leq \frac{1}{2}\left(|\tau|^2\,|p|^{2a}+|p|^{2b}\right),$$

where $a+b=2s-2$. For our estimate we set $2a=2s-4$ which demands that $b=2s-2-a=2s-2-2s+4=2$. In light of this we have the estimate

$$|\tau|\,|p|^{2s-2} \leq \frac{1}{2}\left(|\tau|^2\,|p|^{2s-4}+|p|^4\right),$$

and we are done provided that $2s \geq 4$, or $s \geq 2$.

We now move to establishing

$$\|\partial_t\sigma(x,0)\|_{H^{s-3}} \leq C_t\,\|\sigma\|_{X^s}\,. \tag{A.3}$$

The proof is identical to that presented above save that we must bound

$$\left(1+|p|^2+|\tau|^2\right)^{1/2}\left(1+|p|^{2(s-3)}\right)|\tau|^2,$$

cf., (A.2). All of the terms on the right hand side will be bounded by M provided

$$\begin{aligned}
&|\tau|^2 \le C\,|\tau|^2\,|p|^{2s-4} && \text{requires } s\ge 2\\
&|\tau|^2\,|p|^{2s-6} \le C\,|\tau|^2\,|p|^{2s-4} && \text{requires } s\ge 0\\
&|\tau|^2\,|p| \le C\,|\tau|^2\,|p|^{2s-4} && \text{requires } s\ge 5/2\\
&|\tau|^2\,|p|^{2s-5} \le C\tau^2\,|p|^{2s-4} && \text{requires } s\ge 0\\
&|\tau|^3 \le C\,|\tau|^4\,|p|^{2s-8} && \text{requires } s\ge 4,
\end{aligned}$$

and

$$|\tau|^3\,|p|^{2s-6} \le C\,|\tau|^4\,|p|^{2s-8},$$

which requires more analysis. We note that, from Hölder's Inequality,

$$|\tau|^3\,|p|^{2s-6} = |\tau|^3\,|p|^a\,|p|^b \le \frac{3}{4}\left(|\tau|^3\,|p|^a\right)^{4/3} + \frac{1}{4}\left(|p|^b\right)^4 = \frac{3}{4}\,|\tau|^4\,|p|^{4a/3} + \frac{1}{4}\,|p|^{4b}$$

where $a+b=2s-6$. For our estimate we set $4a/3 = 2s-8$, or $a=(3/2)s-6$, which demands that $b = 2s-6-a = 2s-6-(3/2)s+6 = (1/2)s$, or $4b = 2s$. In light of this we have the estimate

$$|\tau|^3\,|p|^{2s-6} \le \frac{3}{4}\,|\tau|^4\,|p|^{2s-8} + \frac{1}{4}\,|p|^{2s}$$

and we are done provided that $s\ge 4$. □

Appendix B. Products of analytic functions

In this section we collect some identities involving the products of analytic functions in terms of their Taylor series. To begin let suppose that A, B, C, D are analytic functions of ε so that the following Taylor series are convergent

$$D = D(\varepsilon) = \sum_{n=1}^{\infty} D_n\varepsilon^n, \quad E = E(\varepsilon) = \sum_{n=1}^{\infty} E_n\varepsilon^n, \tag{B.1a}$$

$$F = F(\varepsilon) = \sum_{n=1}^{\infty} F_n\varepsilon^n, \quad G = G(\varepsilon) = \sum_{n=1}^{\infty} G_n\varepsilon^n. \tag{B.1b}$$

It is not difficult to see that

$$D(\varepsilon)E(\varepsilon) = \sum_{n=2}^{\infty} [\![DE]\!]_n \varepsilon^n, \quad [\![DE]\!]_n := \sum_{m=1}^{n-1} D_{n-m} E_m, \tag{B.2a}$$

and

$$D(\varepsilon)E(\varepsilon)F(\varepsilon) = \sum_{n=3}^{\infty} [\![DEF]\!]_n \varepsilon^n, \quad [\![DEF]\!]_n := \sum_{m=2}^{n-1} \sum_{\ell=1}^{m-1} D_{n-m} E_{m-\ell} F_\ell, \tag{B.2b}$$

and

$$D(\varepsilon)E(\varepsilon)F(\varepsilon)G(\varepsilon) = \sum_{n=4}^{\infty} [\![DEFG]\!]_n \varepsilon^n, \quad [\![DEFG]\!]_n := \sum_{m=3}^{n-1} \sum_{\ell=1}^{m-1} \sum_{q=1}^{\ell-1} D_{n-m} E_{m-\ell} F_{\ell-q} G_q. \tag{B.2c}$$

For the results above to be true, the quantities $\{D, E, F, G\}$ need not be scalars and may be members of *any* normed linear space, Z. From these expansions we can prove the following fundamental result provided that the norm, $\|\cdot\|_Z$, satisfies the algebra property

$$\|DE\|_Z \le M \|D\|_Z \|E\|_Z, \tag{B.3}$$

for some $M > 0$, e.g., the spaces H^s, X^s, and V^s provided that s is large enough ($s \ge 4$ is certainly sufficient), cf. Lemma 4.1.

Theorem B.1. *Suppose that $D, E, F, G \in Z$, a normed linear space with norm satisfying the algebra property* (B.3). *If D, E, F, G are all analytic in ε with Taylor series expansions* (B.1) *such that*

$$\|D_n\|_Z < C_D \frac{B^{n-1}}{(n+1)^2}, \quad \|E_n\|_Z < C_E \frac{B^{n-1}}{(n+1)^2},$$
$$\|F_n\|_Z < C_F \frac{B^{n-1}}{(n+1)^2}, \quad \|G_n\|_Z < C_G \frac{B^{n-1}}{(n+1)^2},$$

for constants $C_D, C_E, C_F, C_G, B > 0$. Then $DE, DEF, DEFG \in Z$ are all analytic in ε as well, satisfying

$$\left\|[\![DE]\!]_n\right\|_Z < C[D,E] \frac{B^{n-2}}{(n+1)^2}, \quad \left\|[\![DEF]\!]_n\right\|_Z < C[D,E,F] \frac{B^{n-3}}{(n+1)^2}, \tag{B.4a}$$

$$\left\|[\![DEFG]\!]_n\right\|_Z < C[D,E,F,G] \frac{B^{n-4}}{(n+1)^2}, \tag{B.4b}$$

where

$$C[D,E]=C_D C_E M\Sigma,\quad C[D,E,F]=C_D C_E C_F M^2\Sigma,$$
$$C[D,E,F,G]=C_D C_E C_F C_G M^3\Sigma,$$

cf. *Lemma 4.5.*

Proof. The proof is straightforward and we only present it for the final estimate (B.4b). From (B.2c) we have

$$\begin{aligned}
\left\|[\![DEFG]\!]_n\right\|_Z &\le \sum_{m=3}^{n-1}\sum_{\ell=1}^{m-1}\sum_{q=1}^{\ell-1}\left\|D_{n-m}E_{m-\ell}F_{\ell-q}G_q\right\|_Z \\
&\le \sum_{m=3}^{n-1}\sum_{\ell=1}^{m-1}\sum_{q=1}^{\ell-1} M^3\left\|D_{n-m}\right\|_Z\left\|E_{m-\ell}\right\|_Z\left\|F_{\ell-q}\right\|_Z\left\|G_q\right\|_Z \\
&\le \sum_{m=3}^{n-1}\sum_{\ell=1}^{m-1}\sum_{q=1}^{\ell-1} M^3 C_D\frac{B^{n-m-1}}{(n-m+1)^2}C_E\frac{B^{m-\ell-1}}{(m-\ell+1)^2} \\
&\quad\times C_F\frac{B^{\ell-q-1}}{(\ell-q+1)^2}C_G\frac{B^{q-1}}{(q+1)^2} \\
&\le C_D C_E C_F C_G M^3\frac{B^{n-4}}{(n+1)^2} \\
&\quad\times\sum_{m=3}^{n-1}\sum_{\ell=1}^{m-1}\sum_{q=1}^{\ell-1}\frac{(n+1)^2}{(n-m+1)^2(m-\ell+1)^2(\ell-q+1)^2(q+1)^2} \\
&\le C_D C_E C_F C_G M^3\Sigma\frac{B^{n-4}}{(n+1)^2},
\end{aligned}$$

from Lemma 4.5. The estimates (B.4) can readily used in an inductive proof of the analyticity of all of the products DE, DEF, and $DEFG$. □

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