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Numerical Simulation of a Weakly Nonlinear Model for Water Waves with Viscosity

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Abstract The potential flow equations which govern the free-surface motion of an ideal fluid (the water wave problem) are notoriously difficult to solve for a number of reasons. First, they are a classical free-boundary problem where the domain shape is one of the unknowns to be found. Additionally, they are strongly nonlinear (with derivatives appearing in the nonlinearity) without a natural dissipation mechanism so that spurious high-frequency modes are not damped. In this contribution we address the latter of these difficulties using a surface formulation (which addresses the former complication) supplemented with physically-motivated viscous effects recently derived by Dias et al. (Phys. Lett. A 372:1297–1302, 2008). The novelty of our approach is to derive a weakly nonlinear model from the surface formulation of Zakharov (J. Appl. Mech. Tech. Phys. 9:190–194, 1968) and Craig and Sulem (J. Comput. Phys. 108:73–83, 1993), complemented with the viscous effects mentioned above. Our new model is simple to implement while being both faithful to the physics of the problem and extremely stable numerically.

Keywords Water waves · Weak surface viscosity · Weakly nonlinear model · Spectral method · Filtering

1 Introduction

The free-surface evolution of surface ocean waves is important in a wide array of engineering applications from wave-structure interactions in deep-sea oil rig design, to the shoaling and breaking of waves in near-shore regions, to the transport and dispersion of pollutants in lakes, seas, and oceans. The potential flow equations which model this "water wave problem" [10] are notoriously difficult for numerical schemes to simulate and the most successful approaches involve sophisticated integral equation formulations, subtle quadrature rules, and preconditioned iterative solution methods accelerated by, e.g., Fast Multipole methods

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(see [7, 8]). In this paper we propose a new model which is not only simple to implement numerically, but also incorporates a physically motivated dissipation mechanism to overcome some of the difficulties mentioned above.

The computation of these surface water waves is challenging for several reasons. The most important are that the domain of definition of the problem is one of the unknowns, and that the equations are strongly nonlinear (with derivatives appearing in the nonlinearity) without a natural dissipation mechanism to damp the growth of spurious, high-frequency modes. One method for addressing the first difficulty, and reducing the size of the computational domain by a dimension, is to resort to a surface formulation. One way to accomplish this is to utilize surface integrals (for a sampling of the vast literature on this subject see the survey articles of [4] and, from the *Annual Review of Fluid Mechanics* [11, 16–18, 21]). Another approach, which we follow here, is to use the Hamiltonian surface formulation of Zakharov [22] which was augmented and simplified by Craig and Sulem [3] (see also the closely related work of Watson and West [20], West et al. [19], and Milder [12]). The contribution of Craig and Sulem to the formulation was the introduction of the Dirichlet–Neumann operator (DNO)—in this context an operator which inputs surface Dirichlet data for Laplace's equation inside the fluid domain and produces surface Neumann data—together with a perturbative method for its calculation. In this paper we will use this perturbative approach to surface operators to derive a weakly nonlinear model for the water wave problem.

Recently, Dias et al. [5] have generalized the water wave problem to incorporate weak surface viscosity effects. While their derivation is not completely rigorous (e.g., they consider irrotational flows though viscosity will certainly destroy this property), it is correct in the linear wave limit, and they argue that it is a viable model in the case of small viscosity. The reason for putting forward a viscous water wave model is that it is significantly simpler to numerically simulate and mathematically analyze than the full Navier–Stokes equations posed on a moving domain. In this work we take a slightly different point of view to Dias, Dyachenko, and Zakharov's (DDZ's) model: It provides a physically-motivated mechanism for adding dissipation to the water wave equations. This is important since Craig and Sulem's [3] implementation of Zakharov's equations for inviscid flows required significant filtering in order to stabilize their computations. Our new contribution is to argue that it is more natural to consider the DDZ model with very small viscosity for stabilized, inviscid water wave simulations. However, in this initial contribution we further simplify the DDZ equations to include only linear and quadratic contributions thereby constituting a weakly nonlinear model for viscous water waves. This approach has the advantage of capturing nearly all of the essential linear and nonlinear effects seen in mildly nonlinear water waves, while being considerably simpler to implement than the full DDZ equations.

The organization of the paper is as follows: In Sect. 2 we recall the governing equations of the water wave problem and, in Sect. 2.1, the surface formulation due to Zakharov and Craig and Sulem. In Sect. 2.2 we discuss analyticity properties of surface operators which play a crucial role in the surface formulation we employ, in Sect. 2.3 we derive the weakly nonlinear model which is the focus of this paper, and in Sect. 2.4 we discuss a nondimensionalization of free-surface flows. In Sect. 2.5 we derive two classes of exact solutions of our new model in the special cases of linear viscous waves and inviscid traveling waves, respectively. Finally, we present numerical results in Sect. 3 and concluding remarks in Sect. 4.

2 Governing Equations

To begin, we consider the potential flow equations which model the free-surface evolution of a deep, two-dimensional ideal (irrotational, incompressible, inviscid) fluid [10]. For this we define the fluid domain

$$S_\eta := \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid y < \eta(x,t)\},$$

where η measures the deviation of the fluid surface from the quiescent state at $y = 0$. The well-known governing equations of an ideal fluid under the influence of gravity [10] are

$$\Delta\varphi = 0 \quad \text{in } S_\eta \tag{2.1a}$$

$$\partial_y\varphi \to 0 \quad \text{as } y \to -\infty \tag{2.1b}$$

$$\partial_t\eta = \partial_y\varphi - (\partial_x\eta)\partial_x\varphi \quad \text{at } y = \eta \tag{2.1c}$$

$$\partial_t\varphi = -g\eta - \frac{1}{2}(\partial_x\varphi)^2 - \frac{1}{2}(\partial_y\varphi)^2 \quad \text{at } y = \eta, \tag{2.1d}$$

where g is the gravitational constant, and φ is the velocity potential (the velocity can be expressed as $\vec{V} = \nabla\varphi$). This must be supplemented with initial conditions:

$$\eta(x,0) = \eta_0(x), \qquad \varphi(x,y,0) = \varphi_0(x,y), \tag{2.1e}$$

though, from the theory of elliptic partial differential equations [6], the boundary data $\varphi(x, \eta(x,0), 0)$ suffices. Additionally, lateral boundary conditions must be specified, and for this we make the classical choice of periodicity:

$$\eta(x+2\pi,t) = \eta(x,t), \qquad \varphi(x+2\pi,y,t) = \varphi(x,y,t). \tag{2.1f}$$

In a recent contribution [5], Dias, Dyachenko, and Zakharov proposed a modification of (2.1) to take into account some weak effects of surface viscosity:

$$\Delta\varphi = 0 \quad \text{in } S_\eta \tag{2.2a}$$

$$\partial_y\varphi \to 0 \quad y \to -\infty \tag{2.2b}$$

$$\partial_t\eta = \partial_y\varphi + 2\nu\partial_x^2\eta - (\partial_x\eta)\partial_x\varphi \quad \text{at } y = \eta \tag{2.2c}$$

$$\partial_t\varphi = -g\eta - 2\nu\partial_y^2\varphi - \frac{1}{2}(\partial_x\varphi)^2 - \frac{1}{2}(\partial_y\varphi)^2 \quad \text{at } y = \eta, \tag{2.2d}$$

together with initial conditions and periodicity. We point out that in this modification, only (2.2c) and (2.2d) are changed, each with the addition of a *linear* term scaled by the viscosity ν.

2.1 Surface Variables

We now follow the approach of Zakharov [22] and Craig and Sulem [3] and note that to solve (2.2) it is sufficient to find $\{\eta(x,t), \xi(x,t)\}$ where

$$\xi(x,t) := \varphi(x, \eta(x,t), t) \tag{2.3}$$

is the velocity potential at the free surface; the potential *inside* the fluid domain can be found from these boundary values and an appropriate integral formula. It is clear from (2.2) that it will be necessary to produce first and second order derivatives of the velocity potential at the surface in order to close the system of equations for $\{\eta, \xi\}$. With this in mind we define the following maps: Given a solution of the prototype elliptic problem

$$\Delta v = 0 \quad y < \sigma(x) \tag{2.4a}$$

$$\partial_y v \to 0 \quad y \to -\infty \tag{2.4b}$$

$$v = \zeta \quad \text{at } y = \sigma(x), \tag{2.4c}$$

compute

$$X(\sigma)[\zeta] := \partial_x v(x,\sigma), \qquad Y(\sigma)[\zeta] := \partial_y v(x,\sigma), \qquad Z(\sigma)[\zeta] := \partial_y^2 v(x,\sigma), \tag{2.5}$$

which are surface operators closely related to the classical Dirichlet–Neumann operator [3, 14]. In terms of these operators the surface formulation of the water wave equations with viscosity now reads:

$$\partial_t \eta = Y(\eta)[\xi] + 2\nu\partial_x^2\eta - (\partial_x\eta)X(\eta)[\xi] \tag{2.6a}$$

$$\begin{aligned}\partial_t \xi = {} & -g\eta - 2\nu Z(\eta)[\xi] + \frac{1}{2}(Y(\eta)[\xi])^2 - \frac{1}{2}(X(\eta)[\xi])^2 + 2\nu(\partial_x^2\eta)Y(\eta)[\xi] \\ & - (\partial_x\eta)X(\eta)[\xi]Y(\eta)[\xi].\end{aligned} \tag{2.6b}$$

2.2 Analytic Dependence of Surface Integral Operators

An important theorem, which will be useful for our later developments, regarding the operators X, Y, and Z from (2.5) is that they depend analytically upon the surface deformation $\sigma(x)$. More precisely, if we set $\sigma(x) = \varepsilon f(x)$ then the following series converge strongly

$$X(\varepsilon f) = \sum_{n=0}^{\infty} X_n(f)\varepsilon^n, \qquad Y(\varepsilon f) = \sum_{n=0}^{\infty} Y_n(f)\varepsilon^n, \qquad Z(\varepsilon f) = \sum_{n=0}^{\infty} Z_n(f)\varepsilon^n. \tag{2.7}$$

While this result is actually new for all three operators, it is easily derived from our previous work [14] and we therefore omit both the formal statement and proof.

Our aim in this paper is to study a set of model equations of (2.6) valid in the weakly nonlinear regime, $\{\eta, \xi\} \ll 1$, accurate to quadratic order. For this it is crucial to have expressions for the operators X_0, X_1, Y_0, Y_1, Z_0, and Z_1. To find these we use the method of Operator Expansions [3, 13] which we now describe. Consider the harmonic function

$$\varphi_p(x,y) = e^{ipx+|p|y}, \qquad p \in \mathbf{Z}$$

which satisfies (2.4a) and (2.4b) and is 2π-periodic in x. Focusing now upon the operator $Y(\sigma)[\zeta]$ we have, from the definition in (2.5):

$$Y(\sigma)\left[e^{ipx+|p|\sigma}\right] = |p|e^{ipx+|p|\sigma}.$$

Setting $\sigma = \varepsilon f$, substituting in the expansion for Y in (2.7), and using the Taylor series for the exponential we find

$$\left(\sum_{n=0}^{\infty} Y_n(f)\varepsilon^n\right)\left[e^{ipx}\sum_{n=0}^{\infty}(f^n/n!)|p|^n\varepsilon^n\right] = |p|e^{ipx}\sum_{n=0}^{\infty}(f^n/n!)|p|^n\varepsilon^n. \tag{2.8}$$

Equating at order zero we find

$$Y_0(f)[e^{ipx}] = |p|e^{ipx}$$

implying, if we use Fourier multiplier notation,

$$Y_0(f)[e^{ipx}] = |D|e^{ipx},$$

where $D := (1/i)\partial_x$. Recalling that any function of interest to us can be represented through its Fourier series, e.g.

$$\zeta(x) = \sum_{p=-\infty}^{\infty} \hat{\zeta}_p e^{ipx}, \qquad \hat{\zeta}_p = \frac{1}{2\pi}\int_0^{2\pi} \zeta(x)e^{-ipx}\,dx,$$

we conclude that

$$Y_0(f)[\zeta(x)] = |D|\zeta(x). \tag{2.9}$$

At order one in (2.8) we find

$$Y_1(f)[e^{ipx}] + Y_0(f)[f|p|e^{ipx}] = f|p|^2e^{ipx},$$

so that

$$Y_1(f)[e^{ipx}] + Y_0(f)[f|D|e^{ipx}] = f|D|^2e^{ipx}.$$

Again, representing a generic function ζ by its Fourier series

$$Y_1(f)[\zeta(x)] = f|D|^2\zeta(x) - Y_0(f)[f|D|\zeta(x)],$$

using (2.9) we find

$$Y_1(f)[\zeta(x)] = f|D|^2\zeta(x) - |D|[f|D|\zeta(x)]. \tag{2.10}$$

In a similar fashion we can find forms for X_0, X_1, Z_0, and Z_1:

$$X_0(f)[\zeta] = iD\zeta = \partial_x\zeta \tag{2.11a}$$

$$\begin{aligned} X_1(f)[\zeta] &= f(iD)|D|\zeta - (iD)[f|D|\zeta] \\ &= f\partial_x|D|\zeta - \partial_x[f|D|\zeta] = -(\partial_x f)(|D|\zeta) \end{aligned} \tag{2.11b}$$

$$Y_0(f)[\zeta] = |D|\zeta \tag{2.11c}$$

$$Y_1(f)[\zeta] = f|D|^2\zeta - |D|[f|D|\zeta] \tag{2.11d}$$

$$Z_0(f)[\zeta] = |D|^2\zeta \tag{2.11e}$$

$$Z_1(f)[\zeta] = f|D|^3\zeta - |D|^2[f|D|\zeta]. \tag{2.11f}$$

Noting that the zeroth order operators X_0, Y_0, Z_0 are independent of f, we suppress this notation from this point forward.

2.3 Weakly Nonlinear Model Equations

With the expansions of the previous section we can now form a weakly nonlinear approximation to (2.6). If we assume that η and ξ are small (and of the same order) then (2.6) implies:

$$\partial_t \eta = Y_0[\xi] + Y_1(\eta)[\xi] + 2\nu\partial_x^2\eta - (\partial_x\eta)X_0[\xi] + Q^\eta(\eta,\xi)$$
$$\partial_t \xi = -g\eta - 2\nu Z_0[\xi] - 2\nu Z_1(\eta)[\xi] + \frac{1}{2}(Y_0[\xi])^2 - \frac{1}{2}(X_0[\xi])^2 + 2\nu(\partial_x^2\eta)Y_0[\xi] + Q^\xi(\eta,\xi),$$

where Q^η and Q^ξ are of cubic or higher order in $\{\eta,\xi\}$. Using (2.11) and dropping the cubic and higher contributions, we find the weakly nonlinear approximation of (2.6) which we term WWV2 (Water Waves with Viscosity and order of approximation 2):

$$\partial_t \eta = |D|\xi + 2\nu\partial_x^2\eta + \eta|D|^2\xi - |D|[\eta|D|\xi] - (\partial_x\eta)\partial_x\xi \tag{2.12a}$$
$$\partial_t \xi = -g\eta - 2\nu|D|^2\xi - 2\nu\eta|D|^3\xi + 2\nu|D|^2[\eta|D|\xi] + \frac{1}{2}(|D|\xi)^2 - \frac{1}{2}(\partial_x\xi)^2 + 2\nu(\partial_x^2\eta)|D|\xi. \tag{2.12b}$$

2.4 Nondimensionalization

Equations (2.12) can be nondimensionalized using the classical scalings

$$x = \lambda x', \qquad y = \lambda y', \qquad t = \frac{\lambda}{\sqrt{g\lambda}}t', \qquad \eta = a\eta', \qquad \xi = a\sqrt{g\lambda}\xi',$$

where λ denotes a typical wavelength (which we will set to 2π), and a is a typical amplitude. Defining the nondimensional quantities

$$\alpha := \frac{a}{\lambda}, \qquad \beta := \frac{\nu}{\sqrt{g\lambda^3}},$$

it is easy to see that (2.12) transforms to (upon dropping primes)

$$\partial_t \eta = |D|\xi + 2\beta\partial_x^2\eta + \alpha\left\{\eta|D|^2\xi - |D|[\eta|D|\xi] - (\partial_x\eta)\partial_x\xi\right\} \tag{2.13a}$$
$$\partial_t \xi = -\eta - 2\beta|D|^2\xi + \alpha\left\{-2\beta\eta|D|^3\xi + 2\beta|D|^2[\eta|D|\xi] + \frac{1}{2}(|D|\xi)^2 - \frac{1}{2}(\partial_x\xi)^2 + 2\beta(\partial_x^2\eta)|D|\xi\right\}. \tag{2.13b}$$

In our numerical experiments we will vary the nondimensional viscosity β.

2.5 Exact Solutions

Here we present exact solutions of (2.12) in two special cases: Linear waves with viscosity, and nonlinear traveling waves without viscosity. We briefly present these for later use in our numerical convergence studies of Sects. 3.2 and 3.3. To begin, it is easy to see that the (dimensional) linearized water wave equations with viscosity have solution

$$\eta(x,t) = \sum_{p=-\infty}^{\infty} \hat{\eta}_p(t) e^{ipx}, \qquad \xi(x,t) = \sum_{p=-\infty}^{\infty} \hat{\xi}_p(t) e^{ipx},$$

with

$$\begin{pmatrix} \hat{\eta}_p(t) \\ \hat{\xi}_p(t) \end{pmatrix} = e^{-2\nu p^2 t} \begin{pmatrix} \cos(\omega_p t)\hat{\eta}_p(0) + \frac{|p|}{\omega_p}\sin(\omega_p t)\hat{\xi}_p(0) \\ -\frac{\omega_p}{|p|}\sin(\omega_p t)\hat{\eta}_p(0) + \cos(\omega_p t)\hat{\xi}_p(0) \end{pmatrix}, \tag{2.14a}$$

where $\omega_p^2 := g|p|$; in the case $p = 0$:

$$\begin{pmatrix} \hat{\eta}_0(t) \\ \hat{\xi}_0(t) \end{pmatrix} = \begin{pmatrix} \hat{\eta}_0(0) \\ g\hat{\eta}_0(0)t + \hat{\xi}_0(0) \end{pmatrix}. \tag{2.14b}$$

For a second exact solution, we now pursue formulas for the traveling wave solutions of (2.12) when $\nu = 0$ (Note that no traveling wave solutions exist for $\nu \neq 0$). Shifting to a traveling frame moving with speed c, (2.12) ($\nu = 0$) becomes

$$\begin{aligned} \partial_t \eta + c\partial_x \eta &= |D|\xi + \eta|D|^2\xi - |D|[\eta|D|\xi] - (\partial_x\eta)\partial_x\xi \\ \partial_t \xi + c\partial_x \xi &= -g\eta + \frac{1}{2}(|D|\xi)^2 - \frac{1}{2}(\partial_x\xi)^2. \end{aligned}$$

By seeking steady solutions we realize the following governing equations for the traveling waves

$$c\partial_x\eta - |D|\xi = \eta|D|^2\xi - |D|[\eta|D|\xi] - (\partial_x\eta)\partial_x\xi \tag{2.15a}$$

$$c\partial_x\xi + g\eta = \frac{1}{2}(|D|\xi)^2 - \frac{1}{2}(\partial_x\xi)^2, \tag{2.15b}$$

or, symbolically, $Bu = \tilde{R}$, where

$$B = \begin{pmatrix} c\partial_x & -|D| \\ g & c\partial_x \end{pmatrix}, \qquad u = \begin{pmatrix} \eta \\ \xi \end{pmatrix},$$

$$\tilde{R} = \begin{pmatrix} \eta|D|^2\xi - |D|[\eta|D|\xi] - (\partial_x\eta)\partial_x\xi \\ \frac{1}{2}(|D|\xi)^2 - \frac{1}{2}(\partial_x\xi)^2 \end{pmatrix}.$$

We now seek solutions of the traveling wave equation, $Bu = \tilde{R}$, in Taylor series as a function of a waveheight parameter δ:

$$u = u(x;\delta) = \sum_{n=1}^{\infty} u_n(x)\delta^n = \sum_{n=1}^{\infty} \begin{pmatrix} \eta_n(x) \\ \xi_n(x) \end{pmatrix} \delta^n, \tag{2.16}$$

and

$$c = c(\delta) = c_0 + \sum_{n=1}^{\infty} c_n \delta^n. \tag{2.17}$$

With this expansion for $c(\delta)$ in mind we rewrite the traveling wave equations as

$$B_0 u = R, \tag{2.18}$$

where

$$B_0 = \begin{pmatrix} c_0\partial_x & -|D| \\ g & c_0\partial_x \end{pmatrix},$$
$$R = \begin{pmatrix} \eta|D|^2\xi - |D|[\eta|D|\xi] - (\partial_x\eta)\partial_x\xi - (c-c_0)\partial_x\eta \\ \frac{1}{2}(|D|\xi)^2 - \frac{1}{2}(\partial_x\xi)^2 - (c-c_0)\partial_x\xi \end{pmatrix}.$$

Inserting the expansions (2.16) and (2.17) into (2.18) and gathering terms of order $\mathcal{O}(\delta)$ we have

$$B_0 u_1 = 0,$$

or, after switching to Fourier space,

$$\hat{B}_{0,p}\hat{u}_{1,p} = 0 \tag{2.19}$$

where $\hat{u}_{1,p}$ are the Fourier coefficients of $u_1(x)$ and

$$\hat{B}_{0,p} = \begin{pmatrix} c_0(ip) & -|p| \\ g & c_0(ip) \end{pmatrix}.$$

Of course (2.19) will only have non-trivial solutions if one (or more) of the operators $\hat{B}_{0,p}$ are singular which we measure by (the opposite of) the determinant functions

$$\Lambda(c_0, p) := (c_0 p)^2 - g|p|.$$

It is not difficult to show that given a $p_0 \in \Gamma' - \{0\}$, if one chooses c_0 such that $\Lambda(c_0, p_0) = 0$, i.e.

$$c_0 = \frac{\sqrt{g|p_0|}}{p_0},$$

then $\Lambda(c_0, p) \neq 0$ for $p \neq 0, \pm p_0$. In this way we can find solutions

$$\hat{u}_{1,p_0} = \alpha \begin{pmatrix} |p_0| \\ ic_0 p_0 \end{pmatrix}, \qquad \hat{u}_{1,-p_0} = \bar{\hat{u}}_{1,p_0},$$

and $\hat{u}_{1,p} = 0$ for $p \neq \pm p_0$. We note that this condition at $p = 0$ enforces the condition that the velocity potential at the surface satisfies

$$\int \xi(x)\,dx = 0 \tag{2.20}$$

which we now assume.

Having constructed this linear traveling wave at first order, we now insert the expansions (2.16) and (2.17) into (2.18) and see that for orders $n > 1$ we must satisfy

$$B_0 u_n = R_n = \begin{pmatrix} R_n^\eta \\ R_n^\xi \end{pmatrix} - c_{n-1}\begin{pmatrix} \partial_x \eta_1 \\ \partial_x \xi_1 \end{pmatrix}, \tag{2.21}$$

where

$$R_n^\eta = \sum_{l=1}^{n} \left\{ \eta_{n-l}|D|^2\xi_l - |D|[\eta_{n-l}|D|\xi_l] - (\partial_x \eta_{n-l})\partial_x \xi_l \right\} - \sum_{l=2}^{n-1} c_{n-l}\partial_x \eta_l,$$

and

$$R_n^\xi = \sum_{l=1}^{n} \left\{ \frac{1}{2}(|D|\xi_{n-l})(|D|\xi_l) - \frac{1}{2}(\partial_x \xi_{n-l})(\partial_x \xi_l) \right\} - \sum_{l=2}^{n-1} c_{n-l}\partial_x \xi_l.$$

Once again, (2.21) can be rewritten in terms of Fourier coefficients as

$$\hat{B}_{0,p}\hat{u}_{n,p} = \begin{pmatrix} \hat{R}^\eta{}_{n,p} \\ \hat{R}^\xi{}_{n,p} \end{pmatrix} - c_{n-1}\delta_{p,\pm p_0} \begin{pmatrix} (ip)\hat{\eta}_{1,p} \\ (ip)\hat{\xi}_{1,p} \end{pmatrix}, \tag{2.22}$$

where $\delta_{p,q}$ is the Kronecker delta function; note that the final terms involving c_{n-1} only appear at wavenumbers $p = \pm p_0$. To solve (2.22) we separate into three cases:

1. **Case $p \neq 0, \pm p_0$:**
   In this case $\hat{B}_{0,p}$ is invertible and the solution of (2.22) is simply

$$\hat{u}_{n,p} = \frac{-1}{\Lambda(c_0, p)} \begin{pmatrix} (ic_0 p)\hat{R}^\eta{}_{n,p} + |p|\hat{R}^\xi{}_{n,p} \\ -g\hat{R}^\eta{}_{n,p} + (ic_0 p)\hat{R}^\xi{}_{n,p} \end{pmatrix}.$$

2. **Case $p = 0$:**
   In this case (2.22) degenerates to just the second equation:

$$g\hat{\eta}_{n,0} = \hat{R}^\xi{}_{n,0}$$

   which is easily solved. The first equation is consistent because $\hat{R}^\eta{}_{n,0} = 0$ for all n. This can be seen from the computation:

$$\begin{aligned} R^\eta &= \eta|D|^2\xi - |D|[\eta|D|\xi] - (\partial_x\eta)\partial_x\xi \\ &= -\eta\partial_x^2\xi - |D|[\eta|D|\xi] - (\partial_x\eta)\partial_x\xi \\ &= -\partial_x[\eta\partial_x\xi] - |D|[\eta|D|\xi], \end{aligned}$$

   where we have used $|D|^2 = D^2 = (-i\partial_x)^2 = -\partial_x^2$. Since the operators ∂_x and $|D|$ map generic functions to functions with zeroth Fourier coefficient equal to zero, we have $\hat{R}^\eta{}_{n,0} = 0$ as claimed. Regarding $\hat{\xi}_{n,0}$, we simply set it to zero which enforces (2.20).
3. **Case $p = \pm p_0$:**
   For this we select the representative case $p = p_0$. Here the matrix $\hat{B}_{0,p_0}$ is, by design, singular, however, there is a free parameter on the right hand side, c_{n-1}. An easy exercise in Gaussian elimination shows that if

$$c_{n-1} = \frac{g\hat{R}^\eta{}_{n,p_0} - (ic_0 p_0)\hat{R}^\xi{}_{n,p_0}}{g(ip_0)\hat{\eta}_{1,p_0} + (c_0 p_0)p_0\hat{\xi}_{1,p_0}}$$

   then (2.22) is solvable. However, we are left with the issue of uniqueness and for this we follow the approach of Stokes (see [15]) for the full water wave problem, that η_n be

L^2-orthogonal to η_1. As η_1 is only supported at wavenumbers $p = \pm p_0$ this is easily enforced by setting

$$\hat{\eta}_{n,p_0} = 0,$$

and now (2.22) delivers

$$\hat{\xi}_{n,p_0} = \frac{\hat{R^{\xi}}_{n,p_0} - c_{n-1}(ip_0)\hat{\xi}_{1,p_0}}{ic_0 p_0}.$$

We remark that the case $p = -p_0$ can be easily addressed by setting $\hat{u}_{n,-p_0} = \bar{\hat{u}}_{n,p_0}$.

Remark 2.1 Using the procedure outlined above, it is possible to show that, in fact, the expansions (2.16) and (2.17) converge strongly in an appropriate function space. For details of such a proof we refer the interested reader to [15].

3 Numerical Method and Results

In this section we now present results which not only validate our numerical approach to approximating solutions of (2.12), but also display how our model equations can be used for flows with small viscosities. We show how our numerical simulations capture the viscous decay we expect for linear water wave flows from the exact solution we found in Sect. 2.5. We also utilize our new scheme to stabilize computations of inviscid water wave flows with this new, physically motivated dissipation.

3.1 Numerical Method

Briefly, our numerical scheme for approximating solutions of (2.12) is a Fourier spectral collocation method in the spatial variable coupled to a fourth order Runge–Kutta time-stepping scheme [2, 9]. In more detail, we approximate our problem unknowns $\{\eta(x,t), \xi(x,t)\}$ by

$$\eta^{N_x}(x,t) := \sum_{p=-N_x/2}^{N_x/2-1} d_p(t)e^{ipx}, \qquad \xi^{N_x}(x,t) := \sum_{p=-N_x/2}^{N_x/2-1} a_p(t)e^{ipx}, \tag{3.1}$$

where $\{d_p(t), a_p(t)\}$ are approximations to the Fourier coefficients $\{\hat{\eta}_p(t), \hat{\xi}_p(t)\}$. We enforce (2.12) at the equally spaced gridpoints $x_j = (2\pi j)/N_x$ $(j = 0, \dots, N_x - 1)$ and compute derivatives by appealing to (3.1) and the discrete Fourier transform (DFT), accelerated by the FFT algorithm. The Fourier multiplier $|D|$ is computed in a similar manner: After transforming to the Fourier side, via the DFT, we multiply pointwise by $|p|$ before returning to the physical side. Finally, products are computed on the physical side via inverse DFTs and pointwise multiplication, all of which is dealiased. All of this specifies a system of $2 \times N_x$ ordinary differential equations to solve which we approximate with the classical fourth order Runge–Kutta scheme yielding the approximations $\{d_p^{N_t}(t), a_p^{N_t}(t)\}$.

3.2 Spatial Convergence

Before presenting our numerical results, we verify our codes by displaying convergence of our numerically generated solutions to the exact solutions discussed earlier in Sect. 2.5:

Linear viscous water waves and nonlinear inviscid traveling water waves. In this section we focus upon the spatial convergence (with a fixed temporal discretization, $\Delta t = 2.45 \times 10^{-3}$) while in the next section we examine the temporal discretization (with a fixed spatial resolution). To accomplish this we consider the quantities:

$$e_\eta(N_x, N_t, T) := |\eta^{N_x,N_t}(\cdot, T) - \eta_{ex}(\cdot, T)|_{L^\infty} \tag{3.2a}$$

$$e_\xi(N_x, N_t, T) := |\xi^{N_x,N_t}(\cdot, T) - \xi_{ex}(\cdot, T)|_{L^\infty}, \tag{3.2b}$$

where $\{\eta_{ex}(x,t), \xi_{ex}(x,t)\}$ stand for exact solutions (e.g., for linear viscous water waves or traveling nonlinear inviscid water waves) and $\{\eta^{N_x,N_t}(x,t), \xi^{N_x,N_t}(x,t)\}$ are the corresponding numerical approximations.

We note that in all following simulations we have set $g = 1$ and chosen waves of periodicity 2π so that ν can be viewed as the non-dimensional quantity β from Sect. 2.4. Carrying this out with the exact solution of linear water waves, (2.14), with

$$\eta_0(x) = \frac{1}{10}\cos(x), \qquad \xi_0(x) = \frac{1}{10}\sin(x), \tag{3.3}$$

for final times $T = 2, 10$ we have the data presented in Tables 1 and 2, respectively; the results are presented for $\nu = 0, 0.01, 0.1$.

Table 1 Spatial convergence of simulation of solutions to the linearized water wave equations with viscosity with initial conditions, (3.3). The error, (3.2), is measured at time $T = 2$ and compared against the exact solution (2.14)

ν	N_x	e_η	e_ξ
0	16	1.66601×10^{-11}	1.66601×10^{-11}
	32	1.03899×10^{-12}	1.03898×10^{-12}
	64	5.89095×10^{-14}	5.90188×10^{-14}
0.01	16	1.59005×10^{-11}	1.59005×10^{-11}
	32	9.98009×10^{-13}	9.97957×10^{-13}
	64	5.68504×10^{-14}	5.68027×10^{-14}
0.1	16	1.21801×10^{-11}	1.21800×10^{-11}
	32	7.69267×10^{-13}	7.69260×10^{-13}
	64	4.36456×10^{-14}	4.36456×10^{-14}

Table 2 Spatial convergence of simulation of solutions to the linearized water wave equations with viscosity with initial conditions, (3.3). The error, (3.2), is measured at time $T = 10$ and compared against the exact solution (2.14)

ν	N_x	e_η	e_ξ
0	16	8.20554×10^{-11}	8.20558×10^{-11}
	32	5.22295×10^{-12}	5.22374×10^{-12}
	64	1.86613×10^{-13}	1.86566×10^{-13}
0.01	16	6.78144×10^{-11}	6.78143×10^{-11}
	32	4.26790×10^{-12}	4.26798×10^{-12}
	64	1.53369×10^{-13}	1.53350×10^{-13}
0.1	16	1.24571×10^{-11}	1.24572×10^{-11}
	32	7.79649×10^{-13}	7.79680×10^{-13}
	64	3.51351×10^{-14}	3.51837×10^{-14}

Table 3 Spatial convergence of simulation of solutions to the water wave equations with viscosity, (2.12), with $\nu = 0$. The initial conditions and exact solution are provided by the traveling wave solutions derived in Sect. 2.5, see (3.4). The error, (3.2), is measured at times $T = 2$ and $T = 10$

T	N_x	e_η	e_ξ
2	16	1.72501×10^{-12}	1.75794×10^{-12}
	32	1.09069×10^{-13}	1.09654×10^{-13}
	64	6.21920×10^{-15}	6.24397×10^{-15}
10	16	8.22048×10^{-12}	8.26373×10^{-12}
	32	5.24426×10^{-13}	5.26383×10^{-13}
	64	1.87376×10^{-14}	1.88053×10^{-14}

We further test the spatial convergence of our time-stepping algorithm by comparing with approximations of the traveling wave solutions in the inviscid case

$$u^M := \sum_{n=1}^{M} \begin{pmatrix} \eta_n(x) \\ \xi_n(x) \end{pmatrix} \delta^n, \qquad c^M := c_0 + \sum_{n=1}^{M} c_n \delta^n, \tag{3.4}$$

c.f., (2.16)–(2.17), with $M = 20$ and $\delta = 0.01$. The $\{\eta_n(x), \xi_n(x)\}$ are also discretized via a Fourier collocation method as in (3.1) where the resulting $\{d_{n,p}, a_{n,p}\}$ are, of course, independent of t. The results of this convergence study are presented in Table 3 for $T = 2, 10$ (with $\nu = 0$, of course).

3.3 Temporal Convergence

We now investigate the temporal rate of convergence of our scheme by fixing the number of spatial collocation points (at $N_x = 64$) and examining the quantities $\{e_\eta, e_\xi\}$, (3.2) as Δt is refined. Carrying this out with the exact solution of linear water waves, (2.14) (with initial conditions (3.3)), for final times $T = 2, 10$ we have the data presented in Tables 4 and 5, respectively; the results are presented for $\nu = 0, 0.01, 0.1$. If we fit these data to the error estimate:

$$\text{Error} \approx C \Delta t^r,$$

where the well-known theory tells us that r should be 4, using a least-squares procedure we have an experimentally determined value of $\bar{r} = 4.07$ for both η and ξ errors for all three values of ν at $T = 2$, and $\bar{r} = 4.40, 4.40, 4.23$ for $\nu = 0, 0.01, 0.1$ at $T = 10$.

We further test the convergence of our time-stepping algorithm by comparing with the traveling wave solutions in the inviscid case, (3.4) with $M = 20$ and $\delta = 0.01$. These results are presented in Table 6 for $T = 2, 10$ (with $\nu = 0$) and yield rates of convergence $\bar{r} = 4.07$, 4.40 for $T = 2,\ 10$, respectively.

3.4 Numerical Results

We now present numerical results which illustrate the properties of solutions of our model equations, (2.12), and the capabilities of our numerical simulation strategy. In particular, we display the decay rates of solutions to the *nonlinear* WWV2 equations, and then show how our numerical scheme can be used to *stably* compute inviscid surface water waves.

To begin, we note that, from the exact solution formula (2.14), *linear* solutions should decay like $e^{-2\nu p^2 t}$ at wavenumber p. We have numerically simulated such solutions (again using the initial conditions (3.3), so that $p = 1$) and in Table 7 report experimental decay

Table 4 Temporal convergence of simulation of solutions to the linearized water wave equations with viscosity with initial conditions, (3.3). The error, (3.2), is measured at time $T = 2$ and compared against the exact solution (2.14)

ν	Δt	e_η	e_ξ
0	9.82×10^{-3}	1.67×10^{-11}	1.67×10^{-11}
	4.91×10^{-3}	1.04×10^{-12}	1.04×10^{-12}
	2.45×10^{-3}	5.89×10^{-14}	5.90×10^{-14}
0.01	9.82×10^{-3}	1.60×10^{-11}	1.60×10^{-11}
	4.91×10^{-3}	9.99×10^{-13}	9.99×10^{-13}
	2.45×10^{-3}	5.69×10^{-14}	5.68×10^{-14}
0.1	9.82×10^{-3}	1.23×10^{-11}	1.23×10^{-11}
	4.91×10^{-3}	7.69×10^{-13}	7.69×10^{-13}
	2.45×10^{-3}	4.36×10^{-14}	4.36×10^{-14}

Table 5 Temporal convergence of simulation of solutions to the linearized water wave equations with viscosity with initial conditions, (3.3). The error, (3.2), is measured at time $T = 10$ and compared against the exact solution (2.14)

ν	Δt	e_η	e_ξ
0	9.82×10^{-3}	8.33×10^{-11}	8.33×10^{-11}
	4.91×10^{-3}	5.22×10^{-12}	5.22×10^{-12}
	2.45×10^{-3}	1.87×10^{-13}	1.87×10^{-14}
0.01	9.82×10^{-3}	6.83×10^{-11}	6.83×10^{-11}
	4.91×10^{-3}	4.28×10^{-12}	4.28×10^{-12}
	2.45×10^{-3}	1.53×10^{-13}	1.53×10^{-13}
0.1	9.82×10^{-3}	1.25×10^{-11}	1.25×10^{-11}
	4.91×10^{-3}	7.80×10^{-13}	7.80×10^{-13}
	2.45×10^{-3}	3.51×10^{-14}	3.51×10^{-14}

Table 6 Temporal convergence of simulation of solutions to the water wave equations with viscosity, (2.12), with $\nu = 0$. The initial conditions and exact solution are provided by the traveling wave solutions derived in Sect. 2.5, see (3.4). The error, (3.2), is measured at times $T = 2$ and $T = 10$

T	Δt	e_η	e_ξ
2	9.82×10^{-3}	1.75×10^{-12}	1.76×10^{-12}
	4.91×10^{-3}	1.09×10^{-13}	1.10×10^{-13}
	2.45×10^{-3}	6.22×10^{-15}	6.24×10^{-15}
10	9.82×10^{-3}	8.36×10^{-12}	8.39×10^{-12}
	4.91×10^{-3}	5.24×10^{-13}	5.26×10^{-13}
	2.45×10^{-3}	1.87×10^{-14}	1.88×10^{-14}

rates. We see that within a very small tolerance (e.g. 10^{-5}) the theoretical decay rate is realized. Additionally, we have evolved the traveling waveforms, (3.4), in the fully *nonlinear* WWV2 equations and report in Table 8 our results. We see how strong the effects of viscosity can be as these *nonlinear* solutions also decay at roughly the rate expected for linear solutions.

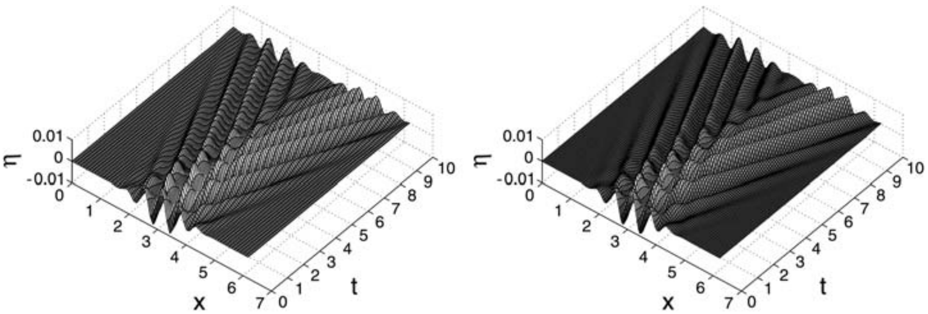
These results, while interesting in their own right, also suggest a new strategy for evolving *inviscid* surface water waves in a stable way. As noted in publications such as [3], the faithful computation of these waves is quite delicate and filtering is typically required to ensure that solutions do not blow up. The reason for this can be seen in the energy conserving nature of the equations (implying no natural energy dissipation mechanism) coupled to

Table 7 Rate of decay of the amplitude of simulated solutions of the linearized water wave equations with viscosity with initial conditions, (3.3). These amplitudes are measured at times $T = 2$ and $T = 10$

T	ν	$\bar{\nu}(\eta)$	$\bar{\nu}(\xi)$
2	0	-3.33×10^{-4}	-3.33×10^{-4}
	0.01	-2.03×10^{-2}	-2.03×10^{-2}
	0.1	-2.00×10^{-1}	-2.00×10^{-1}
10	0	-1.78×10^{-5}	-1.78×10^{-5}
	0.01	-2.00×10^{-2}	-2.00×10^{-2}
	0.1	-2.00×10^{-1}	-2.00×10^{-1}

Table 8 Rate of decay of the amplitude of simulated solutions of the water wave equation with viscosity, (2.12). The initial conditions are provided by the traveling wave solutions derived in Sect. 2.5, see (3.4). These amplitudes are measured at times $T = 2$ and $T = 10$

T	ν	$\bar{\nu}(\eta)$	$\bar{\nu}(\xi)$
2	0	-3.39×10^{-4}	-1.51×10^{-4}
	0.01	-2.05×10^{-2}	-2.01×10^{-2}
	0.1	-2.01×10^{-1}	-2.00×10^{-1}
10	0	-1.78×10^{-5}	-8.80×10^{-6}
	0.01	-2.01×10^{-2}	-2.00×10^{-2}
	0.1	-2.00×10^{-1}	-2.00×10^{-1}



(a) $N_x = 64$ (b) $N_x = 128$

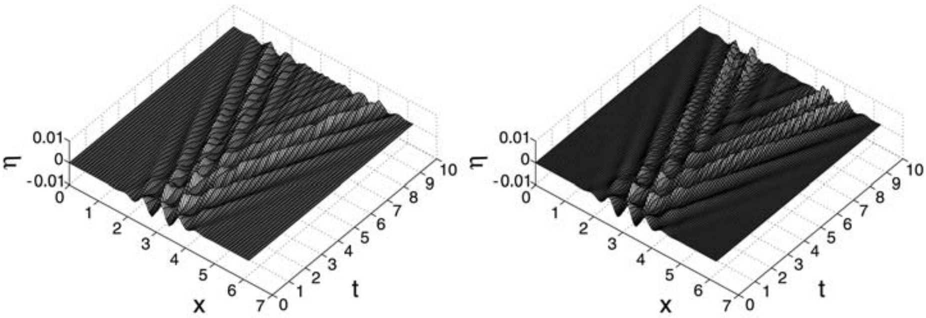
Fig. 1 Evolution of modulated cosine initial condition (3.5) ($A = 0.01$) in the water wave equations with viscosity, (2.12), with $\nu = 0$. Solution was evolved to $T = 10$ with $\Delta t = \Delta x/10$ for $N_x = 64$ and $N_x = 128$

very strong nonlinearities. Of course our new set of equations circumvent this first challenge with the introduction of viscous dissipation terms. Thus, it seems natural to consider the possibility of approximating *inviscid* water waves by solving *slightly* viscous equations.

We have carried out this program for the modulated cosine profile

$$\eta_0(x) = A\cos(10x)e^{-(4/3)(x-L/2)^2}, \qquad \xi_0(x) = 0, \tag{3.5}$$

proposed by Craig and Sulem [3]. To study the evolution of this profile we have chosen the same physical parameter values as those given in [3], namely $L = 2\pi$, $A = 0.01$, and final time $T = 10$. For this configuration we were able to satisfactorily evolve the initial conditions (3.5) without need for any filtering or viscosity (see Fig. 1), with $N_x = 64, 128$ and $\Delta t = \Delta x/10$.



(a) $N_x = 64$ (b) $N_x = 128$

Fig. 2 Evolution of modulated cosine initial condition (3.5) ($A = 0.045$) in the water wave equations with viscosity, (2.12), with $\nu > 0$. Solution was evolved to $T = 10$ with $\Delta t = \Delta x/10$ for $N_x = 64$ and $N_x = 128$ with viscosities $\nu = 2.4 \times 10^{-5}$ and $\nu = 1.095 \times 10^{-4}$, respectively

However, if A is increased to a value of $A = 0.045$ we found that with a moderate number of Fourier collocation points, $N_x = 64$, and a reasonable time-step, $\Delta t = \Delta x/10$, we were unable to resolve a believable solution. To make these ideas more precise we note that the inviscid version of our model equations have a conserved energy (Hamiltonian)

$$H := \frac{1}{2}\int_0^{2\pi} \xi \,\{Y_0[\xi] + Y_1(\eta)[\xi] - (\partial_x \eta) X_0[\xi]\} + g\eta^2 \, dx, \tag{3.6}$$

which can be derived from a weakly nonlinear expansion of the well-known energy for fully nonlinear water waves [3]. To measure the integrity of our solutions we measure the relative change in this energy from the initial to the final time:

$$e_H := \frac{H(t=T) - H(t=0)}{H(t=T)}.$$

In the case mentioned above ($A = 0.045$, $N_x = 64$) this relative error is approximately 0.35, while this quantity is unchanged if the time-step is reduced by a factor of 10. If the number of collocation points is increased to $N_x = 128$ then, for both $\Delta t = \Delta x/10$ and $\Delta t = \Delta x/100$, the solution blows up shortly after $t = 2$. By contrast, if we select $\nu = 2.4 \times 10^{-5}$ with $N_x = 64$ and $\Delta t = \Delta x/10$, then we can produce a solution which not only looks quite reasonable, see Fig. 2a, but also produces a relative energy error of $e_H \approx 7 \times 10^{-3}$, under 1%. If we select $\nu = 1.095 \times 10^{-4}$ with $N_x = 128$ and $\Delta t = \Delta x/10$, then we can compute the solution depicted in Fig. 2b with $e_H \approx 8 \times 10^{-2}$.

In a similar fashion we also investigated the slightly more nonlinear case $A = 0.05$. Here, regardless of our choice of N_x (64 or 128) or our time-step ($\Delta x/10$ or $\Delta x/100$) we were unable to obtain a finite solution at $T = 10$ using our code with $\nu = 0$. In this case filtering of some sort is *required*. However, if we set $\nu = 5.5 \times 10^{-5}$ then, again, we found a physically reasonable solution (Fig. 3a) with a relative energy error of $e_H \approx 6 \times 10^{-2}$, just over 6%. If we refine to $N_x = 128$ with $\Delta t = \Delta x/10$ then with $\nu = 1.9365 \times 10^{-4}$ we find a solution, Fig. 3b, with $e_H \approx 0.38$. While not really a very satisfactory solution, it at least provides a reasonable profile without finite-time blow-up.

While these simulations do show the promise of filtering inviscid water wave simulations with small viscous effects, the values of ν chosen were quite specific. In general we found

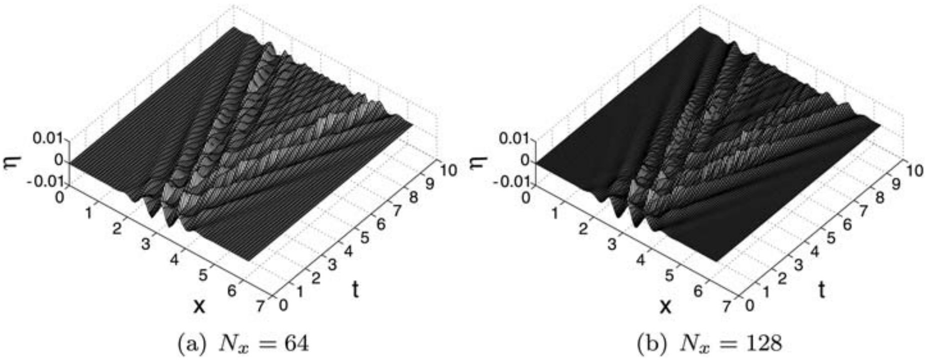


Fig. 3 Evolution of modulated cosine initial condition (3.5) ($A = 0.05$) in the water wave equations with viscosity, (2.12), with $\nu > 0$. Solution was evolved to $T = 10$ with $\Delta t = \Delta x/10$ for $N_x = 64$ and $N_x = 128$ with viscosities $\nu = 5.5 \times 10^{-5}$ and $\nu = 1.9365 \times 10^{-4}$, respectively

that values much larger than the ones chosen resulted in solutions which were overly damped and had energies tending to zero quite rapidly. On the other hand, if ν were chosen much smaller than those reported above, oftentimes solutions would blow up significantly before $T = 10$.

Remark 3.1 If we return to the original derivations of Dias et al. [5] we recall that the viscosity used is the kinematic viscosity which, for water at 15 degrees Celsius, is reported by Acheson [1] to be

$$\nu_{\text{water}} \approx 10^{-2}\ \text{cm/s}^2 = 10^{-6}\ \text{m/s}^2.$$

This, upon nondimensionalization, yields

$$\beta = \frac{\nu}{\sqrt{g\lambda^3}} \approx 3.2 \times 10^{-6}$$

if we use $\lambda = 1$ m. Thus, the values for which our scheme performs well are all in a close neighborhood of the kinematic viscosity appropriate for water.

4 Conclusions

In this paper we have taken the water wave equations with small viscosity effects of Dias et al. [5] and restated them in terms of the boundary quantities advocated by Zakharov [22] for a Hamiltonian formulation of the water wave problem, namely the surface shape and the surface velocity potential. Upon analyzing the relevant surface integral operators (related to the Dirichlet–Neumann operator), we used their analyticity properties to derive a new, second order weakly nonlinear model with small viscosity. We then outlined a new Fourier spectral collocation method for their numerical simulation and verified the accuracy and high-order convergence of this scheme. Finally, we displayed some preliminary results on the utilization of *very* weakly viscous water wave flows for the stable numerical simulation of very nonlinear water waves.

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