PHILOSOPHICAL TRANSACTIONS A

royalsocietypublishing.org/journal/rsta

Research

Article submitted to journal

Subject Areas:
35Q60, 78A60.

Keywords:
High–Order Perturbation Methods, Nonlinear Kerr Media, Dirichlet–Neumann Operators.

Author for correspondence:
David P. Nicholls
e-mail: davidn@uic.edu

A High–Order Perturbation Method for Analyzing the Dirichlet–Neumann Operator for a Nonlinear Kerr Medium

David P. Nicholls[1]

[1] Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
Chicago, IL 60607

It has recently been realized that illumination by intensely powerful radiation is not the only path to a nonlinear optical response by a given material. As demonstrated by Capretti *et al* for a layer of Indium Tin Oxide, strong nonlinear effects can be observed in a material for illuminating fields of quite moderate strength in a neighborhood of the wavelengths which render it an Epsilon Near Zero material. Inspired by these observations we introduce, discuss, and analyze a rather different formulation of the governing equations for the Capretti experiment with a view towards robust and highly accurate numerical simulation. In contrast to volumetric algorithms which are greatly disadvantaged for the piecewise homogeneous geometries we consider, surface methods provide optimal performance as they only consider interfacial unknowns. In this contribution we study an interfacial approach which is based upon Dirichlet–Neumann Operators (DNOs). We show that, for a layer of nonlinear Kerr medium, the DNO is not only well–defined, but also analytic with respect to all of its independent variables. Our method of proof is perturbative in nature and suggests several new avenues of investigation, including stable numerical simulation, and how one would include the effects of periodic deformations of the layer interfaces into both theory and numerical simulation of the resulting DNOs.

THE ROYAL SOCIETY
PUBLISHING

1. Introduction

With the invention of the laser in the 1960s it became clear that the nonlinear response of common materials to illumination by electromagnetic radiation could no longer be ignored for many phenomena of applied interest [1]. The list of fascinating and crucial applications of Nonlinear Optics is, by now, extensive and certainly beyond the scope of this short contribution, so we refer the interested reader to the survey texts of Moloney and Newell [1] and Boyd [2] for further details. We do point out the more recent realization that illumination by intensely powerful radiation is not the only path to a nonlinear response by a given material. As demonstrated by Capretti *et al* [3] for a layer of Indium Tin Oxide (ITO), strong nonlinear effects can be observed in a material for illuminating fields of quite moderate strength in a neighborhood of the wavelengths which render it an Epsilon Near Zero (ENZ) material (i.e., where the real part of the permittivity is nearly zero). These effects were subsequently examined in detail by Alam, De Leon, and Boyd [4] and others in the following years. Inspired by these observations we now introduce, discuss, and analyze a rather different formulation of the governing equations for the Capretti experiment [3] with a view towards robust and highly accurate numerical simulation.

On the topic of the numerical simulation of electromagnetic scattering problems we recommend the survey paper of Gallinet, Butet, and Martin [5]. Among the wide array of techniques available to the practitioner, volumetric approaches are very popular, particularly the Finite Difference Time Domain (FDTD) method [6], the Finite Element Method (FEM) [7], the Discontinous Galerkin (DG) method [7–9], the Volume Integral Method (VIM) [10,11], the Spectral Element Method [12], and the Spectral Method [13,14]. Regarding periodic structures, we mention the Fourier Modal Method (FMM) [15] which is popular in the engineering literature and appears to be related to the Rigorous Coupled Wave Analysis (RCWA) [16–18]. (We note that, ironically, the RCWA has only recently been demonstrated to be rigorous [19,20].) Of particular note are the studies of the optical Kerr effect (where nonlinear effects are important due to high intensities) in [21–23]. Finally, we note recent results on multiscale scattering in nonlinear Kerr media [24], higher–order FEM for nonlinear Helmholtz equations [25], and inverse problems in nonlinear Kerr media [26].

However, these volumetric approaches are greatly disadvantaged for the piecewise homogeneous geometries which appear in many crucial applications. Surface methods, by contrast, provide optimal performance for such structures as they only consider unknowns at the layer interfaces. Among such algorithms, Surface Integral Methods (SIMs) such as Boundary Integral Methods (BIMs) and Boundary Element Methods (BEMs), stand out [27–31]. We also point out the imporant recent work of Barnett and collaborators [32–34] and Bruno and collaborators [35–37] which have rendered these methods more widely applicable by ingeniously and effectively addressing some of the challenges faced by naive implementations of BIM/BEM.

In this contribution we investigate a somewhat different interfacial approach which is based upon Dirichlet–Neumann Operators (DNOs) and related quantities. As we have shown in previous work [38], the problem of computing the field scattered by a multiply layered medium can be reduced to one of recovering the scattered field *traces* at the layer interfaces. Once the relevant Dirichlet and Neumann traces have been discovered, the fields anywhere inside any of the material layers can be readily recovered from an appropriate integral formula [27]. The set of unknowns can be further reduced to the Dirichlet traces alone with the introduction of DNOs which map these quantities to their Neumann counterparts. In this paper we show that, for a layer of Kerr medium, the DNO is not only well–defined, but also analytic with respect to all of its independent variables. Our method of proof is perturbative in nature and suggests several new avenues of investigation, including stable numerical simulation, and how one would include the effects of periodic deformations of the layer interfaces into both theory and numerical simulation of the resulting DNOs.

The paper is organized as follows. The governing equations and their interfacial reformulation in terms of DNOs is given in § 2. Our high–order perturbation approach to establishing analyticity of the DNO is outlined in § 3. In § 4 we define function spaces which are required by our rigorous results, given in § 5. We close with a discussion of future directions in § 6.

2. Governing equations

The equations governing the propagation of electromagnetic radiation through any material are specified by the time harmonic Maxwell equations

$$\mathrm{curl}[\mathbf{E}] = i\omega\mu_0\mathbf{H}, \quad \mathrm{curl}[\mathbf{H}] = -i\omega\mathbf{D}, \tag{2.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, $\mathbf{D}$ is the displacement vector, ω is the frequency of the radiation, and μ_0 is the free–space permeability [39]. The displacement is given by

$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P},$$

where ϵ_0 is the free–space permittivity, and $\mathbf{P}$ is the polarization vector. The latter can be expressed as the sum of linear and nonlinear components

$$\mathbf{P} = \mathbf{P}^{\mathrm{L}} + \mathbf{P}^{\mathrm{NL}}, \quad \mathbf{P}^{\mathrm{L}} = \epsilon_0\chi^{(1)}\mathbf{E},$$

where $\chi^{(1)}$ is the linear electric susceptibility. For a linear material $\mathbf{P}^{\mathrm{NL}} \approx 0$, while in a nonlinear Kerr medium

$$\mathbf{P}^{\mathrm{NL}} = \epsilon_0\chi^{(2)}\mathbf{E}\mathbf{E} + \epsilon_0\chi^{(3)}\mathbf{E}\mathbf{E}\mathbf{E} + \dots,$$

and $\chi^{(j)}$ is the j–th order component of the electric susceptibility tensor [2,40]. We can combine Maxwell's equations (2.1) into a single equation for the electric field

$$\mathrm{curl}[\mathrm{curl}[\mathbf{E}]] = \mathrm{curl}[i\omega\mu_0\mathbf{H}] = i\omega\mu_0(-i\omega\mathbf{D}) = \omega^2\mu_0\left(\epsilon_0\mathbf{E} + \epsilon_0\chi^{(1)}\mathbf{E} + \mathbf{P}^{\mathrm{NL}}\right).$$

Using the identity

$$\mathrm{curl}[\mathrm{curl}[\mathbf{E}]] = -\Delta\mathbf{E} + \nabla\mathrm{div}\,[\mathbf{E}],$$

and the fact that, in the absence of sources, $\mathrm{div}\,[\mathbf{E}] = 0$, we find

$$\Delta\mathbf{E} + k_0^2 n_0^2\mathbf{E} + \omega^2\mu_0\mathbf{P}^{\mathrm{NL}} = 0,$$

where $k_0^2 = \omega^2\mu_0\epsilon_0 = \omega^2/c_0^2$ and $n_0^2 = 1 + \chi^{(1)}$.

We now specialize to the case of Transverse Electric (TE) polarization where $\mathbf{E} = (0, v(x,z), 0)^T$, and centrosymmetric and isotropic materials [1] (so that $\chi^{(2)} \equiv 0$). Further assuming that higher order contributions to the electric susceptibility tensor are negligible ($\chi^{(j)} \approx 0$, $j \geq 4$) so that [40]

$$\mathbf{P}_y^{\mathrm{NL}} = \epsilon_0 n_0 n_2 \left|\mathbf{E}\right|^2\mathbf{E}, \quad n_2 = \frac{3\chi_{2222}^{(3)}}{n_0\epsilon_0} = \frac{3\chi_{2222}^{(3)}\mu_0 c_0^2}{n_0},$$

and using well–known manipulations, we find that v must satisfy

$$\Delta v + k_0^2 n_0^2\left(1 + \frac{n_2}{n_0}\left|v\right|^2\right)v = 0.$$

We now consider a triply layered medium consisting of a finite–thickness layer of nonlinear Kerr medium mounted between two semi–infinite layers of linear ($n_2 \equiv 0$) dielectric materials. The Kerr medium occupies the domain $\{-h < z < h\}$, while the linear dielectric media are found in the regions $\{z > h\}$ and $\{z < -h\}$. Following our previous developments in [38] (§ 2) we find that the governing equations can be written entirely in terms of the interfacial quantities

$$U(x) := u(x,h), \quad V^h(x) := v(x,h), \quad V^{-h}(x) := v(x,-h), \quad W(x) := w(x,-h),$$

$$\tilde{U}(x) := -\partial_z u(x,h), \quad \tilde{V}^h(x) := \partial_z v(x,h), \quad \tilde{V}^{-h}(x) := -\partial_z v(x,-h), \quad \tilde{W}(x) := \partial_z w(x,-h),$$

where $\{u, v, w\}$ are the scattered fields in the upper, middle, and lower layers, respectively. The quantities $\{U, V^h, V^{-h}, W\}$ and $\{\tilde{U}, \tilde{V}^h, \tilde{V}^{-h}, \tilde{W}\}$ are the Dirichlet and Neumann traces,

royalsocietypublishing.org/journal/rsta Phil. Trans. R. Soc0000000

respectively. As we saw in [38], the continuity conditions provide two constraints (one Dirchlet and one Neumann) at each of the two interfaces, $\{z = \pm h\}$, leaving us with only four equations for the eight unknowns. However, the Dirichlet/Neumann pairs are *not* independent and their close relationship can be quantified via the Dirichlet–Neumann Operators (DNOs)

$$G : U \to \tilde{U}, \quad H : \begin{pmatrix} V^h \\ V^{-h} \end{pmatrix} \to \begin{pmatrix} \tilde{V}^h \\ \tilde{V}^{-h} \end{pmatrix}, \quad J : W \to \tilde{W}.$$

With these the governing equations become [38]

$$\begin{pmatrix} G + H^{h,h} & H^{h,-h} \\ H^{-h,h} & H^{-h,-h} + J \end{pmatrix} \begin{pmatrix} U \\ V^{-h} \end{pmatrix} = \begin{pmatrix} -\psi + H^{hh}[\zeta] \\ H^{-h,h}[\zeta] \end{pmatrix},$$

where

$$H = \begin{pmatrix} H^{h,h} & H^{h,-h} \\ H^{-h,h} & H^{-h,-h} \end{pmatrix}, \quad \zeta := -u^{\text{inc}}(x,h), \quad \psi(x) := -\partial_z u^{\text{inc}}(x,h),$$

and the plane–wave incidence (of incidence angle θ) is of the form

$$u^{\text{inc}}(x,z) = e^{i\alpha x - i\gamma z}, \quad \alpha = n_0 k_0 \sin(\theta), \quad \gamma = n_0 k_0 \cos(\theta).$$

While the operators G and J have been extensively studied [38,41,42], the operator H has yet to be investigated for a nonlinear Kerr medium and we begin that program here.

For the flat–interface geometry we have specified thus far, the problem is one–dimensional so that $v = v(z)$ and the data $\{V^h, V^{-h}, \tilde{V}^h, \tilde{V}^{-h}\}$ are constants. However, with an eye towards future developments with periodically *perturbed* interfaces,

$$z = \pm h + g^{\pm}(x), \quad g^{\pm}(x+d) = g^{\pm}(x),$$

we consider a genuinely two–dimensional problem where $v = v(x,z)$ and $\{V^h(x), V^{-h}(x), \tilde{V}^h(x), \tilde{V}^{-h}(x)\}$ are α–quasiperiodic functions of x. With these considerations in mind, we observe that, due to existence and uniqueness demands, the solution v must be α–quasiperiodic [43]

$$v(x+d,z) = e^{i\alpha d} v(x,z).$$

We note that, while an arbitrary function of an α–quasiperiodic function is no longer α–quasiperiodic, the combination $|v|^2 v = v\bar{v}v$ retains this property as

$$\begin{aligned} v(x+d,z)\overline{v(x+d,z)}v(x+d,z) &= e^{i\alpha d} v(x,z) e^{-i\alpha d}\overline{v(x,z)} e^{i\alpha d} v(x,z) \\ &= e^{i\alpha d} v(x,z)\overline{v(x,z)} v(x,z). \end{aligned}$$

Gathering all of this, we have the following governing equations in the nonlinear Kerr medium

$$\Delta v + k_0^2 n_0^2 \left(1 + \frac{n_2}{n_0}|v|^2\right) v = 0, \qquad -h < z < h, \tag{2.2a}$$

$$v(x,h) = V^h(x), \qquad z = h, \tag{2.2b}$$

$$v(x,-h) = V^{-h}(x), \qquad z = -h, \tag{2.2c}$$

$$v(x+d,z) = e^{i\alpha d} v(x,z), \tag{2.2d}$$

for the object of our study, the DNO

$$H : \begin{pmatrix} V^h(x) \\ V^{-h}(x) \end{pmatrix} \to \begin{pmatrix} \tilde{V}^h(x) \\ \tilde{V}^{-h}(x) \end{pmatrix}. \tag{2.3}$$

3. High–Order Perturbation Approach

We now set

$$V^h(x) = \delta P(x), \quad V^{-h}(x) = \delta Q(z), \quad \delta \in \mathbf{R},$$

and seek a *real analytic* solution of (2.2) of the form

$$v = v(x,z;\delta) = \sum_{m=1}^{\infty} v_m(x,z)\delta^m, \quad \delta \in \mathbf{R}. \tag{3.1}$$

It is not difficult to see that, at order m, we must solve

$$\Delta v_m(x,z) + k_0^2 n_0^2 v_m(x,z) = F_m(x,z), \qquad -h < z < h, \tag{3.2a}$$

$$v_m(x,h) = \delta_{m,1} P(x), \qquad z = h, \tag{3.2b}$$

$$v_m(x,-h) = \delta_{m,1} Q(x), \qquad z = -h, \tag{3.2c}$$

$$v_m(x+d,z) = e^{i\alpha d} v_m(x,z), \tag{3.2d}$$

for $m \geq 1$, where $\delta_{m,n}$ is the Kronecker delta, and

$$F_m(x,z) = -k_0^2 n_0 n_2 \sum_{\ell=2}^{m-1} \sum_{j=1}^{\ell-1} v_{m-\ell}(x,z)\overline{v_{\ell-j}(x,z)} v_j(x,z). \tag{3.2e}$$

From this it is clear that we can expand the DNO, (2.3),

$$H\left[\begin{pmatrix} P \\ Q \end{pmatrix}\right] = H(\delta)\left[\begin{pmatrix} P \\ Q \end{pmatrix}\right] = \sum_{m=1}^{\infty} H_m \left[\begin{pmatrix} P \\ Q \end{pmatrix}\right] \delta^m, \tag{3.3}$$

and deduce that

$$H_m\left[\begin{pmatrix} P \\ Q \end{pmatrix}\right] = \begin{pmatrix} \partial_z v_m(x,h) \\ -\partial_z v_m(x,-h) \end{pmatrix}. \tag{3.4}$$

Since $F_1 \equiv 0$, at order $m = 1$ we can compute the solution of (3.2) explicitly, beginning with the Helmholtz equation [44], (3.2a),

$$v_1(x,z) = \sum_{p=-\infty}^{\infty} \left\{ a_p e^{i\gamma_p(z-h)} + b_p e^{-i\gamma_p(z+h)} \right\} e^{i\alpha_p x},$$

where

$$\alpha_p = \alpha + \frac{2\pi p}{d}, \quad \gamma_p = \sqrt{n_0^2 k_0^2 - \alpha_p^2}, \quad \operatorname{Im}\{\gamma_p\} \geq 0.$$

The boundary conditions (3.2b) and (3.2c) demand that

$$\begin{pmatrix} 1 & \Gamma_p \\ \Gamma_p & 1 \end{pmatrix} \begin{pmatrix} a_p \\ b_p \end{pmatrix} = \begin{pmatrix} \hat{P}_p \\ \hat{Q}_p \end{pmatrix}, \quad \Gamma_p := e^{-2i\gamma_p h}, \quad \hat{P}_p = \frac{1}{d}\int_0^d P(x) e^{-i\alpha_p x}\, dx,$$

requiring

$$a_p = \frac{\hat{P}_p - \Gamma_p \hat{Q}_p}{1 - \Gamma_p^2}, \quad b_p = \frac{\hat{Q}_p - \Gamma_p \hat{P}_p}{1 - \Gamma_p^2},$$

giving

$$v_1(x,z) = \sum_{p=-\infty}^{\infty} \left\{ \left(\frac{\hat{P}_p - \Gamma_p \hat{Q}_p}{1 - \Gamma_p^2} \right) e^{i\gamma_p(z-h)} + \left(\frac{\hat{Q}_p - \Gamma_p \hat{P}_p}{1 - \Gamma_p^2} \right) e^{-i\gamma_p(z+h)} \right\} e^{i\alpha_p x}. \tag{3.5}$$

Remark 3.1. We make the important point that a unique solution will only exist if $\Gamma_p^2 \neq 1$, i.e.,

$$h \neq \frac{n\pi}{\gamma_p}, \quad \forall\, n, p \in \mathbf{Z}. \tag{3.6}$$

This is not a *physical* singularity but rather a *mathematical* one due to the nature of the unknowns we have selected, namely the Dirichlet and Neumann data. This is a well–known problem and one fix is to use impedance data [45–47] which we could certainly pursue here. For the sake of simplicity of formulation we do not make this change and simply avoid these "singularities." Please see the survey paper [48] for more details about these more sophisticated Domain Decomposition Methods with impedance data.

Since $F_2 \equiv 0$ we can use the same procedure to find the solution at $m = 2$. As the Dirichlet data is identically zero at both $z = \pm h$, we deduce that $v_2 \equiv 0$. For $m \geq 3$ we have $F_m \not\equiv 0$ and must resort to numerical simulation as convenient analytical expressions are no longer available.

4. Function Spaces

We now work to show that the expansion (3.1) can be justified rigorously, namely that we can produce estimates of the form

$$\|v_m\|_X \leq C \frac{B^{m-1}}{m^2}, \quad m \geq 1, \quad C, B > 0,$$

for an appropriate function space X. Clearly, from (3.4), once this is established we have the analyticity of the DNO which is the object of our study. Our proof roughly follows the strategy employed by Nicholls & Reitich [49] to establish joint analyticity of traveling wave solutions of the nonlinear water waves problem. We begin by recalling that any laterally α–quasiperiodic L^2 function can be expressed

$$v(x,z) = \sum_{p=-\infty}^{\infty} \hat{v}_p(z) e^{i\alpha_p x}, \quad \hat{v}_p(z) = \frac{1}{d} \int_0^d v(x,z) e^{-i\alpha_p x}\, dx,$$

and we define the classical Sobolev spaces [31,50], $s \in \mathbf{Z}^+$,

$$H^s(\Omega) = \left\{ v(x,z) \in L^2(\Omega) \mid \|v\|_{H^s} < \infty \right\}, \quad \Omega := (0,d) \times (-h,h),$$

where

$$\|v\|_{H^s}^2 = \sum_{r=0}^{s} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2(s-r)} \int_{-h}^{h} \left| \partial_z^r \hat{v}_p(z) \right|^2 dz, \qquad \langle p \rangle^2 := 1 + |p|^2.$$

We point out that if $f = f(x)$ depends on x alone then we can define the interfacial spaces [31,50], $s \in \mathbf{R}^+$,

$$H^s(\Gamma) = \left\{ f(x) \in L^2(\Gamma) \mid \|f\|_{H^s} < \infty \right\}, \quad \Gamma := (0,d),$$

where

$$\|f\|_{H^s}^2 = \sum_{p=-\infty}^{\infty} \langle p \rangle^{2s} \left| \hat{f}_p \right|^2.$$

We require the following classical algebra property [50–52].

Lemma 4.1. *Given an integer $s > 1$ there exists a constant $\tilde{M} = \tilde{M}(s)$ such that if $f \in H^s(\Gamma)$ and $u, v \in H^s(\Omega)$ then*

$$\|fu\|_{H^s} \leq \tilde{M} \|f\|_{H^s} \|u\|_{H^s}, \quad \|uv\|_{H^s} \leq \tilde{M} \|u\|_{H^s} \|v\|_{H^s}.$$

Additionally, we will have need of the following combinatorial result found in Friedman & Reitich [53] (Equation (9.2)).

Lemma 4.2. *Given any integers $a, c \geq 0$ we have*

$$\binom{a}{b}\binom{c}{d} \leq \binom{a+c}{b+d}, \quad 0 \leq b \leq a, \quad 0 \leq d \leq c. \tag{4.1}$$

Proof. We work by induction in a. When $a=0$ we verify (4.1) trivially as $\binom{0}{0}=1$. We now assume (4.1) for all $0\le a\le A-1$ and $c\ge 0$, and, using the well–known equality for $n\ge 1$,

$$\binom{n}{m}=\binom{n-1}{m-1}+\binom{n-1}{m},\quad 0\le m\le n,$$

we examine

$$\begin{aligned}\binom{A}{b}\binom{c}{d}&=\left\{\binom{A-1}{b-1}+\binom{A-1}{b}\right\}\binom{c}{d}\\&=\binom{A-1}{b-1}\binom{c}{d}+\binom{A-1}{b}\binom{c}{d}.\end{aligned}$$

By the inductive hypothesis we have

$$\begin{aligned}\binom{A}{b}\binom{c}{d}&\le\binom{A-1+c}{b-1+d}+\binom{A-1+c}{b+d}\\&=\binom{(A+c)-1}{(b+d)-1}+\binom{(A+c)-1}{(b+d)}=\binom{A+c}{b+d},\end{aligned}$$

and we are done. □

We conclude with the observation that the *joint* analyticity results we seek for the field, v, and DNO, H, will make very stringent demands on the Dirichlet data, $\{P,Q\}$, namely that they themselves be real analytic. More precisely we make the following definition of the analytic class $C^{\omega}_{s}(\Gamma)$.

Definition 4.3. A function f is a member of the space $C^{\omega}_{s}(\Gamma)$ if it is real analytic and satisfies the estimate

$$\left\|\frac{\partial_x^q}{q!}f\right\|_{H^s}\le C_f\frac{A^q}{(q+1)^2},\quad q\ge 0,$$

for some $C_f,A>0$. This notation is meant to indicate the space of real analytic functions, C^{ω}, with radius of analyticity (characterized by A) measured in the H^s norm.

5. Analyticity

We are now in a position to establish joint analyticity of the solution $v(x,z)$ of (2.2), expressed with the formula (3.1), by proving the following theorem.

Theorem 5.1. *Provided that (3.6) holds, given an integer $s>1$, if $P,Q\in C^{\omega}_{s+3/2}(\Gamma)$ and*

$$\left\|\frac{\partial_x^q}{q!}P\right\|_{H^{s+3/2}}\le C_P\frac{A^q}{(q+1)^2},\quad\left\|\frac{\partial_x^q}{q!}Q\right\|_{H^{s+3/2}}\le C_Q\frac{A^q}{(q+1)^2},\quad q\ge 0,\tag{5.1}$$

for some $C_P,C_Q,A>0$, then there exists a unique solution $v_m\in C^{\omega}(\Omega)$ of (3.2) satisfying

$$\left\|\frac{\partial_x^q\partial_z^r}{(q+r)!}v_m\right\|_{H^{s+2}}\le C\frac{A^q}{(q+1)^2}\frac{D^r}{(r+1)^2}\frac{B^{m-1}}{m^2},\quad q,r\ge 0,\quad m\ge 1,\tag{5.2}$$

for some $C,D,B>0$.

To accomplish this we conduct an induction in the order of the z–derivative, r, which requires the following result which establishes the estimate for $r=0$.

Theorem 5.2. *Provided that* (3.6) *holds, given an integer* $s>1$, *if* $P,Q \in C^{\omega}_{s+3/2}(\Gamma)$ *and*

$$\left\| \frac{\partial_x^q}{q!} P \right\|_{H^{s+3/2}} \le C_P \frac{A^q}{(q+1)^2}, \quad \left\| \frac{\partial_x^q}{q!} Q \right\|_{H^{s+3/2}} \le C_Q \frac{A^q}{(q+1)^2}, \quad q \ge 0, \tag{5.3}$$

for some $C_P, C_Q, A > 0$, *then there exists a unique solution* $v_m \in C^{\omega}(\Omega)$ *of* (3.2) *satisfying*

$$\left\| \frac{\partial_x^q}{q!} v_m \right\|_{H^{s+2}} \le C \frac{A^q}{(q+1)^2} \frac{B^{m-1}}{m^2}, \quad q \ge 0, \quad m \ge 1, \tag{5.4}$$

for some $C, B > 0$.

The proof of this result is dependent on the following: A recursive estimate (Lemma 5.3) and an elliptic existence and regularity result (Theorem 5.4) which we now present.

Lemma 5.3. *Given an integer* $s>1$, *suppose that*

$$\left\| \frac{\partial_x^q}{q!} v_m \right\|_{H^{s+2}} \le C \frac{A^q}{(q+1)^2} \frac{B^{m-1}}{m^2}, \quad q \ge 0, \quad 1 \le m \le M-1,$$

for some constants $C, A, B > 0$. *Then, there exists a constant* $C_1 > 0$ *such that*

$$\left\| \frac{\partial_x^q}{q!} F_M \right\|_{H^s} \le C C_1 \frac{A^q}{(q+1)^2} \frac{B^{M-3}}{M^2}, \quad q \ge 0,$$

for F_M *given in* (3.2*e*).

Proof. From (3.2*e*) we have

$$\frac{\partial_x^q}{q!} F_M = -k_0^2 n_0 n_2 \sum_{\ell=2}^{M-1} \sum_{j=1}^{\ell-1} \sum_{\sigma=0}^{q} \sum_{\tau=0}^{\sigma} \frac{\partial_x^{q-\sigma}}{(q-\sigma)!} v_{M-\ell} \frac{\partial_x^{\sigma-\tau}}{(\sigma-\tau)!} \overline{v_{\ell-j}} \frac{\partial_x^{\tau}}{\tau!} v_j.$$

Now we estimate

$$\begin{aligned} \left\| \frac{\partial_x^q}{q!} F_M \right\|_{H^s} &\le k_0^2 \left| n_0 n_2 \right| \sum_{\ell=2}^{M-1} \sum_{j=1}^{\ell-1} \sum_{\sigma=0}^{q} \sum_{\tau=0}^{\sigma} \left\| \frac{\partial_x^{q-\sigma}}{(q-\sigma)!} v_{M-\ell} \frac{\partial_x^{\sigma-\tau}}{(\sigma-\tau)!} \overline{v_{\ell-j}} \frac{\partial_x^{\tau}}{\tau!} v_j \right\|_{H^s} \\ &\le k_0^2 \left| n_0 n_2 \right| \sum_{\ell=2}^{M-1} \sum_{j=1}^{\ell-1} \sum_{\sigma=0}^{q} \sum_{\tau=0}^{\sigma} \tilde{M}^2 \left\| \frac{\partial_x^{q-\sigma}}{(q-\sigma)!} v_{M-\ell} \right\|_{H^s} \left\| \frac{\partial_x^{\sigma-\tau}}{(\sigma-\tau)!} \overline{v_{\ell-j}} \right\|_{H^s} \left\| \frac{\partial_x^{\tau}}{\tau!} v_j \right\|_{H^s}. \end{aligned}$$

Using the hypotheses of the Lemma we continue

$$\begin{aligned} \left\| \frac{\partial_x^q}{q!} F_M \right\|_{H^s} &\le k_0^2 \left| n_0 n_2 \right| \sum_{\ell=2}^{M-1} \sum_{j=1}^{\ell-1} \sum_{\sigma=0}^{q} \sum_{\tau=0}^{\sigma} \tilde{M}^2 C^3 \frac{A^{q-\sigma}}{(q-\sigma+1)^2} \frac{B^{M-\ell-1}}{(M-\ell)^2} \\ &\quad \times \frac{A^{\sigma-\tau}}{(\sigma-\tau+1)^2} \frac{B^{\ell-j-1}}{(\ell-j)^2} \frac{A^{\tau}}{(\tau+1)^2} \frac{B^{j-1}}{j^2} \\ &\le k_0^2 \left| n_0 n_2 \right| \tilde{M}^2 C^3 \frac{A^q}{(q+1)^2} \frac{B^{M-3}}{M^2} \\ &\quad \times \sum_{\sigma=0}^{q} \sum_{\tau=0}^{\sigma} \frac{(q+1)^2}{(q-\sigma+1)^2 (\sigma-\tau+1)^2 (\tau+1)^2} \\ &\quad \times \sum_{\ell=2}^{M-1} \sum_{j=1}^{\ell-1} \frac{M^2}{(M-\ell)^2 (\ell-j)^2 j^2} \\ &\le C k_0^2 \left| n_0 n_2 \right| \tilde{M}^2 C^2 S^2 \frac{A^q}{(q+1)^2} \frac{B^{M-3}}{M^2}, \end{aligned}$$

where [49]

$$S := \max\left\{\sum_{\sigma=0}^{q}\sum_{\tau=0}^{\sigma}\frac{(q+1)^2}{(q-\sigma+1)^2(\sigma-\tau+1)^2(\tau+1)^2}, \sum_{\ell=2}^{M-1}\sum_{j=1}^{\ell-1}\frac{M^2}{(M-\ell)^2(\ell-j)^2 j^2}\right\} < \infty.$$

We are done provided that we choose

$$C_1 > k_0^2 \left|n_0 n_2\right| \tilde{M}^2 C^2 S^2.$$

□

We now state without proof the elliptic regularity result [49,50,54].

Theorem 5.4. *Provided that (3.6) holds, consider any positive integer $s \geq 0$, if $F \in H^s(\Omega)$ and $P, Q \in H^{s+3/2}(\Gamma)$ then there exists a unique solution v of*

$$\Delta v(x,z) + k_0^2 n_0^2 v(x,z) = F(x,z), \qquad -h < z < h, \tag{5.5a}$$

$$v(x,h) = P(x), \qquad z = h, \tag{5.5b}$$

$$v(x,-h) = Q(x), \qquad z = -h, \tag{5.5c}$$

$$v(x+d,z) = e^{i\alpha d} v(x,z), \tag{5.5d}$$

such that, for a constant $C_e > 0$,

$$\|v\|_{H^{s+2}} \leq C_e \left\{\|F\|_{H^s} + \|P\|_{H^{s+3/2}} + \|Q\|_{H^{s+3/2}}\right\}. \tag{5.6}$$

At this stage we can now present the proof of Theorem 5.2.

Proof. (Theorem 5.2.) We work by induction on the perturbation order, m. By applying the operator $\partial_x^q/q!$ to (3.2) we find

$$\Delta \frac{\partial_x^q}{q!} v_m(x,z) + k_0^2 n_0^2 \frac{\partial_x^q}{q!} v_m(x,z) = \frac{\partial_x^q}{q!} F_m(x,z), \qquad -h < z < h,$$

$$\frac{\partial_x^q}{q!} v_m(x,h) = \delta_{m,1} \frac{\partial_x^q}{q!} P(x), \qquad z = h,$$

$$\frac{\partial_x^q}{q!} v_m(x,-h) = \delta_{m,1} \frac{\partial_x^q}{q!} Q(x), \qquad z = -h,$$

$$\frac{\partial_x^q}{q!} v_m(x+d,z) = e^{i\alpha d} \frac{\partial_x^q}{q!} v_m(x,z).$$

In the case $m = 1$ we have $F_1 \equiv 0$ and can invoke Theorem 5.4 to realize

$$\left\|\frac{\partial_x^q}{q!} v_1\right\|_{H^{s+2}} \leq C_e \left\{\left\|\frac{\partial_x^q}{q!} P\right\|_{H^{s+3/2}} + \left\|\frac{\partial_x^q}{q!} Q\right\|_{H^{s+3/2}}\right\} \leq C_e \left\{C_P + C_Q\right\} \frac{A^q}{(q+1)^2},$$

and we have used the hypotheses (5.3). We are done if we choose

$$C := C_e \left\{C_P + C_Q\right\}.$$

We now assume (5.4) for $1 \leq m \leq M-1$ and seek to establish the estimate when $m = M$. Since $M > 1$ we can apply Theorem 5.4 (with $P \equiv Q \equiv 0$) to the problem for $(\partial_x^q/q!) v_m$ above to find

$$\left\|\frac{\partial_x^q}{q!} v_M\right\|_{H^{s+2}} \leq C_e \left\|\frac{\partial_x^q}{q!} F_M\right\|_{H^s}.$$

From Lemma 5.3 we have

$$\left\|\frac{\partial_x^q}{q!} v_M\right\|_{H^{s+2}} \leq C_e C C_1 \frac{A^q}{(q+1)^2} \frac{B^{M-3}}{M^2},$$

and we are done provided that

$$B > \sqrt{C_e C_1}.$$

□

We now move to the proof of Theorem 5.1 which we conduct by an induction in the order of the z derivative, r. This we will accomplish by an induction in the perturbation order, m, which requires the following result.

Theorem 5.5. *Provided that (3.6) holds, given an integer $s > 1$, if $P, Q \in C^{\omega}_{s+3/2}(\Gamma)$ and*

$$\left\| \frac{\partial_x^q}{q!} P \right\|_{H^{s+3/2}} \leq C_P \frac{A^q}{(q+1)^2}, \quad \left\| \frac{\partial_x^q}{q!} Q \right\|_{H^{s+3/2}} \leq C_Q \frac{A^q}{(q+1)^2}, \quad q \geq 0, \tag{5.7}$$

for some $C_P, C_Q, A > 0$, then there exists a unique solution $v_1 \in C^{\omega}(\Omega)$ of (3.2) satisfying

$$\left\| \frac{\partial_x^q \partial_z^r}{(q+r)!} v_1 \right\|_{H^{s+2}} \leq C \frac{A^q}{(q+1)^2} \frac{D^r}{(r+1)^2}, \quad q, r \geq 0, \tag{5.8}$$

for some $C, D > 0$.

Proof. This is a straightforward consequence of Theorem 5.4 (existence and uniqueness) applied to the exact solution formula for $v_1(x,z)$, (3.5), and the hypotheses (5.7). □

We also require a recursive estimate very much in the spirit of Lemma 5.3.

Lemma 5.6. *Given an integer $s > 1$, suppose that*

$$\left\| \frac{\partial_x^q \partial_z^r}{(q+r)!} v_m \right\|_{H^{s+2}} \leq C \frac{A^q}{(q+1)^2} \frac{D^r}{(r+1)^2} \frac{B^{m-1}}{m^2}, \quad q \geq 0, \quad m \geq 1, \quad 1 \leq r \leq R-1,$$

and

$$\left\| \frac{\partial_x^q \partial_z^R}{(q+R)!} v_m \right\|_{H^{s+2}} \leq C \frac{A^q}{(q+1)^2} \frac{D^R}{(R+1)^2} \frac{B^{m-1}}{m^2}, \quad q \geq 0, \quad 1 \leq m \leq M-1,$$

for some constants $C, A, D, B > 0$. Then, there exists a constant $C_2 > 0$ such that

$$\left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R)!} F_M \right\|_{H^{s+1}} \leq C C_2 \frac{A^q}{(q+1)^2} \frac{D^{R-1}}{(R+1)^2} \frac{B^{M-3}}{M^2}, \quad q \geq 0,$$

for F_M given in (3.2e).

Proof. From (3.2e) we have

$$\frac{\partial_x^q \partial_z^r}{(q+r)!} F_M = -k_0^2 n_0 n_2 \sum_{\ell=2}^{M-1} \sum_{j=1}^{\ell-1} \sum_{\sigma=0}^{q} \sum_{\tau=0}^{\sigma} \sum_{\mu=0}^{r} \sum_{\nu=0}^{\mu} C_{q,\sigma,\tau,r,\mu,\nu} \left(\frac{\partial_x^{q-\sigma} \partial_z^{r-\mu}}{(q-\sigma+r-\mu)!} v_{M-\ell} \right) \left(\frac{\partial_x^{\sigma-\tau} \partial_z^{\mu-\nu}}{(\sigma-\tau+\mu-\nu)!} \overline{v_{\ell-j}} \right) \left(\frac{\partial_x^{\tau} \partial_z^{\nu}}{(\tau+\nu)!} v_j \right),$$

where

$$C_{q,\sigma,\tau,r,\mu,\nu} = \frac{q!r!(q-\sigma+r-\mu)!(\sigma-\tau+\mu-\nu)!(\tau+\nu)!}{(q+r)!(q-\sigma)!(r-\mu)!(\sigma-\tau)!(\mu-\nu)!\tau!\nu!} = \frac{\binom{q-\sigma+r-\mu}{q-\sigma}\binom{\sigma-\tau+\mu-\nu}{\sigma-\tau}\binom{\tau+\nu}{\tau}}{\binom{q+r}{q}}.$$

royalsocietypublishing.org/journal/rsta Phil. Trans. R. Soc0000000

Using (4.1) from Lemma 4.2 we have

$$
\begin{aligned}
C_{q,\sigma,\tau,r,\mu,\nu} &\leq \frac{1}{\binom{q+r}{q}} \binom{(q-\sigma+r-\mu)+(\sigma-\tau+\mu-\nu)}{(q-\sigma)+(\sigma-\tau)} \binom{\tau+\nu}{\tau} \\
&= \frac{1}{\binom{q+r}{q}} \binom{q+r-\tau-\nu}{q-\tau} \binom{\tau+\nu}{\tau} \\
&\leq \frac{1}{\binom{q+r}{q}} \binom{(q+r-\tau-\nu)+(\tau+\nu)}{(q-\tau)+\tau} = \frac{\binom{q+r}{q}}{\binom{q+r}{q}} = 1.
\end{aligned}
$$

With this we estimate

$$
\begin{aligned}
\left\| \frac{\partial_x^q \partial_z^r}{(q+r)!} F_M \right\|_{H^{s+1}} &\leq k_0^2 \left| n_0 n_2 \right| \sum_{\ell=2}^{M-1} \sum_{j=1}^{\ell-1} \sum_{\sigma=0}^{q} \sum_{\tau=0}^{\sigma} \sum_{\mu=0}^{r} \sum_{\nu=0}^{\mu} \\
&\qquad \left\| \left(\frac{\partial_x^{q-\sigma} \partial_z^{r-\mu}}{(q-\sigma+r-\mu)!} v_{M-\ell} \right) \left(\frac{\partial_x^{\sigma-\tau} \partial_z^{\mu-\nu}}{(\sigma-\tau+\mu-\nu)!} \overline{v_{\ell-j}} \right) \left(\frac{\partial_x^{\tau} \partial_z^{\nu}}{(\tau+\nu)!} v_j \right) \right\|_{H^{s+1}} \\
&\leq k_0^2 \left| n_0 n_2 \right| \sum_{\ell=2}^{M-1} \sum_{j=1}^{\ell-1} \sum_{\sigma=0}^{q} \sum_{\tau=0}^{\sigma} \sum_{\mu=0}^{r} \sum_{\nu=0}^{\mu} \tilde{M}^2 \left\| \frac{\partial_x^{q-\sigma} \partial_z^{r-\mu}}{(q-\sigma+r-\mu)!} v_{M-\ell} \right\|_{H^{s+1}} \\
&\qquad \times \left\| \frac{\partial_x^{\sigma-\tau} \partial_z^{\mu-\nu}}{(\sigma-\tau+\mu-\nu)!} \overline{v_{\ell-j}} \right\|_{H^{s+1}} \left\| \frac{\partial_x^{\tau} \partial_z^{\nu}}{(\tau+\nu)!} v_j \right\|_{H^{s+1}}.
\end{aligned}
$$

Using the hypotheses of the Lemma we continue

$$
\begin{aligned}
\left\| \frac{\partial_x^q \partial_z^r}{(q+r)!} F_M \right\|_{H^{s+1}} &\leq k_0^2 \left| n_0 n_2 \right| \tilde{M}^2 \sum_{\ell=2}^{M-1} \sum_{j=1}^{\ell-1} \sum_{\sigma=0}^{q} \sum_{\tau=0}^{\sigma} \sum_{\mu=0}^{r} \sum_{\nu=0}^{\mu} \\
&\qquad C \frac{A^{q-\sigma}}{(q-\sigma+1)^2} \frac{D^{r-\mu}}{(r-\mu+1)^2} \frac{B^{M-\ell-1}}{(M-\ell)^2} \\
&\qquad \times C \frac{A^{\sigma-\tau}}{(\sigma-\tau+1)^2} \frac{D^{\mu-\nu}}{(\mu-\nu+1)^2} \frac{B^{\ell-j-1}}{(\ell-j)^2} \\
&\qquad \times C \frac{A^{\tau}}{(\tau+1)^2} \frac{D^{\nu}}{(\nu+1)^2} \frac{B^{j-1}}{j^2} \\
&\leq k_0^2 \left| n_0 n_2 \right| C^3 \tilde{M}^2 \frac{A^q}{(q+1)^2} \frac{D^r}{(r+1)^2} \frac{B^{M-3}}{M^2} \\
&\qquad \sum_{\sigma=0}^{q} \sum_{\tau=0}^{\sigma} \left(\frac{(q+1)^2}{(q-\sigma+1)^2 (\sigma-\tau+1)^2 (\tau+1)^2} \right) \\
&\qquad \sum_{\mu=0}^{r} \sum_{\nu=0}^{\mu} \left(\frac{(r+1)^2}{(r-\mu+1)^2 (\mu-\nu+1)^2 (\nu+1)^2} \right) \\
&\qquad \sum_{\ell=2}^{M-1} \sum_{j=1}^{\ell-1} \left(\frac{M^2}{(M-\ell)^2 (\ell-j)^2 j^2} \right) \\
&\leq C C^2 k_0^2 \left| n_0 n_2 \right| \tilde{M}^2 S^3 \frac{A^q}{(q+1)^2} \frac{D^r}{(r+1)^2} \frac{B^{M-3}}{M^2}.
\end{aligned}
$$

royalsocietypublishing.org/journal/rsta Phil. Trans. R. Soc0000000

Setting $r = R - 1$ we find

$$\left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R-1)!} F_M \right\|_{H^{s+1}} \leq C C^2 k_0^2 \left| n_0 n_2 \right| \tilde{M}^2 S^3 \frac{A^q}{(q+1)^2} \frac{D^{R-1}}{R^2} \frac{B^{M-3}}{M^2}$$
$$\leq C \left(C^2 k_0^2 \left| n_0 n_2 \right| \tilde{M}^2 S^3 \frac{(R+1)^2}{R^2} \right) \frac{A^q}{(q+1)^2} \frac{D^{R-1}}{(R+1)^2} \frac{B^{M-3}}{M^2}.$$

Since, for $q + R \geq 1$,

$$\frac{1}{(q+R)!} = \frac{1}{(q+R)} \frac{1}{(q+R-1)!} \leq \frac{1}{(q+R-1)!},$$

we are done provided that we choose

$$C_2 > 4 C^2 k_0^2 \left| n_0 n_2 \right| \tilde{M}^2 S^3 \geq C^2 k_0^2 \left| n_0 n_2 \right| \tilde{M}^2 S^3 \frac{(R+1)^2}{R^2},$$

since $(R+1)/R \leq 2$ when $R \geq 1$. □

At last, we can now prove Theorem 5.1.

Proof. (Theorem 5.1.) We work by induction on the order of the z derivative, r, and note that the case $r = 0$ (any $q \geq 0$ and any $m \geq 1$) is addressed by Theorem 5.2. We now assume

$$\left\| \frac{\partial_x^q \partial_z^r}{(q+r)!} v_m \right\|_{H^{s+2}} \leq C \frac{A^q}{(q+1)^2} \frac{D^r}{(r+1)^2} \frac{B^{m-1}}{m^2}, \quad q \geq 0, \quad m \geq 1, \quad 0 \leq r \leq R-1,$$

and seek to prove

$$\left\| \frac{\partial_x^q \partial_z^R}{(q+R)!} v_m \right\|_{H^{s+2}} \leq C \frac{A^q}{(q+1)^2} \frac{D^R}{(R+1)^2} \frac{B^{m-1}}{m^2}, \quad q \geq 0, \quad m \geq 1.$$

We accomplish this with a second induction on m and note that the case $m = 1$ is addressed by Theorem 5.5. So, we further assume that

$$\left\| \frac{\partial_x^q \partial_z^R}{(q+R)!} v_m \right\|_{H^{s+2}} \leq C \frac{A^q}{(q+1)^2} \frac{D^R}{(R+1)^2} \frac{B^{m-1}}{m^2}, \quad q \geq 0, \quad 1 \leq m \leq M-1,$$

and seek to prove

$$\left\| \frac{\partial_x^q \partial_z^R}{(q+R)!} v_M \right\|_{H^{s+2}} \leq C \frac{A^q}{(q+1)^2} \frac{D^R}{(R+1)^2} \frac{B^{M-1}}{M^2}, \quad q \geq 0.$$

For this we proceed with

$$\left\| \frac{\partial_x^q \partial_z^R}{(q+R)!} v_M \right\|_{H^{s+2}} \leq \left\| \frac{\partial_x^q \partial_z^R}{(q+R)!} v_M \right\|_{H^{s+1}} + \left\| \frac{\partial_x^q \partial_z^R}{(q+R)!} \partial_x v_M \right\|_{H^{s+1}} + \left\| \frac{\partial_x^q \partial_z^R}{(q+R)!} \partial_z v_M \right\|_{H^{s+1}}$$
$$\leq \left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R)!} v_M \right\|_{H^{s+2}} + \left\| \frac{\partial_x^{q+1} \partial_z^{R-1}}{(q+R)!} v_M \right\|_{H^{s+2}} + \left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R)!} \partial_z^2 v_M \right\|_{H^{s+1}}.$$

Using the fact that v_M satisfies the inhomogeneous Helmholtz equation (3.2*a*) so that

$$\partial_z^2 v_M = -\partial_x^2 v_M - k_0^2 n_0^2 v_M + F_M,$$

rsta.royalsocietypublishing.org Phil. Trans. R. Soc0000000

we continue

$$
\begin{aligned}
\left\| \frac{\partial_x^q \partial_z^R}{(q+R)!} v_M \right\|_{H^{s+2}} \leq & \left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R)!} v_M \right\|_{H^{s+2}} + \left\| \frac{\partial_x^{q+1} \partial_z^{R-1}}{(q+R)!} v_M \right\|_{H^{s+2}} + \left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R)!} \partial_x^2 v_M \right\|_{H^{s+1}} \\
& + \left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R)!} k_0^2 n_0^2 v_M \right\|_{H^{s+1}} + \left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R)!} F_M \right\|_{H^{s+1}}, \\
\leq & \left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R)!} v_M \right\|_{H^{s+2}} + \left\| \frac{\partial_x^{q+1} \partial_z^{R-1}}{(q+R)!} v_M \right\|_{H^{s+2}} + \left\| \frac{\partial_x^{q+1} \partial_z^{R-1}}{(q+R)!} v_M \right\|_{H^{s+2}} \\
& + \left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R)!} k_0^2 n_0^2 v_M \right\|_{H^{s+2}} + \left\| \frac{\partial_x^q \partial_z^{R-1}}{(q+R)!} F_M \right\|_{H^{s+1}}.
\end{aligned}
$$

Using the inductive hypotheses and Lemma 5.6 we find

$$
\begin{aligned}
\left\| \frac{\partial_x^q \partial_z^R}{(q+R)!} v_M \right\|_{H^{s+2}} \leq & \frac{C}{(q+R)} \frac{A^q}{(q+1)^2} \frac{D^{R-1}}{R^2} \frac{B^{M-1}}{M^2} + 2C \frac{A^{q+1}}{(q+2)^2} \frac{D^{R-1}}{R^2} \frac{B^{M-1}}{M^2} \\
& + k_0^2 \left| n_0 \right|^2 \frac{C}{q+R} \frac{A^q}{(q+1)^2} \frac{D^{R-1}}{R^2} \frac{B^{M-1}}{M^2} \\
& + C C_2 \frac{A^q}{(q+1)^2} \frac{D^{R-1}}{(R+1)^2} \frac{B^{M-3}}{M^2} \\
\leq & C \frac{R^2}{(R+1)^2} \left(1 + 2A + k_0^2 \left| n_0 \right|^2 \right) \frac{A^q}{(q+1)} \frac{D^{R-1}}{(R+1)^2} \frac{B^{M-1}}{M^2} \\
& + C C_2 \frac{A^q}{(q+1)^2} \frac{D^{R-1}}{(R+1)^2} \frac{B^{M-3}}{M^2},
\end{aligned}
$$

and we are done provided that

$$
D > \max \left\{ 2 \left(1 + 2A + k_0^2 \left| n_0 \right|^2 \right), \frac{2C_2}{B^2} \right\}.
$$

□

It is clear from (3.3) & (3.4) that a direct corollary of the joint analyticity result, Theorem 5.1, is the joint analyticity of the DNO itself.

Theorem 5.7. *Provided that (3.6) holds, given an integer $s > 1$, if $P, Q \in C^{\omega}_{s+3/2}$ and*

$$
\left\| \frac{\partial_x^q}{q!} P \right\|_{H^{s+3/2}} \leq C_P \frac{A^q}{(q+1)^2}, \quad \left\| \frac{\partial_x^q}{q!} Q \right\|_{H^{s+3/2}} \leq C_Q \frac{A^q}{(q+1)^2}, \quad q \geq 0,
$$

for some $C_P, C_Q, A > 0$, then

$$
H_m^{h,h}[P], H_m^{h,-h}[Q], H_m^{-h,h}[P], H_m^{-h,-h}[Q] \in C^{\omega}_{s+1/2}(\Gamma), \quad m \geq 1,
$$

and

$$
\begin{aligned}
\max \Bigg\{ & \left\| \frac{\partial_x^q}{q!} H_m^{h,h}[P] \right\|_{H^{s+1/2}}, \left\| \frac{\partial_x^q}{q!} H_m^{h,-h}[Q] \right\|_{H^{s+1/2}}, \\
& \left\| \frac{\partial_x^q}{q!} H_m^{-h,h}[P] \right\|_{H^{s+1/2}}, \left\| \frac{\partial_x^q}{q!} H_m^{-h,-h}[Q] \right\|_{H^{s+1/2}} \Bigg\} \\
& \leq K \frac{A^q}{(q+1)^2} \frac{B^{m-1}}{m^2}, \quad q \geq 0, \quad m \geq 1,
\end{aligned}
$$

for some $K > 0$. *Furthermore, this implies that the operators* H_m *satisfy*

$$\|H_m\|_{H^{s+3/2}\times H^{s+3/2}\to H^{s+1/2}\times H^{s+1/2}} \le K\frac{B^{m-1}}{m^2}, \quad m \ge 1.$$

6. Future Directions

While we have established the boundedness and analyticity of the DNO for a layer of nonlinear Kerr medium with flat interfaces, it is clear that there is more work to be done, and our method of proof provides guidance towards accomplishing this. To start, our goal is to devise stable and high accuracy numerical methods for the approximation of the DNO in this setting. In future work we will study the truncated sums

$$v^M(x,z) := \sum_{m=1}^{M} v_m(x,z)\delta^m, \quad H^M(x,z) := \sum_{m=1}^{M} H_m\delta^m,$$

where the $\{v_m, H_m\}$ satisfy (3.2) and (3.4), respectively. In order to solve the boundary value problem (3.2) numerically we will appeal to a Fourier/Chebyshev approach [13,14] which has been successfully brought to bear on related problems in the past [41,42]. Beyond this, we believe that we will be able to address the question of layers of nonlinear Kerr media with periodically *perturbed* interfaces, $z = \pm h + g_{\pm h}(x)$, $g_{\pm h}(x+d) = g_{\pm h}(x)$. A natural approach based upon our previous work [41,42,55] is perturbative in nature and posits the forms $g_{\pm h}(x) = \varepsilon f_{\pm h}(x)$. With this, the field and DNO could be *jointly* expanded in δ and ε,

$$v(x,z) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} v_{m,n}(x,z)\varepsilon^n\delta^m, \quad H = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} H_{m,n}\varepsilon^n\delta^m,$$

recursions derived for the $\{v_{m,n}, H_{m,n}\}$, and the relevant boundary value problems estimated and numerically solved.

Acknowledgements. D.P.N. gratefully acknowledges support from the National Science Foundation through grant No. DMS–2111283.

References

1. Moloney J, Newell A. 2004 *Nonlinear optics*. Westview Press. Advanced Book Program, Boulder, CO.
2. Boyd RW. 2008 *Nonlinear optics*. Elsevier/Academic Press, Amsterdam third edition.
3. Capretti A, Wang Y, Engheta N, Negro LD. 2015 Enhanced third-harmonic generation in Si-compatible epsilon-near-zero indium tin oxide nanolayers. *Opt. Lett.* **40**, 1500–1503.
4. Alam MZ, Leon ID, Boyd RW. 2016 Large optical nonlinearity of indium tin oxide in its epsilon-near-zero region. *Science* **352**, 795–797.
5. Gallinet B, Butet J, Martin OJF. 2015 Numerical methods for nanophotonics: Standard problems and future challenges. *Laser and Photonics Reviews* **9**, 577–603.
6. Taflove A, Hagness SC. 2000 *Computational electrodynamics: the finite-difference time-domain method*. Artech House, Inc., Boston, MA second edition.
7. Jin J. 2002 *The finite element method in electromagnetics*. Wiley-Interscience [John Wiley & Sons], New York second edition.
8. Hesthaven JS, Warburton T. 2008 *Nodal discontinuous Galerkin methods* vol. 54*Texts in Applied Mathematics*. New York: Springer. Algorithms, analysis, and applications.
9. Busch K, König M, Niegemann J. 2011 Discontinuous Galerkin methods in nanophotonics. *Laser & Photonics Reviews* **5**, 773–809.
10. Draine BT, Flatau PJ. 1994 Discrete-Dipole Approximation For Scattering Calculations. *J. Opt. Soc. Am. A* **11**, 1491–1499.
11. Martin OJF, Piller NB. 1998 Electromagnetic scattering in polarizable backgrounds. *Phys. Rev. E* **58**, 3909–3915.

12. Deville MO, Fischer PF, Mund EH. 2002 *High-order methods for incompressible fluid flow* vol. 9*Cambridge Monographs on Applied and Computational Mathematics*. Cambridge: Cambridge University Press.
13. Gottlieb D, Orszag SA. 1977 *Numerical analysis of spectral methods: theory and applications*. Philadelphia, Pa.: Society for Industrial and Applied Mathematics. CBMS-NSF Regional Conference Series in Applied Mathematics, No. 26.
14. Shen J, Tang T, Wang LL. 2011 *Spectral methods* vol. 41*Springer Series in Computational Mathematics*. Springer, Heidelberg. Algorithms, analysis and applications.
15. Kim H, Park J, Lee B. 2012 *Fourier modal method and its applications in computational nanophotonics*. CRC Press, Boca Raton, FL.
16. Moharam MG, Gaylord TK. 1981 Rigorous coupled-wave analysis of planar-grating diffraction. *J. Opt. Soc. Am.* **71**, 811–818.
17. Moharam MG, Pommet DA, Grann EB, Gaylord TK. 1995 Stable implementation of the rigorous coupled-wave analysis for surface-relief gratings: enhanced transmittance matrix approach. *J. Opt. Soc. Am. A* **12**, 1077–1086.
18. Lalanne P, Morris GM. 1996 Highly improved convergence of the coupled-wave method for TM polarization. *J. Opt. Soc. Am. A* **13**, 779–784.
19. Civiletti BJ, Lakhtakia A, Monk PB. 2020 Analysis of the rigorous coupled wave approach for s-polarized light in gratings. *J. Comput. Appl. Math.* **368**, 112478, 19.
20. Civiletti BJ, Lakhtakia A, Monk PB. 2021 Analysis of the rigorous coupled wave approach for p-polarized light in gratings. *J. Comput. Appl. Math.* **386**, Paper No. 113235, 21.
21. Bonod N, Popov E, Nevière M. 2005 Fourier factorization of nonlinear Maxwell equations in periodic media: application to the optical Kerr effect. *Optics Communications* **244**, 389–398.
22. Bonnefois JJ, Guida G, Priou A, Nevière M, Popov E. 2006 Simulation of two-dimensional Kerr photonic crystals via fast Fourier factorization. *J. Opt. Soc. Am. A* **23**, 842–847.
23. Bej S, Tervo J, Svirko YP, Turunen J. 2014 Modeling the optical Kerr effect in periodic structures by the linear Fourier modal method. *J. Opt. Soc. Am. B* **31**, 2371–2378.
24. Maier R, Verfürth B. 2022 Multiscale scattering in nonlinear Kerr-type media. *Math. Comp.* **91**, 1655–1685.
25. Verfürth B. 2024 Higher-order finite element methods for the nonlinear Helmholtz equation. *J. Sci. Comput.* **98**, Paper No. 66, 24.
26. Griesmaier R, Knöller M, Mandel R. 2022 Inverse medium scattering for a nonlinear Helmholtz equation. *J. Math. Anal. Appl.* **515**, Paper No. 126356, 27.
27. Colton D, Kress R. 2013 *Inverse acoustic and electromagnetic scattering theory* vol. 93*Applied Mathematical Sciences*. Springer, New York third edition.
28. Linton CM. 1998 The Green's function for the two-dimensional Helmholtz equation in periodic domains. *J. Eng. Math.* **33**, 377–402.
29. Meier A, Arens T, Chandler-Wilde S, Kirsch A. 2000 A Nyström method for a class of integral equations on the real line with applications to scattering by diffraction gratings and rough surfaces. *J. Int Equ. Appl.* **12**, 281–321.
30. Reitich F, Tamma K. 2004 State–of–the–art, trends, and directions in computational electromagnetics. *CMES Comput. Model. Eng. Sci.* **5**, 287–294.
31. Kress R. 2014 *Linear integral equations*. New York: Springer-Verlag third edition.
32. Barnett A, Greengard L. 2011 A new integral representation for quasi-periodic scattering problems in two dimensions. *BIT Numerical Mathematics* **51**, 67–90.
33. Lai J, Kobayashi M, Barnett A. 2015 A fast and robust solver for the scattering from a layered periodic structure containing multi-particle inclusions. *J. Comput. Phys.* **298**, 194–208.
34. Cho MH, Barnett A. 2015 Robust fast direct integral equation solver for quasi–periodic scattering problems with a large number of layers. *Optics Express* **23**, 1775–1799.
35. Bruno O, Sei A, Caponi M. 2002 High-order high-frequency solutions of rough surface scattering problems. *Radio Science* **37**, 1–13.
36. Bruno OP, Pérez-Arancibia C. 2017 Windowed Green function method for the Helmholtz equation in the presence of multiply layered media. *Proc. A.* **473**, 20170161, 20.
37. Bruno OP, Lyon M, Pérez-Arancibia C, Turc C. 2016 Windowed Green function method for layered-media scattering. *SIAM J. Appl. Math.* **76**, 1871–1898.
38. Nicholls DP. 2012 Three–Dimensional Acoustic Scattering by Layered Media: A Novel Surface Formulation with Operator Expansions Implementation. *Proceedings of the Royal Society of London, A* **468**, 731–758.

39. Jackson JD. 1975 *Classical electrodynamics*. New York: John Wiley & Sons Inc. second edition.
40. Bruno OP, Reitich F. 1994 Maxwell equations in a nonlinear Kerr medium. *Proc. Roy. Soc. London Ser. A* **447**, 65–76.
41. Nicholls DP, Reitich F. 2004a Shape Deformations in Rough Surface Scattering: Cancellations, Conditioning, and Convergence. *J. Opt. Soc. Am. A* **21**, 590–605.
42. Nicholls DP, Reitich F. 2004b Shape Deformations in Rough Surface Scattering: Improved Algorithms. *J. Opt. Soc. Am. A* **21**, 606–621.
43. Bao G, Li P. [2022] ©2022 *Maxwell's equations in periodic structures* vol. 208*Applied Mathematical Sciences*. Springer, Singapore; Science Press Beijing, Beijing.
44. Yeh P. 2005 *Optical waves in layered media* vol. 61. Wiley-Interscience.
45. Després B. 1991a *Méthodes de décomposition de domaine pour les problèmes de propagation d'ondes en régime harmonique. Le théorème de Borg pour l'équation de Hill vectorielle*. Institut National de Recherche en Informatique et en Automatique (INRIA), Rocquencourt. Thèse, Université de Paris IX (Dauphine), Paris, 1991.
46. Després B. 1991b Domain decomposition method and the Helmholtz problem. In *Mathematical and numerical aspects of wave propagation phenomena (Strasbourg, 1991)* , pp. 44–52. SIAM, Philadelphia, PA.
47. Nicholls DP. 2018 Stable, High–Order Computation of Impedance–Impedance Operators for Three–Dimensional Layered Media Simulations. *Proc. Roy. Soc. Lond., A* **474**, 20170704.
48. Collino F, Ghanemi S, Joly P. 2000 Domain decomposition method for harmonic wave propagation: a general presentation. *Comput. Methods Appl. Mech. Engrg.* **184**, 171–211. Vistas in domain decomposition and parallel processing in computational mechanics.
49. Nicholls DP, Reitich F. 2005 On Analyticity of Traveling Water Waves. *Proc. Roy. Soc. Lond., A* **461**, 1283–1309.
50. Evans LC. 2010 *Partial differential equations*. Providence, RI: American Mathematical Society second edition.
51. Folland GB. 1976 *Introduction to partial differential equations*. Princeton, N.J.: Princeton University Press. Preliminary informal notes of university courses and seminars in mathematics, Mathematical Notes.
52. Nicholls DP, Reitich F. 2001 A new approach to analyticity of Dirichlet-Neumann operators. *Proc. Roy. Soc. Edinburgh Sect. A* **131**, 1411–1433.
53. Friedman A, Reitich F. 2001 Symmetry-breaking bifurcation of analytic solutions to free boundary problems: an application to a model of tumor growth. *Trans. Amer. Math. Soc.* **353**, 1587–1634 (electronic).
54. Ladyzhenskaya OA, Ural'tseva NN. 1968 *Linear and quasilinear elliptic equations*. New York: Academic Press.
55. Nicholls DP, Taber M. 2008 Joint Analyticity and Analytic Continuation for Dirichlet–Neumann Operators on Doubly Perturbed Domains. *J. Math. Fluid Mech.* **10**, 238–271.