SUPPLEMENTARY MATERIAL

A HIGH–ORDER PERTURBATION OF ENVELOPES (HOPE) METHOD FOR VECTOR ELECTROMAGNETIC SCATTERING BY PERIODIC INHOMOGENEOUS MEDIA: JOINT ANALYTICITY *

DAVID P. NICHOLLS † AND LIET VO ‡

Appendix A. Proof of the Elliptic Estimate.

In this appendix we provide the proof of the elliptic estimate which has been so crucial to all of our developments. We restate it here for convenience.

THEOREM A.1. *Given any integer* $s \geq 0$, *if* $(\omega, \bar{\epsilon}) \in \mathcal{P}$, $F \in H^s(S_v)$, $\text{div}\,[F] \in H^{s+1}(S_v)$, $Q \in H^{s+1/2}(\Gamma)$, *and* $R \in H^{s+1/2}(\Gamma)$, *then there exists a unique solution of*

$$\mathcal{L}_0 v = F, \quad \text{in } S_v, \tag{A.1a}$$

$$-\partial_z v - T_u[v] = Q \quad \text{at } \Gamma_h, \tag{A.1b}$$

$$\partial_z v - T_w[v] = R \quad \text{at } \Gamma_{-h}, \tag{A.1c}$$

$$v(x + d_x, y + d_y, z) = e^{i\alpha d_x + i\beta d_y} v(x, y, z), \tag{A.1d}$$

satisfying

$$\|v\|_{H^{s+2}} \leq C_e \left(\|F\|_{H^s} + \|\text{div}\,[F]\|_{H^{s+1}} + \|Q\|_{H^{s+1/2}} + \|R\|_{H^{s+1/2}} \right), \tag{A.2}$$

where $C_e > 0$ *is a constant.*

We will focus on establishing this result in the case $s = 0$ as the case $s > 0$ follows in analogous fashion. To begin, we recall that if the functions v, F, Q, R from the theorem satisfy quasiperiodic boundary conditions, then they can each be expanded in (generalized) Fourier series, e.g.,

$$v(x, y, z) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \hat{v}_{p,q}(z) e^{i\alpha_p x + i\beta_q y}, \quad \hat{v}_{p,q}(z) = \begin{pmatrix} \hat{v}^x_{p,q}(z) \\ \hat{v}^y_{p,q}(z) \\ \hat{v}^z_{p,q}(z) \end{pmatrix}. \tag{A.3}$$

In terms of these expansions we have the following restatement of the governing equations (A.1).

LEMMA A.2. *Let* v *be the solution of* (A.1). *Under the assumptions of Theorem A.1,* $\hat{v}^x_{p,q}$ *and* $\hat{v}^y_{p,q}$ *satisfy the following systems of two point boundary value problems*

$$\partial_z^2 \hat{v}^x_{p,q} + (\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{v}^x_{p,q} = H_{p,q}(z), \quad -h < z < h, \tag{A.4a}$$

$$\partial_z \hat{v}^x_{p,q}(h) - i\gamma^{(u)}_{p,q} \hat{v}^x_{p,q}(h) = -\hat{Q}^x_{p,q}, \tag{A.4b}$$

$$\partial_z \hat{v}^x_{p,q}(-h) + i\gamma^{(w)}_{p,q} \hat{v}^x_{p,q}(-h) = \hat{R}^x_{p,q}, \tag{A.4c}$$

*D.P.N. gratefully acknowledges support from the National Science Foundation through Grant No. DMS–2111283.

†Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, U.S.A. (davidn@uic.edu).

‡Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago, Chicago, IL 60607, U.S.A. (lietvo@uic.edu).

and

$$\partial_z^2 \hat{v}^y_{p,q} + (\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{v}^y_{p,q} = L_{p,q}(z), \qquad -h < z < h, \tag{A.5a}$$

$$\partial_z \hat{v}^y_{p,q}(h) - i\gamma^{(u)}_{p,q} \hat{v}^y_{p,q}(h) = -\hat{Q}^y_{p,q}, \tag{A.5b}$$

$$\partial_z \hat{v}^y_{p,q}(-h) + i\gamma^{(w)}_{p,q} \hat{v}^y_{p,q}(-h) = \hat{R}^y_{p,q}, \tag{A.5c}$$

where

$$\gamma^{(\bar{\epsilon})}_{p,q} := \begin{cases} \sqrt{\bar{\epsilon} k_0^2 - \alpha_p^2 - \beta_q^2}, & \alpha_p^2 + \beta_q^2 \leq \bar{\epsilon} k_0^2, \\ i\sqrt{\alpha_p^2 + \beta_q^2 - \bar{\epsilon} k_0^2} := i\bar{\gamma}_{p,q}, & \alpha_p^2 + \beta_q^2 > \bar{\epsilon} k_0^2, \end{cases}$$

and

$$\gamma^{(m)}_{p,q} := \begin{cases} \sqrt{\epsilon^{(m)} k_0^2 - \alpha_p^2 - \beta_q^2}, & \alpha_p^2 + \beta_q^2 \leq \epsilon^{(m)} k_0^2, \\ i\sqrt{\alpha_p^2 + \beta_q^2 - \epsilon^{(m)} k_0^2} := i\bar{\gamma}^{(m)}_{p,q}, & \alpha_p^2 + \beta_q^2 > \epsilon^{(m)} k_0^2, \end{cases}$$

for $m \in \{u, v\}$*, and* $\bar{\gamma}_{p,q}, \bar{\gamma}^{(m)}_{p,q} \in \mathbf{R}^+$*, and*

$$H_{p,q}(z) := \frac{\alpha_p \beta_q}{k_0^2 \bar{\epsilon}} \hat{F}^y_{p,q} + \frac{\alpha_p^2 - k_0^2 \bar{\epsilon}}{k_0^2 \bar{\epsilon}} \hat{F}^x_{p,q} - \frac{i\alpha_p}{k_0^2 \bar{\epsilon}} \partial_z \hat{F}^z_{p,q}, \tag{A.6a}$$

$$L_{p,q}(z) := \frac{\alpha_p \beta_q}{k_0^2 \bar{\epsilon}} \hat{F}^x_{p,q} + \frac{\beta_q^2 - k_0^2 \bar{\epsilon}}{k_0^2 \bar{\epsilon}} \hat{F}^y_{p,q} - \frac{i\beta_{p,q}}{k_0^2 \bar{\epsilon}} \partial_z \hat{F}^z_{p,q}. \tag{A.6b}$$

Furthermore, we can compute $\hat{v}^z_{p,q}$ *from these as*

$$\hat{v}^z_{p,q} = -\frac{1}{(\gamma^{(\bar{\epsilon})}_{p,q})^2} \hat{F}^z_{p,q} + \frac{i\alpha_p}{(\gamma^{(\bar{\epsilon})}_{p,q})^2} \partial_z \hat{v}^x_{p,q} + \frac{i\beta_q}{(\gamma^{(\bar{\epsilon})}_{p,q})^2} \partial_z \hat{v}^y_{p,q}. \tag{A.7}$$

Proof. We begin with the observation that

$$\text{curl}\,[\text{curl}\,[v]] = -\Delta v + \nabla \text{div}\,[v],$$

so that

$$\mathcal{L}_0[v] = -\Delta v + \nabla \text{div}\,[v] - k_0^2 \bar{\epsilon}.$$

Next we apply $\mathcal{L}_0$ to the expansion (A.3),

$$\mathcal{L}_0[v] = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \mathcal{L}_0[\hat{v}_{p,q}(z) e^{i\alpha_p x + i\beta_q y}],$$

which requires

$$-\Delta \hat{v}^j_{p,q} e^{i\alpha_p x + i\beta_q y} = \left\{ (\alpha_p^2 + \beta_q^2) \hat{v}^j_{p,q} - \partial_z^2 \hat{v}^j_{p,q} \right\} e^{i\alpha_p x + i\beta_q y}, \quad j \in \{x, y, z\},$$

and

$$\text{div}\,\left[\hat{v}_{p,q} e^{i\alpha_p x + i\beta_q y}\right] = \left\{ (i\alpha_p) \hat{v}^x_{p,q} + (i\beta_q) \hat{v}^y_{p,q} + \partial_z \hat{v}^z_{p,q} \right\} e^{i\alpha_p x + i\beta_q y},$$

and

$$\partial_x \text{div}\,\left[\hat{v}_{p,q} e^{i\alpha_p x + i\beta_q y}\right] = \left\{ -\alpha_p^2 \hat{v}^x_{p,q} - \alpha_p \beta_q \hat{v}^y_{p,q} + (i\alpha_p) \partial_z \hat{v}^z_{p,q} \right\} e^{i\alpha_p x + i\beta_q y},$$

$$\partial_y \text{div} \left[\hat{v}_{p,q} e^{i\alpha_p x + i\beta_q y}\right] = \left\{-\alpha_p \beta_q \hat{v}^x_{p,q} - \beta_q^2 \hat{v}^y_{p,q} + (i\beta_q)\partial_z \hat{v}^z_{p,q}\right\} e^{i\alpha_p x + i\beta_q y},$$
$$\partial_z \text{div} \left[\hat{v}_{p,q} e^{i\alpha_p x + i\beta_q y}\right] = \left\{(i\alpha_p)\partial_z \hat{v}^x_{p,q} + (i\beta_q)\partial_z \hat{v}^y_{p,q} + \partial_z^2 \hat{v}^z_{p,q}\right\} e^{i\alpha_p x + i\beta_q y}.$$

From these (A.1a) demands that

$$-\alpha_p \beta_q \hat{v}^y_{p,q} + \beta_q^2 \hat{v}^x_{p,q} - \partial_z^2 \hat{v}^x_{p,q} + (i\alpha_p)\partial_z \hat{v}^z_{p,q} - k_0^2 \bar{\epsilon} \hat{v}^x_{p,q} = \hat{F}^x_{p,q}, \tag{A.8a}$$
$$-\alpha_p \beta_q \hat{v}^x_{p,q} + \alpha_p^2 \hat{v}^y_{p,q} - \partial_z^2 \hat{v}^y_{p,q} + (i\beta_q)\partial_z \hat{v}^z_{p,q} - k_0^2 \bar{\epsilon} \hat{v}^y_{p,q} = \hat{F}^y_{p,q}, \tag{A.8b}$$
$$-(\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{v}^z_{p,q} + (i\alpha_p)\partial_z \hat{v}^x_{p,q} + (i\beta_q)\partial_z \hat{v}^y_{p,q} = \hat{F}^z_{p,q}. \tag{A.8c}$$

If we now multiply (A.8a) by β_q and (A.8b) by α_p, and subtract them we obtain

$$-\beta_q(\partial_z^2 \hat{v}^x_{p,q} + (\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{v}^x_{p,q}) + \alpha_p(\partial_z^2 \hat{v}^y_{p,q} + (\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{v}^y_{p,q}) = \beta_q \hat{F}^x_{p,q} - \alpha_p \hat{F}^y_{p,q}.$$

Furthermore, dividing (A.8c) by $(\gamma^{(\bar{\epsilon})}_{p,q})^2$ and then differentiating the result with respect to z, we obtain

$$\partial_z \hat{v}^z_{p,q} = \frac{1}{(\gamma^{(\bar{\epsilon})}_{p,q})^2} \left((i\alpha_p)\partial_z^2 \hat{v}^x_{p,q} + (i\beta_q)\partial_z^2 \hat{v}^y_{p,q} - \partial_z \hat{F}^z_{p,q}\right).$$

Substituting this into (A.8b) we obtain

$$\alpha_p\beta_q(\partial_z^2 \hat{v}^x_{p,q} + (\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{v}^x_{p,q}) - (\alpha_p^2 - k_0^2\bar{\epsilon})(\partial_z^2 \hat{v}^y_{p,q} + (\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{v}^y_{p,q}) = -(i\beta_q)\partial_z \hat{F}^z_{p,q} - (\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{F}^y_{p,q}.$$

If we denote

$$U := \partial_z^2 \hat{v}^x_{p,q} + (\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{v}^x_{p,q}, \quad W := \partial_z^2 \hat{v}^y_{p,q} + (\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{v}^y_{2n},$$

then we find a system of equations for U and W

$$-\beta_q U + \alpha_p W = \beta_q \hat{F}^x_{p,q} - \alpha_p \hat{F}^y_{p,q},$$
$$\alpha_p\beta_q U - (\alpha_p^2 - k_0^2\bar{\epsilon})W = -(i\beta_q)\partial_z \hat{F}^z_{p,q} - (\gamma^{(\bar{\epsilon})}_{p,q})^2 \hat{F}^y_{p,q}.$$

Solving this system gives us

$$U = \frac{\alpha_p\beta_q}{k_0^2\bar{\epsilon}} \hat{F}^y_{p,q} + \frac{\alpha_p^2 - k_0^2\bar{\epsilon}}{k_0^2\bar{\epsilon}} \hat{F}^x_{p,q} - \frac{i\alpha_p}{k_0^2\bar{\epsilon}} \partial_z \hat{F}^z_{p,q} = H_{p,q}(z),$$
$$W = \frac{\alpha_p\beta_q}{k_0^2\bar{\epsilon}} \hat{F}^x_{p,q} + \frac{\beta_q^2 - k_0^2\bar{\epsilon}}{k_0^2\bar{\epsilon}} \hat{F}^y_{p,q} - \frac{i\beta_q}{k_0^2\bar{\epsilon}} \partial_z \hat{F}^z_{p,q} = L_{p,q}(z).$$

The proof is complete. ☐

Lemma A.3. *The unique solutions* $\hat{v}^x_{p,q}$ *and* $\hat{v}^y_{p,q}$ *of* (A.8a) *and* (A.8b) *are*

$$\hat{v}^x_{p,q}(z) = -\hat{Q}^x_{p,q}\phi_h(z;p,q) - \hat{R}^x_{p,q}\phi_{-h}(z;p,q)\frac{2\bar{\gamma}_{p,q}}{D} - I_h[H_{p,q}](z) - I_{-h}[H_{p,q}](z),$$
$$\hat{v}^y_{p,q}(z) = -\hat{Q}^y_{p,q}\phi_h(z;p,q) - \hat{R}^y_{p,q}\phi_{-h}(z;p,q)\frac{2\bar{\gamma}_{p,q}}{D} - I_h[L_{p,q}](z) - I_{-h}[L_{p,q}](z),$$

where

$$\phi_h(z;p,q) := \left(\frac{\bar{\gamma}_{p,q} + \bar{\gamma}^{(w)}_{p,q}}{D}\right) e^{\bar{\gamma}_{p,q}(z+h)} + \left(\frac{\bar{\gamma}_{p,q} - \bar{\gamma}^{(w)}_{p,q}}{D}\right) e^{-\bar{\gamma}_{p,q}(z+h)},$$

$$\phi_{-h}(z;p,q) := \left(\frac{\bar{\gamma}_{p,q} - \bar{\gamma}_{p,q}^{(u)}}{2\bar{\gamma}_{p,q}}\right) e^{\bar{\gamma}_{p,q}(z-h)} + \left(\frac{\bar{\gamma}_{p,q} + \bar{\gamma}_{p,q}^{(u)}}{2\bar{\gamma}_{p,q}}\right) e^{-\bar{\gamma}_{p,q}(z-h)},$$

$$I_h[\zeta](z) := \int_z^h \phi_h(z;p,q)\phi_{-h}(s;p,q)\zeta(s)\,ds,$$

$$I_{-h}[\zeta](z) := \int_{-h}^z \phi_{-h}(z;p,q)\phi_h(s;p,q)\zeta(s)\,ds,$$

$$D := (\bar{\gamma}_{p,q} + \bar{\gamma}_{p,q}^{(w)})(\bar{\gamma}_{p,q} + \bar{\gamma}_{p,q}^{(u)})e^{2\bar{\gamma}_{p,q}h} - (\bar{\gamma}_{p,q} - \bar{\gamma}_{p,q}^{(u)})(\bar{\gamma}_{p,q} - \bar{\gamma}_{p,q}^{(w)})e^{-2\bar{\gamma}_{p,q}h}.$$

With this we are ready to give the proof of Theorem A.1. As stated above, we provide a detailed proof for the estimate (A.2) in the case when $s = 0$.

Proof. [Theorem A.1] To start we recall that

$$\|v\|_{H^2}^2 := \sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left(\langle(p,q)\rangle^4\int_{-h}^h |\hat{v}_{p,q}(z)|^2\ dz + \langle(p,q)\rangle^2\int_{-h}^h |\partial_z\hat{v}_{p,q}(z)|^2\ dz + \int_{-h}^h \left|\partial_z^2\hat{v}_{p,q}(z)\right|^2\ dz\right).$$

We point out that, since $\bar{\epsilon} \in \mathbf{R}^+$, the indices in the double sum on (p,q) can be divided into two sets: The *propagating modes* which are defined by

$$\bar{\mathbf{P}} := \left\{(p,q) \in \mathbf{Z}^2 \mid \alpha_p^2 + \beta_q^2 \le \bar{\epsilon}k_0^2\right\},$$

and the *evanescent modes* specified by

$$\bar{\mathbf{E}} := \left\{(p,q) \in \mathbf{Z}^2 \mid \alpha_p^2 + \beta_q^2 > \bar{\epsilon}k_0^2\right\}.$$

The former is of *finite* size and gives *complex* $\bar{\gamma}_{p,q} = \gamma_{p,q}^{(\bar{\epsilon})}/i$, while the latter is *unbounded* and features $\bar{\gamma}_{p,q} = \gamma_{p,q}^{(\bar{\epsilon})}/i$ real and positive. From Lemma A.3 we observe that $\{\hat{v}_{p,q}(z), \partial_z\hat{v}_{p,q}(z), \partial_z^2\hat{v}_{p,q}(z)\}$ are all bounded on $-h < z < h$ so that there exists a constant $K_0 > 0$ such that, among the propagating modes,

$$\max\left\{\|\hat{v}_{p,q}\|_{L^2}, \|\partial_z\hat{v}_{p,q}\|_{L^2}, \left\|\partial_z^2\hat{v}_{p,q}\right\|_{L^2}\right\} < K_0, \quad \forall\ (p,q) \in \bar{\mathbf{P}}.$$

As there are only a finite number of these, we can estimate all of them uniformly by

$$\max_{(p,q)\in\bar{\mathbf{P}}}\left\{\left|\hat{Q}_{p,q}\right|, \left|\hat{R}_{p,q}\right|, \left\|\hat{H}_{p,q}\right\|_{L^2}, \left\|\hat{L}_{p,q}\right\|_{L^2}\right\}.$$

For this reason we restrict our subsequent developments to the evanescent modes (where the $\bar{\gamma}_{p,q}$ are real and positive) which requires a careful asymptotic study as $|(p,q)|$ grows.

We begin by estimating $\hat{v}_{p,q}^x$. If we denote

$$a_{p,q} = \bar{\gamma}_{p,q} + \bar{\gamma}_{p,q}^{(w)}, \quad b_{p,q} = \bar{\gamma}_{p,q} - \bar{\gamma}_{p,q}^{(w)}, \quad c_{p,q} = \bar{\gamma}_{p,q} - \bar{\gamma}_{p,q}^{(u)}, \quad d_{p,q} = \bar{\gamma}_{p,q} + \bar{\gamma}_{p,q}^{(u)},$$

then,

$$\phi_h(z;p,q) = \frac{a_{p,q}}{D}e^{\bar{\gamma}_{p,q}(z+h)} + \frac{b_{p,q}}{D}e^{-\bar{\gamma}_{p,q}(z+h)},$$

$$\phi_{-h}(z;p,q) = \frac{c_{p,q}}{2\bar{\gamma}_{p,q}} e^{\bar{\gamma}_{p,q}(z-h)} + \frac{d_{p,q}}{2\bar{\gamma}_{p,q}} e^{-\bar{\gamma}_{p,q}(z-h)},$$
$$D = a_{p,q} d_{p,q} e^{2\bar{\gamma}_{p,q}h} - b_{p,q} c_{p,q} e^{-2\bar{\gamma}_{p,q}h},$$

and $\hat{v}^x_{p,q}(z) = \sum_{j=1}^{10} S_j(z)$, where

$$S_1(z) = -\hat{Q}^x_{p,q} \frac{a_{p,q}}{D} e^{\bar{\gamma}_{p,q}(z+h)}, \quad S_2(z) = -\hat{Q}^x_{p,q} \frac{b_{p,q}}{D} e^{-\bar{\gamma}_{p,q}(z+h)},$$
$$S_3(z) = -\hat{R}^x_{p,q} \frac{c_{p,q}}{2\bar{\gamma}_{p,q}} e^{\bar{\gamma}_{p,q}(z-h)}, \quad S_4(z) = -\hat{R}^x_{p,q} \frac{d_{p,q}}{2\bar{\gamma}_{p,q}} e^{-\bar{\gamma}_{p,q}(z-h)},$$
$$S_5(z) = -\frac{a_{p,q} c_{p,q}}{2\bar{\gamma}_{p,q} D} \int_{-h}^{h} e^{\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s)\, ds,$$
$$S_6(z) = -\frac{b_{p,q} d_{p,q}}{2\bar{\gamma}_{p,q} D} \int_{-h}^{h} e^{-\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s)\, ds,$$
$$S_7(z) = -\frac{a_{p,q} d_{p,q}}{2\bar{\gamma}_{p,q} D} \int_{z}^{h} e^{\bar{\gamma}_{p,q}(z-s+2h)} H_{p,q}(s)\, ds,$$
$$S_8(z) = -\frac{b_{p,q} c_{p,q}}{2\bar{\gamma}_{p,q} D} \int_{-h}^{z} e^{\bar{\gamma}_{p,q}(z-s-2h)} H_{p,q}(s)\, ds,$$
$$S_9(z) = -\frac{b_{p,q} c_{p,q}}{2\bar{\gamma}_{p,q} D} \int_{z}^{h} e^{-\bar{\gamma}_{p,q}(z-s+2h)} H_{p,q}(s)\, ds,$$
$$S_{10}(z) = -\frac{a_{p,q} d_{p,q}}{2\bar{\gamma}_{p,q} D} \int_{-h}^{z} e^{-\bar{\gamma}_{p,q}(z-s-2h)} H_{p,q}(s)\, ds.$$

To estimate $\left\| \hat{v}^x_{p,q} \right\|_{L^2}$ one must address each of these ten terms individually and use

(A.9)
$$\left\| \hat{v}^x_{p,q} \right\|_{L^2} \le \sum_{j=1}^{10} T_j, \quad T_j := \left\| S_j \right\|_{L^2}.$$

For brevity we provide details on two of these, T_1 and T_5. For the former we begin

$$\begin{aligned} T_1 &\le \left| \hat{Q}^x_{p,q} \right| \left(\left| \frac{a_{p,q}}{D} \right|^2 \int_{-h}^{h} e^{2\bar{\gamma}_{p,q}(z+h)}\, dz \right)^{1/2} \\ &= \left| \hat{Q}^x_{p,q} \right| \left(\left| \frac{a_{p,q}}{D} \right|^2 \frac{e^{4\bar{\gamma}_{p,q}h} - 1}{2\bar{\gamma}_{p,q}} \right)^{1/2} \\ &= \frac{\left| \hat{Q}^x_{p,q} \right|}{\sqrt{2\bar{\gamma}_{p,q}}\, |d_{p,q}|} \left(\frac{(e^{4\bar{\gamma}_{p,q}h} - 1)^{1/2}}{\left| e^{2\bar{\gamma}_{p,q}h} - \frac{b_{p,q} c_{p,q}}{a_{p,q} d_{p,q}} e^{-2\bar{\gamma}_{p,q}h} \right|} \right). \end{aligned}$$

Since

$$\lim_{|(p,q)| \to \infty} \frac{(e^{4\bar{\gamma}_{p,q}h} - 1)^{1/2}}{\left| e^{2\bar{\gamma}_{p,q}h} - \frac{b_{p,q} c_{p,q}}{a_{p,q} d_{p,q}} e^{-2\bar{\gamma}_{p,q}h} \right|} = \lim_{|(p,q)| \to \infty} \frac{(1 - e^{-4\bar{\gamma}_{p,q}h})^{1/2}}{\left| 1 - \frac{b_{p,q} c_{p,q}}{a_{p,q} d_{p,q}} e^{-4\bar{\gamma}_{p,q}h} \right|} = 1,$$

there exists a constant $C > 0$ such that

$$\frac{(e^{4\bar{\gamma}_{p,q}h} - 1)^{1/2}}{\left| e^{2\bar{\gamma}_{p,q}h} - \frac{b_{p,q} c_{p,q}}{a_{p,q} d_{p,q}} e^{-2\bar{\gamma}_{p,q}h} \right|} \le C \quad \forall\ (p,q) \in \mathbf{Z}^2,$$

therefore,

$$T_1 \leq C \frac{\left|\hat{Q}^x_{p,q}\right|}{\sqrt{2\bar{\gamma}_{p,q}}\,|d_{p,q}|}.$$

For the latter, by using Hölder's inequality, we obtain

$$\begin{aligned}
T_5 &= \left|\frac{a_{p,q}c_{p,q}}{2\bar{\gamma}_{p,q}D}\right| \left(\int_{-h}^{h} \left|\int_{-h}^{h} e^{\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s)\,ds\right|^2 \,dz\right)^{1/2} \\
&\leq \left|\frac{a_{p,q}c_{p,q}}{2\bar{\gamma}_{p,q}D}\right| \left(\int_{-h}^{h} \left(\int_{-h}^{h} e^{2\bar{\gamma}_{p,q}(z+s)}\,ds\right)\left(\int_{-h}^{h} |H_{p,q}(s)|^2\,ds\right)\,dz\right)^{1/2} \\
&= \left|\frac{a_{p,q}c_{p,q}}{2\bar{\gamma}_{p,q}D}\right| \left(\int_{-h}^{h} \frac{e^{2\bar{\gamma}_{p,q}(z+h)} - e^{2\bar{\gamma}_{p,q}(z-h)}}{2\bar{\gamma}_{p,q}}\,dz\right)^{1/2} \|H_{p,q}\|_{L^2} \\
&= \left|\frac{a_{p,q}c_{p,q}}{2\bar{\gamma}_{p,q}D}\right| \left(\frac{e^{4\bar{\gamma}_{p,q}h} + e^{-4\bar{\gamma}_{p,q}h} - 2}{4\bar{\gamma}^2_{p,q}}\right)^{1/2} \|H_{p,q}\|_{L^2} \\
&= \frac{(e^{4\bar{\gamma}_{p,q}h} + e^{-4\bar{\gamma}_{p,q}h} - 2)^{1/2}}{\left|e^{2\bar{\gamma}_{p,q}h} - \frac{b_{p,q}c_{p,q}}{a_{p,q}d_{p,q}}\,e^{-2\bar{\gamma}_{p,q}h}\right|}\,\frac{|c_{p,q}|\,\|H_{p,q}\|_{L^2}}{4\bar{\gamma}^2_{p,q}\,|d_{p,q}|} \\
&= \frac{(1 + e^{-8\bar{\gamma}_{p,q}h} - 2e^{-4\bar{\gamma}_{p,q}h})^{1/2}}{\left|1 - \frac{b_{p,q}c_{p,q}}{a_{p,q}d_{p,q}}\,e^{-4\bar{\gamma}_{p,q}h}\right|}\,\frac{|c_{p,q}|\,\|H_{p,q}\|_{L^2}}{4\bar{\gamma}^2_{p,q}\,|d_{p,q}|} \leq C\frac{\|H_{p,q}\|_{L^2}}{4\bar{\gamma}^2_{p,q}\,|d_{p,q}|},
\end{aligned}$$

where the last inequality was obtained by using the fact that

$$\lim_{|(p,q)|\to\infty} \frac{(1 + e^{-8\bar{\gamma}_{p,q}h} - 2e^{-4\bar{\gamma}_{p,q}h})^{1/2}}{\left|1 - \frac{b_{p,q}c_{p,q}}{a_{p,q}d_{p,q}}\,e^{-4\bar{\gamma}_{p,q}h}\right|} = 1.$$

Substituting the estimates of $T_1, \ldots, T_{10}$ into (A.9) we obtain

$$\begin{aligned}
\left\|\hat{v}^x_{p,q}\right\|_{L^2} &\leq C\frac{\left|\hat{Q}^x_{p,q}\right|}{\sqrt{\bar{\gamma}_{p,q}}\,|d_{p,q}|} + C\frac{\left|\hat{R}^x_{p,q}\right|}{\sqrt{\bar{\gamma}_{p,q}}\,|a_{p,q}|} + C\frac{\|H_{p,q}\|_{L^2}}{\bar{\gamma}^2_{p,q}\,|d_{p,q}|} + C\frac{\|H_{p,q}\|_{L^2}}{\bar{\gamma}^2_{p,q}} \\
&\leq C\frac{\left|\hat{Q}^x_{p,q}\right|}{\sqrt{\bar{\gamma}_{p,q}}\,|d_{p,q}|} + C\frac{\left|\hat{R}^x_{p,q}\right|}{\sqrt{\bar{\gamma}_{p,q}}\,|a_{p,q}|} + C\frac{\|H_{p,q}\|_{L^2}}{\bar{\gamma}^2_{p,q}}.
\end{aligned} \tag{A.10}$$

Similarly, we obtain

$$\left\|\hat{v}^y_{p,q}\right\|_{L^2} \leq C\frac{\left|\hat{Q}^y_{p,q}\right|}{\sqrt{\bar{\gamma}_{p,q}}\,|d_{p,q}|} + C\frac{\left|\hat{R}^y_{p,q}\right|}{\sqrt{\bar{\gamma}_{p,q}}\,|a_{p,q}|} + C\frac{\|L_{p,q}\|_{L^2}}{\bar{\gamma}^2_{p,q}}. \tag{A.11}$$

Next, we estimate $\hat{v}^z_{p,q}$ and from (A.7) we obtain

$$\begin{aligned}
\hat{v}^z_{p,q} &= \frac{1}{\bar{\gamma}^2_{p,q}}\hat{F}^z_{p,q} - \frac{i\alpha_p}{\bar{\gamma}^2_{p,q}}\partial_z\hat{v}^x_{p,q} - \frac{i\beta_q}{\bar{\gamma}^2_{p,q}}\partial_z\hat{v}^y_{p,q} \\
&= \frac{1}{\bar{\gamma}^2_{p,q}}\hat{F}^z_{p,q} - \frac{i\alpha_p}{\bar{\gamma}^2_{p,q}}\left[-\hat{Q}^x_{p,q}\partial_z\phi_h - \hat{R}^x_{p,q}\partial_z\phi_{-h}\frac{2\bar{\gamma}_{p,q}}{D} - \partial_z I_h[H_{p,q}] - \partial_z I_{-h}[H_{p,q}]\right]
\end{aligned}$$

$$- \frac{i\beta_q}{\bar{\gamma}_{p,q}^2} \left[-\hat{Q}^y_{p,q} \partial_z \phi_h - \hat{R}^y_{p,q} \partial_z \phi_{-h} \frac{2\bar{\gamma}_{p,q}}{D} - \partial_z I_h[L_{p,q}] - \partial_z I_{-h}[L_{p,q}] \right]$$

$$= \frac{1}{\bar{\gamma}_{p,q}^2} \hat{F}^z_{p,q} + \frac{i\partial_z \phi_h}{\bar{\gamma}_{p,q}^2} \left(\alpha_p \hat{Q}^x_{p,q} + \beta_q \hat{Q}^y_{p,q} \right) + \frac{2i\partial_z \phi_{-h}}{\bar{\gamma}_{p,q} D} \left(\alpha_p \hat{R}^x_{p,q} + \beta_q \hat{R}^y_{p,q} \right)$$

$$+ \frac{i\alpha_p}{\bar{\gamma}_{p,q}^2} \left(\partial_z I_h[H_{p,q}] + \partial_z I_{-h}[H_{p,q}] \right) + \frac{i\beta_q}{\bar{\gamma}_{p,q}^2} \left(\partial_z I_h[L_{p,q}] + \partial_z I_{-h}[L_{p,q}] \right).$$

Therefore,

$$\left\| \hat{v}^z_{p,q} \right\|_{L^2} = \left(\int_{-h}^{h} \left| \hat{v}^z_{p,q}(z) \right|^2 \, dz \right)^{1/2} \le \frac{1}{\bar{\gamma}_{p,q}^2} \left\| \hat{F}^z_{p,q} \right\|_{L^2} + \sum_{j=1}^{4} I_j,$$

where

$$I_1 := \frac{\left| \alpha_p \hat{Q}^x_{p,q} + \beta_q \hat{Q}^y_{p,q} \right|}{\bar{\gamma}_{p,q}^2} \left(\int_{-h}^{h} \left| \partial_z \phi_h \right|^2 \, dz \right)^{1/2},$$

$$I_2 := \frac{2 \left| \alpha_p \hat{R}^x_{p,q} + \beta_q \hat{R}^y_{p,q} \right|}{\bar{\gamma}_{p,q} \left| D \right|} \left(\int_{-h}^{h} \left| \partial_z \phi_{-h} \right|^2 \, dz \right)^{1/2},$$

$$I_3 := \frac{|\alpha_p|}{\bar{\gamma}_{p,q}^2} \left(\int_{-h}^{h} \left| \partial_z I_h[H_{p,q}] + \partial_z I_{-h}[H_{p,q}] \right|^2 \, dz \right)^{1/2},$$

$$I_4 := \frac{|\beta_q|}{\bar{\gamma}_{p,q}^2} \left(\int_{-h}^{h} \left| \partial_z I_h[L_{p,q}] + \partial_z I_{-h}[L_{p,q}] \right|^2 \, dz \right)^{1/2}.$$

All four of these I_j must be estimated, but we focus on I_1 and I_3 to streamline our presentation. To start,

$$\partial_z \phi_h(z; p, q) = \frac{a_{p,q} \bar{\gamma}_{p,q}}{D} e^{\bar{\gamma}_{p,q}(z+h)} - \frac{b_{p,q} \bar{\gamma}_{p,q}}{D} e^{-\bar{\gamma}_{p,q}(z+h)},$$

so, by the Hölder Inequality, we have, cancelling a factor of $\bar{\gamma}_{p,q}$,

$$I_1 \le \frac{\left| \alpha_p \hat{Q}^x_{p,q} + \beta_q \hat{Q}^y_{p,q} \right|}{\bar{\gamma}_{p,q}} \left(\left| \frac{a_{p,q}}{D} \right| \left\| e^{\bar{\gamma}(z+h)} \right\|_{L^2} + \left| \frac{b_{p,q}}{D} \right| \left\| e^{-\bar{\gamma}(z+h)} \right\|_{L^2} \right).$$

The discrete Cauchy–Schwartz inequality tells us that

$$\frac{\left| \alpha_p \hat{Q}^x_{p,q} + \beta_q \hat{Q}^y_{p,q} \right|}{\bar{\gamma}_{p,q}} \le \frac{\sqrt{\alpha_p^2 + \beta_q^2} \left| \hat{Q}_{p,q} \right|}{\bar{\gamma}_{p,q}} \le C \left| \hat{Q}_{p,q} \right|,$$

where the final inequality comes from

$$\lim_{|(p,q)| \to \infty} \frac{\sqrt{\alpha_p^2 + \beta_q^2}}{\bar{\gamma}_{p,q}} = 1.$$

So, we continue,

$$I_1 \le C \left| \hat{Q}_{p,q} \right| \left\{ \left| \frac{a_{p,q}}{D} \right| \left(\frac{e^{4\bar{\gamma}_{p,q} h - 1}}{2\bar{\gamma}_{p,q}} \right)^{1/2} + \left| \frac{b_{p,q}}{D} \right| \left(\frac{1 - e^{-4\bar{\gamma}_{p,q}}}{2\bar{\gamma}_{p,q}} \right)^{1/2} \right\}$$

$$\leq C \frac{\left|\hat{Q}_{p,q}\right|}{\sqrt{2\bar{\gamma}_{p,q}}} \left\{ \frac{1}{|d_{p,q}|} \frac{(1-e^{-4\bar{\gamma}_{p,q}h})^{1/2}}{\left|1 - \frac{b_{p,q}c_{p,q}}{a_{p,q}d_{p,q}} e^{-4\bar{\gamma}_{p,q}h}\right|} \right.$$
$$\left. + \frac{|b_{p,q}|}{|a_{p,q}d_{p,q}|\, e^{2\bar{\gamma}_{p,q}h}} \frac{(1-e^{-4\bar{\gamma}_{p,q}h})^{1/2}}{\left|1 - \frac{b_{p,q}c_{p,q}}{a_{p,q}d_{p,q}} e^{-4\bar{\gamma}_{p,q}h}\right|} \right\}.$$

Since

$$\lim_{|(p,q)|\to\infty} \frac{(1-e^{-4\bar{\gamma}_{p,q}h})^{1/2}}{\left|1 - \frac{b_{p,q}c_{p,q}}{a_{p,q}d_{p,q}} e^{-4\bar{\gamma}_{p,q}h}\right|} = 1,$$

we have that

$$I_1 \leq C \frac{\left|\hat{Q}_{p,q}\right|}{\sqrt{2\bar{\gamma}_{p,q}}\, |d_{p,q}|}.$$

Continuing, we have

$$\partial_z I_h[H_{p,q}](z) = \partial_z \phi_h(z) \int_z^h \phi_{-h}(s) H_{p,q}(s)\, ds - \phi_h(z)\phi_{-h}(z) H_{p,q}(z),$$
$$\partial_z I_{-h}[H_{p,q}](z) = \partial_z \phi_{-h}(z) \int_{-h}^z \phi_h(s) H_{p,q}(s)\, ds + \phi_{-h}(z)\phi_h(z) H_{p,q}(z).$$

so that

$$\begin{aligned}
\text{(A.12)}\quad & \partial_z I_h[H_{p,q}](z) + \partial_z I_{-h}[H_{p,q}](z) \\
&= \int_z^h \partial_z \phi_h(z)\phi_{-h}(s) H_{p,q}(s)\, ds + \int_{-h}^z \partial_z \phi_{-h}(z)\phi_h(s) H_{p,q}(s)\, ds \\
&= \frac{a_{p,q}c_{p,q}}{2D} \int_z^h e^{\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s)\, ds + \frac{a_{p,q}d_{p,q}}{2D} \int_z^h e^{-\bar{\gamma}_{p,q}(s-z-2h)} H_{p,q}(s)\, ds \\
&- \frac{b_{p,q}c_{p,q}}{2D} \int_z^h e^{\bar{\gamma}_{p,q}(s-z-2h)} H_{p,q}(s)\, ds - \frac{b_{p,q}d_{p,q}}{2D} \int_z^h e^{-\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s)\, ds \\
&+ \frac{a_{p,q}c_{p,q}}{2D} \int_{-h}^z e^{\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s)\, ds + \frac{b_{p,q}c_{p,q}}{2D} \int_{-h}^z e^{-\bar{\gamma}_{p,q}(s-z+2h)} H_{p,q}(s)\, ds \\
&- \frac{a_{p,q}d_{p,q}}{2D} \int_{-h}^z e^{\bar{\gamma}_{p,q}(s-z+2h)} H_{p,q}(s)\, ds - \frac{b_{p,q}d_{p,q}}{2D} \int_{-h}^z e^{-\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s)\, ds \\
&= \frac{a_{p,q}c_{p,q}}{2D} \int_{-h}^h e^{\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s)\, ds - \frac{b_{p,q}d_{p,q}}{2D} \int_{-h}^h e^{-\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s)\, ds \\
&+ \frac{a_{p,q}d_{p,q}}{2D} \int_z^h e^{-\bar{\gamma}_{p,q}(s-z-2h)} H_{p,q}(s)\, ds + \frac{b_{p,q}c_{p,q}}{2D} \int_{-h}^z e^{-\bar{\gamma}_{p,q}(s-z+2h)} H_{p,q}(s)\, ds \\
&- \frac{b_{p,q}c_{p,q}}{2D} \int_z^h e^{\bar{\gamma}_{p,q}(s-z-2h)} H_{p,q}(s)\, ds - \frac{a_{p,q}d_{p,q}}{2D} \int_{-h}^z e^{\bar{\gamma}_{p,q}(s-z+2h)} H_{p,q}(s)\, ds.
\end{aligned}$$

From this we see that the estimates of $T_5, \dots, T_{10}$ can be used to estimate I_3. In fact, by using the triangle inequality, we obtain

$$I_3 := \frac{|\alpha_p|}{\bar{\gamma}_{p,q}^2} \left(\int_{-h}^h \left|\partial_z I_h[H_{p,q}] + \partial_z I_{-h}[H_{p,q}]\right|^2 \, dz \right)^{1/2}$$

$$\leq \frac{|\alpha_p|}{\bar{\gamma}_{p,q}} (T_5 + T_6 + T_7 + T_8 + T_9 + T_{10}) \leq C \frac{\|H_{p,q}\|_{L^2}}{\bar{\gamma}_{p,q}^2},$$

where

$$\lim_{|(p,q)| \to \infty} \frac{|\alpha_p|}{\bar{\gamma}_{p,q}} = 1,$$

was used to obtain the last inequality above. From all of this we find

$$\text{(A.13)} \quad \left\|\hat{v}_{p,q}^z\right\|_{L^2} \leq \frac{1}{\bar{\gamma}_{p,q}^2} \left\|\hat{F}_{p,q}^z\right\|_{L^2} + C \frac{\left|\hat{Q}_{p,q}\right|}{\sqrt{\bar{\gamma}_{p,q}}\,|d_{p,q}|} + C \frac{\left|\hat{R}_{p,q}\right|}{\sqrt{\bar{\gamma}_{p,q}}\,|a_{p,q}|} + C \frac{\|H_{p,q}\|_{L^2}}{\bar{\gamma}_{p,q}^2} + C \frac{\|L_{p,q}\|_{L^2}}{\bar{\gamma}_{p,q}^2}.$$

From the estimates (A.10), (A.11), and (A.13) we find

$$\begin{aligned} \|\hat{v}_{p,q}\|_{L^2}^2 &= \left\|\hat{v}_{p,q}^x\right\|_{L^2}^2 + \left\|\hat{v}_{p,q}^y\right\|_{L^2}^2 + \left\|\hat{v}_{p,q}^z\right\|_{L^2}^2 \\ &\leq C \left(\frac{\left|\hat{Q}_{p,q}\right|^2}{\bar{\gamma}_{p,q}^3} + \frac{\left|\hat{R}_{p,q}\right|^2}{\bar{\gamma}_{p,q}^3} \right) + C \left(\frac{\|H_{p,q}\|_{L^2}^2}{\bar{\gamma}_{p,q}^4} + \frac{\|L_{p,q}\|_{L^2}^2}{\bar{\gamma}_{p,q}^4} \right) + \frac{\left\|\hat{F}_{p,q}^z\right\|_{L^2}^2}{\bar{\gamma}_{p,q}^4}. \end{aligned}$$

In addition, from (A.6a) and (A.6b), we have

$$\begin{aligned} H_{p,q}(z) &= \frac{-i\alpha_p}{k_0^2 \bar{\epsilon}} (i\alpha_p \hat{F}_{p,q}^x + i\beta_q \hat{F}_{p,q}^y + \partial_z \hat{F}_{p,q}^z) - \hat{F}_{p,q}^x, \\ L_{p,q}(z) &= \frac{-i\beta_q}{k_0^2 \bar{\epsilon}} (i\alpha_p \hat{F}_{p,q}^x + i\beta_q \hat{F}_{p,q}^y + \partial_z \hat{F}_{p,q}^z) - \hat{F}_{p,q}^y, \end{aligned}$$

so that

$$\begin{aligned} \|\hat{v}_{p,q}\|_{L^2}^2 \leq{}& C \left(\frac{\left|\hat{Q}_{p,q}\right|^2}{\bar{\gamma}_{p,q}^3} + \frac{\left|\hat{R}_{p,q}\right|^2}{\bar{\gamma}_{p,q}^3} \right) + C \frac{|\alpha_p|^2}{\bar{\gamma}_{p,q}^4} \left\| i\alpha_p \hat{F}_{p,q}^x + i\beta_q \hat{F}_{p,q}^y + \partial_z \hat{F}_{p,q}^x \right\|_{L^2}^2 \\ &+ C \frac{\left\|\hat{F}_{p,q}^x\right\|_{L^2}^2 + \left\|\hat{F}_{p,q}^y\right\|_{L^2}^2 + \left\|\hat{F}_{p,q}^z\right\|_{L^2}^2}{\bar{\gamma}_{p,q}^4}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q) \rangle^4 \|\hat{v}_{p,q}\|_{L^2}^2 \leq C \Bigg(& \|Q\|_{H^{1/2}}^2 + \|R\|_{H^{1/2}}^2 + \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q) \rangle^0 \left\|\hat{F}_{p,q}\right\|_{L^2}^2 \\ & + \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q) \rangle^2 \left\| i\alpha_p \hat{F}_{p,q}^x + i\beta_q \hat{F}_{p,q}^y + \partial_z \hat{F}_{p,q}^z \right\|_{L^2}^2 \Bigg). \end{aligned} \quad \text{(A.14)}$$

Next, we estimate

$$\sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q) \rangle^2 \|\partial_z \hat{v}_{p,q}\|_{L^2}^2 .$$

For any integer $j \geq 0$, we have

$$\partial_z^j \phi_h = \bar{\gamma}_{p,q}^j \left(\frac{a_{p,q}}{D} e^{\bar{\gamma}_{p,q}(z+h)} + (-1)^j \frac{b_{p,q}}{D} e^{-\bar{\gamma}_{p,q}(z+h)} \right),$$
$$\partial_z^j \phi_{-h} = \bar{\gamma}_{p,q}^j \left(\frac{c_{p,q}}{2\bar{\gamma}_{p,q}} e^{\bar{\gamma}_{p,q}(z-h)} + (-1)^j \frac{d_{p,q}}{2\bar{\gamma}_{p,q}} e^{-\bar{\gamma}_{p,q}(z-h)} \right),$$

and, from the Helmholtz equation,

$$\partial_z^2 \phi_h = \bar{\gamma}_{p,q}^2 \phi_h, \quad \partial_z^2 \phi_{-h} = \bar{\gamma}_{p,q}^2 \phi_{-h}.$$

From Lemma A.2, we also notice that, for $j \geq 0$,

$$\partial_z^j \hat{v}_{p,q}^x(z) = -\hat{Q}_{p,q}^x \partial_z^j \phi_h(z;p,q) - \hat{R}_{p,q}^x \partial_z^j \phi_{-h}(z;p,q) \frac{2\bar{\gamma}_{p,q}}{D}$$
$$- \partial_z^j (I_h[H_{p,q}](z) + I_{-h}[H_{p,q}](z)).$$

Therefore, from (A.12), we obtain

$$\begin{aligned}
\partial_z \hat{v}_{p,q}^x = \bar{\gamma}_{p,q} \Bigg[& -\hat{Q}_{p,q}^x \frac{a_{p,q}}{D} e^{\bar{\gamma}_{p,q}(z+h)} + \hat{Q}_{p,q}^x \frac{b_{p,q}}{D} e^{-\bar{\gamma}_{p,q}(z+h)} \\
& -\hat{R}_{p,q}^x \frac{c_{p,q}}{D} e^{\bar{\gamma}_{p,q}(z-h)} + \hat{R}_{p,q}^x \frac{d_{p,q}}{D} e^{-\bar{\gamma}_{p,q}(z-h)} \Bigg] \\
& - \frac{a_{p,q} c_{p,q}}{2D} \int_{-h}^h e^{\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s) \, ds + \frac{b_{p,q} d_{p,q}}{2D} \int_{-h}^h e^{-\bar{\gamma}_{p,q}(z+s)} H_{p,q}(s) \, ds \\
& + \frac{a_{p,q} d_{p,q}}{2D} \int_z^h e^{\bar{\gamma}_{p,q}(z-s+2h)} H_{p,q}(s) \, ds + \frac{b_{p,q} c_{p,q}}{2D} \int_{-h}^z e^{\bar{\gamma}_{p,q}(z-s-2h)} H_{p,q}(s) \, ds \\
& - \frac{b_{p,q} c_{p,q}}{2D} \int_z^h e^{-\bar{\gamma}_{p,q}(z-s+2h)} H_{p,q}(s) \, ds - \frac{a_{p,q} d_{p,q}}{2D} \int_{-h}^z e^{-\bar{\gamma}_{p,q}(z-s-2h)} H_{p,q}(s) \, ds.
\end{aligned}$$

We observe that the explicit form of $\partial_z \hat{v}_{p,q}^x$ is quite similar to that of $\hat{v}_{p,q}^x$ so we can use the estimates for $T_1, \ldots, T_{10}$ to estimate $\partial_z \hat{v}_{p,q}^x$. In particular we find

$$\begin{aligned}
\left\| \partial_z \hat{v}_{p,q}^x \right\|_{L^2} & \leq \bar{\gamma}_{p,q} (T_1 + \ldots + T_{10}) \\
& \leq C \left(\frac{\left| \hat{Q}_{p,q}^x \right|}{\sqrt{\bar{\gamma}_{p,q}}} + \frac{\left| \hat{R}_{p,q}^x \right|}{\sqrt{\bar{\gamma}_{p,q}}} \right) + C \frac{\|H_{p,q}\|_{L^2}}{\bar{\gamma}_{p,q}}.
\end{aligned}$$

Similarly, we also obtain

$$\begin{aligned}
\left\| \partial_z \hat{v}_{p,q}^y \right\|_{L^2} & \leq \bar{\gamma}_{p,q} (T_1 + \ldots + T_{10}) \\
& \leq C \left(\frac{\left| \hat{Q}_{p,q}^y \right|}{\sqrt{\bar{\gamma}_{p,q}}} + \frac{\left| \hat{R}_{p,q}^y \right|}{\sqrt{\bar{\gamma}_{p,q}}} \right) + C \frac{\|L_{p,q}\|_{L^2}}{\bar{\gamma}_{p,q}}.
\end{aligned}$$

Next, by using (A.4a) and (A.5a), we have

$$\partial_z^2 \hat{v}_{p,q}^x = \bar{\gamma}_{p,q}^2 \hat{v}_{p,q}^x + H_{p,q}(z), \quad \partial_z^2 \hat{v}_{p,q}^y = \bar{\gamma}_{p,q}^2 \hat{v}_{p,q}^y + L_{p,q}(z). \tag{A.15}$$

Therefore,

$$\begin{aligned}\partial_z \hat{v}^z_{p,q} &= \frac{1}{\bar{\gamma}^2_{p,q}} \partial_z \hat{F}^z_{p,q} - \frac{i\alpha_p}{\bar{\gamma}^2_{p,q}} \partial_z^2 \hat{v}^x_{p,q} - \frac{i\beta_q}{\bar{\gamma}^2_{p,q}} \partial_z^2 \hat{v}^y_{p,q} \\ &= \frac{1}{\bar{\gamma}^2_{p,q}} \partial_z \hat{F}^z_{p,q} - i\alpha_p \hat{v}^x_{p,q} - \frac{i\alpha_p}{\bar{\gamma}^2_{p,q}} H_{p,q}(z) - i\beta_q \hat{v}^y_{p,q} - \frac{i\beta_q}{\bar{\gamma}^2_{p,q}} L_{p,q}(z).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\left\|\partial_z \hat{v}^z_{p,q}\right\|_{L^2} \leq \frac{1}{\bar{\gamma}^2_{p,q}} \left\|\partial_z \hat{F}^z_{p,q}\right\|_{L^2} + |\alpha_p| \left\|\hat{v}^x_{p,q}\right\|_{L^2} + |\beta_q| \left\|\hat{v}^y_{p,q}\right\|_{L^2} \\ + \frac{|\alpha_p|}{\bar{\gamma}^2_{p,q}} \left\|H_{p,q}\right\|_{L^2} + \frac{|\beta_q|}{\bar{\gamma}^2_{p,q}} \left\|L_{p,q}\right\|_{L^2}.\end{aligned}$$

So, recalling the estimates of $\hat{v}^x_{p,q}$ and $\hat{v}^y_{2n}$ in (A.10) and (A.11), we obtain

$$\left\|\partial_z \hat{v}^z_{p,q}\right\|_{L^2} \leq \frac{1}{\bar{\gamma}^2_{p,q}} \left\|\partial_z \hat{F}^z_{p,q}\right\|_{L^2} + C\left(\frac{\left|\hat{Q}_{p,q}\right|}{\bar{\gamma}_{p,q}} + \frac{\left|\hat{R}_{p,q}\right|}{\sqrt{\bar{\gamma}_{p,q}}}\right) + C\left(\frac{\left\|H_{p,q}\right\|_{L^2}}{\bar{\gamma}_{p,q}} + \frac{\left\|L_{p,q}\right\|_{L^2}}{\bar{\gamma}_{p,q}}\right).$$

Combining the estimates of $\partial_z \hat{v}^x_{p,q}$, $\partial_z \hat{v}^y_{p,q}$, and $\partial_z \hat{v}^z_{p,q}$ gives

$$\begin{aligned}\left\|\partial_z \hat{v}_{p,q}\right\|^2_{L^2} &= \left\|\partial_z \hat{v}^x_{p,q}\right\|^2_{L^2} + \left\|\partial_z \hat{v}^y_{p,q}\right\|^2_{L^2} + \left\|\partial_z \hat{v}^z_{p,q}\right\|^2_{L^2} \\ &\leq C\left(\frac{\left|\hat{Q}_{p,q}\right|^2}{\bar{\gamma}_{p,q}} + \frac{\left|\hat{R}_{p,q}\right|^2}{\bar{\gamma}_{p,q}}\right) + C\left(\frac{\left\|H_{p,q}\right\|^2_{L^2}}{\bar{\gamma}^2_{p,q}} + \frac{\left\|L_{p,q}\right\|^2_{L^2}}{\bar{\gamma}^2_{p,q}}\right) \\ &\quad + C\frac{1}{\bar{\gamma}^4_{p,q}} \left\|\partial_z \hat{F}^z_{p,q}\right\|^2_{L^2},\end{aligned}$$

which results in

$$\begin{aligned}\sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q)\rangle^2 \left\|\partial_z \hat{v}_{p,q}\right\|^2_{L^2} &\leq C\left(\|Q\|^2_{H^{1/2}} + \|R\|^2_{H^{1/2}}\right) \\ &\quad + C\sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left(\|H_{p,q}\|^2_{L^2} + \|L_{p,q}\|^2_{L^2}\right) \\ &\quad + C\sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q)\rangle^{-2} \left\|\partial_z \hat{F}^z_{p,q}\right\|^2_{L^2} \\ &\leq C\Bigg(\|Q\|^2_{H^{1/2}} + \|R\|^2_{H^{1/2}} + \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q)\rangle^0 \left\|\hat{F}_{p,q}\right\|^2_{L^2} \\ &\quad + \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q)\rangle^2 \left\|i\alpha_p \hat{F}^x_{p,q} + i\beta_q \hat{F}^y_{p,q} + \partial_z \hat{F}^z_{p,q}\right\|^2_{L^2} \\ &\quad + \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \langle (p,q)\rangle^{-2} \left\|\partial_z \hat{F}^z_{p,q}\right\|^2_{L^2}\Bigg). \qquad \text{(A.16)}\end{aligned}$$

We conclude with the estimate of the final sum

$$\sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \left\| \partial_z^2 \hat{v}_{p,q} \right\|_{L^2}^2 .$$

From (A.15) we have

$$\begin{aligned} \left\| \partial_z^2 \hat{v}^x_{p,q} \right\|_{L^2} &\leq \bar{\gamma}_{p,q}^2 \left\| \hat{v}^x_{p,q} \right\|_{L^2} + \left\| H_{p,q} \right\|_{L^2} , \\ \left\| \partial_z^2 \hat{v}^y_{p,q} \right\|_{L^2} &\leq \bar{\gamma}_{p,q}^2 \left\| \hat{v}^y_{p,q} \right\|_{L^2} + \left\| L_{p,q} \right\|_{L^2} , \end{aligned}$$

and the estimates of $\hat{v}^x_{p,q}$ and $\hat{v}^y_{p,q}$ give the following

$$\begin{aligned} \left\| \partial_z^2 \hat{v}^x_{p,q} \right\|_{L^2} &\leq C \sqrt{\bar{\gamma}_{p,q}} \left(\left| \hat{Q}^x_{p,q} \right| + \left| \hat{R}^x_{p,q} \right| \right) + C \left\| H_{p,q} \right\|_{L^2} , \\ \left\| \partial_z^2 \hat{v}^y_{p,q} \right\|_{L^2} &\leq C \sqrt{\bar{\gamma}_{p,q}} \left(\left| \hat{Q}^y_{p,q} \right| + \left| \hat{R}^y_{p,q} \right| \right) + C \left\| L_{p,q} \right\|_{L^2} . \end{aligned}$$

To conclude, we require an estimate of $\partial_z^2 \hat{v}^z_{p,q}$, and, by using (A.15) again, we have

$$\begin{aligned} \partial_z^2 \hat{v}^z_{p,q} &= \frac{1}{\bar{\gamma}_{p,q}^2} \partial_z^2 \hat{F}^z_{p,q} - \frac{i\alpha_p}{\bar{\gamma}_{p,q}^2} \partial_z^3 \hat{v}^x_{p,q} - \frac{i\beta_q}{\bar{\gamma}_{p,q}^2} \partial_z^3 \hat{v}^y_{p,q} \\ &= \frac{1}{\bar{\gamma}_{p,q}^2} \partial_z^2 \hat{F}^z_{p,q} - i\alpha_p \partial_z \hat{v}^x_{p,q} - i\beta_q \partial_z \hat{v}^y_{p,q} - \frac{i\alpha_p}{\bar{\gamma}_{p,q}^2} \partial_z H_{p,q}(z) - \frac{i\beta_q}{\bar{\gamma}_{p,q}^2} \partial_z L_{p,q}(z). \end{aligned}$$

From the estimates of $\partial_z \hat{v}^x_{p,q}$ and $\partial_z \hat{v}^y_{p,q}$ we can bound $\partial_z^2 \hat{v}^z_{p,q}$. More specifically,

$$\begin{aligned} \left\| \partial_z^2 \hat{v}^z_{p,q} \right\|_{L^2} \leq{}& \frac{1}{\bar{\gamma}_{p,q}^2} \left\| \partial_z^2 \hat{F}^z_{p,q} \right\|_{L^2} + C \left(\sqrt{\bar{\gamma}_{p,q}} \left| \hat{Q}_{p,q} \right| + \sqrt{\bar{\gamma}_{p,q}} \left| \hat{R}_{p,q} \right| \right) \\ &+ C \left(\left\| H_{p,q} \right\|_{L^2} + \left\| L_{p,q} \right\|_{L^2} \right) + C \left(\frac{\left\| \partial_z H_{p,q} \right\|_{L^2}}{\bar{\gamma}_{p,q}} + \frac{\left\| \partial_z L_{p,q} \right\|_{L^2}}{\bar{\gamma}_{p,q}} \right). \end{aligned}$$

From this we obtain

$$\begin{aligned} \left\| \partial_z^2 \hat{v}_{p,q} \right\|_{L^2}^2 \leq{}& \frac{C}{\bar{\gamma}_{p,q}^4} \left\| \partial_z^2 \hat{F}^z_{p,q} \right\|_{L^2}^2 + C \left(\bar{\gamma}_{p,q} \left| \hat{Q}_{p,q} \right|^2 + \bar{\gamma}_{p,q} \left| \hat{R}_{p,q} \right|^2 \right) \\ &+ C \left(\left\| H_{p,q} \right\|_{L^2}^2 + \left\| L_{p,q} \right\|_{L^2}^2 \right) + C \left(\frac{\left\| \partial_z H_{p,q} \right\|_{L^2}^2}{\bar{\gamma}_{p,q}^2} + \frac{\left\| \partial_z L_{p,q} \right\|_{L^2}^2}{\bar{\gamma}_{p,q}^2} \right) \\ \leq{}& \frac{C}{\bar{\gamma}_{p,q}^4} \left\| \partial_z \hat{F}^z_{p,q} \right\|_{L^2}^2 + C \left(\bar{\gamma}_{p,q} \left| \hat{Q}_{p,q} \right|^2 + \bar{\gamma}_{p,q} \left| \hat{R}_{p,q} \right|^2 \right) \\ &+ C \left| \alpha_p \right|^2 \left\| i\alpha_p \hat{F}^x_{p,q} + i\beta_q \hat{F}^y_{p,q} + \partial_z \hat{F}^z_{p,q} \right\|_{L^2}^2 \\ &+ C \left(\left\| \hat{F}^x_{p,q} \right\|_{L^2}^2 + \left\| \hat{F}^y_{p,q} \right\|_{L^2}^2 \right) \\ &+ C \left\| i\alpha_p \partial_z \hat{F}^x_{p,q} + i\beta_q \partial_z \hat{F}^y_{p,q} + \partial_z^2 \hat{F}^z_{p,q} \right\|_{L^2}^2 \\ &+ \frac{C}{\bar{\gamma}_{p,q}^2} \left(\left\| \partial_z \hat{F}^x_{p,q} \right\|_{L^2}^2 + \left\| \partial_z \hat{F}^y_{p,q} \right\|_{L^2}^2 \right), \end{aligned}$$

which produces

$$\begin{aligned}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\langle(p,q)\rangle^0\left\|\partial_z^2\hat{v}_{p,q}\right\|_{L^2}^2 \leq C\Bigg(&\|Q\|_{H^{1/2}}^2+\|R\|_{L^2}^2+\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left\|\hat{F}_{p,q}\right\|_{L^2}^2\\ &+\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\langle(p,q)\rangle^2\left\|\hat{F}^x_{p,q}\right\|_{L^2}^2+\left\|\hat{F}^y_{p,q}\right\|_{L^2}^2\\ &+\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left\|i\alpha_p\partial_z\hat{F}^x_{p,q}+i\beta_q\partial_z\hat{F}^y_{p,q}+\partial_z^2\hat{F}^z_{p,q}\right\|_{L^2}^2\\ &+\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\langle(p,q)\rangle^{-2}\left\|\partial_z\hat{F}_{p,q}\right\|_{L^2}^2\\ &+\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\langle(p,q)\rangle^{-4}\left\|\partial_z^2\hat{F}^z_{p,q}\right\|_{L^2}^2\Bigg).\end{aligned} \tag{A.17}$$

Moreover,

$$\text{div}\,[F]=\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}(i\alpha_p\hat{F}^x_{p,q}+i\beta_q\hat{F}^y_{p,q}+\partial_z\hat{F}^z_{p,q})e^{i\alpha_p x+i\beta_q y},$$

so

$$\begin{aligned}\|\text{div}\,[F]\|_{H^1}^2=&\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\langle(p,q)\rangle^2\left\|i\alpha_p\hat{F}^x_{p,q}+i\beta_q\hat{F}^y_{p,q}+\partial_z\hat{F}^z_{p,q}\right\|_{L^2}^2\\ &+\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\langle(p,q)\rangle^0\left\|\partial_z(i\alpha_p\hat{F}^x_{p,q}+i\beta_q\hat{F}^y_{p,q}+\partial_z\hat{F}^z_{p,q})\right\|_{L^2}^2.\end{aligned}$$

Finally, we combine all the estimates from (A.14), (A.16), and (A.17) to obtain

$$\begin{aligned}\|v\|_{H^2}^2 \leq\; & C\left(\|Q\|_{H^{1/2}}^2+\|R\|_{H^{1/2}}^2\right)\\ &+C\left(\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\langle(p,q)\rangle^2\left\|i\alpha_p\hat{F}^x_{p,q}+i\beta_q\hat{F}^y_{p,q}+\partial_z\hat{F}^z_{p,q}\right\|_{L^2}^2\right.\\ &\left.+\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\left\|\partial_z(i\alpha_p\hat{F}^x_{p,q}+i\beta_q\hat{F}^y_{p,q}+\partial_z\hat{F}^z_{p,q})\right\|_{L^2}^2\right)\\ &+C\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}\langle(p,q)\rangle^{-4}\left(\langle(p,q)\rangle^4\left\|\hat{F}_{p,q}\right\|_{L^2}^2+\langle(p,q)\rangle^2\left\|\partial_z\hat{F}_{p,q}\right\|_{L^2}^2+\left\|\partial_z^2\hat{F}_{p,q}\right\|_{L^2}^2\right)\\ \leq\; & C\left(\|Q\|_{H^{1/2}}^2+\|R\|_{H^{1/2}}^2+\|\text{div}\,[F]\|_{H^1}^2+\|F\|_{H^2}^2\right).\end{aligned}$$

With this the proof is complete. □

Appendix B. Numerical Analysis.

At this point we can conduct a numerical analysis for our HOPE scheme. We recall the recursions

$$\text{curl}\,[\text{curl}\,[E_\ell]]-\bar{\epsilon}k_0^2E_\ell=-\bar{\epsilon}k_0^2\mathcal{E}E_{\ell-1}, \qquad \text{in } S_v, \tag{B.1a}$$

$$-\partial_z E_\ell - T_u[E_\ell] = \delta_{\ell,0}\phi, \qquad \text{at } \Gamma_h, \tag{B.1b}$$

$$\partial_z E_\ell - T_w[E_\ell] = 0, \qquad \text{at } \Gamma_{-h}, \tag{B.1c}$$

$$E_\ell(x + d_x, y + d_y, z) = \exp(i\alpha d_x + i\beta d_y) E_\ell(x, y, z), \tag{B.1d}$$

and note that when $\ell = 0$ we can solve this system explicitly via separation of variables. Therefore, in our analysis we simply focus on numerical approximations of solutions of (B.1) when $\ell \geq 1$. First, we consider the (generalized) Fourier series representation of the solution of (B.1)

$$E_\ell(x, y, z) = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \hat{E}_{\ell,p,q}(z) e^{i\alpha_p x + i\beta_q y}.$$

Inserting these forms into (B.1) it is not difficult to see that the three components of $\hat{E}_{\ell,p,q}$,

$$\hat{E}_{\ell,p,q} = \begin{pmatrix} \hat{E}^x_{\ell,p,q} \\ \hat{E}^y_{\ell,p,q} \\ \hat{E}^z_{\ell,p,q} \end{pmatrix}$$

satisfy

$$\partial_z^2 \hat{E}^x_{\ell,p,q} - \bar{\gamma}_{p,q}^2 \hat{E}^x_{\ell,p,q} = H_{\ell,p,q}, \qquad -h < z < h, \tag{B.2a}$$

$$\partial_z \hat{E}^x_{\ell,p,q} - i\gamma^{(u)}_{p,q} \hat{E}^x_{\ell,p,q} = 0, \qquad z = h, \tag{B.2b}$$

$$\partial_z \hat{E}^x_{p,q,\ell} + i\gamma^{(w)}_{p,q} \hat{E}^x_{\ell,p,q} = 0, \qquad z = -h, \tag{B.2c}$$

and

$$\partial_z^2 \hat{E}^y_{\ell,p,q} - \bar{\gamma}_{p,q}^2 \hat{E}^y_{\ell,p,q} = L_{\ell,p,q}, \qquad -h < z < h, \tag{B.3a}$$

$$\partial_z \hat{E}^y_{\ell,p,q} - i\gamma^{(u)}_{p,q} \hat{E}^y_{\ell,p,q} = 0, \qquad z = h, \tag{B.3b}$$

$$\partial_z \hat{E}^y_{p,q,\ell} + i\gamma^{(w)}_{p,q} \hat{E}^y_{\ell,p,q} = 0, \qquad z = -h, \tag{B.3c}$$

and

$$\hat{E}^z_{\ell,p,q} = \frac{1}{\bar{\gamma}_{p,q}^2} \hat{F}^z_{\ell,p,q} - \frac{i\alpha_p}{\bar{\gamma}_{p,q}^2} \partial_z \hat{E}^x_{\ell,p,q} - \frac{i\beta_q}{\bar{\gamma}_{p,q}^2} \partial_z \hat{E}^y_{\ell,p,q}, \tag{B.4}$$

where $H_{\ell,p,q}$ and $L_{\ell,p,q}$ are defined in (A.6a) and (A.6b) in terms of

$$\hat{F}_{\ell,p,q} = \begin{pmatrix} \hat{F}^x_{\ell,p,q} \\ \hat{F}^y_{\ell,p,q} \\ \hat{F}^z_{\ell,p,q} \end{pmatrix}.$$

As (B.2) and (B.3) are effectively identical, we focus our numerical analysis upon the former. For simplicity, we suppress all superscript, x, and subscripts, (ℓ, p, q), and consider the following weak formulation of (B.2):

$$\text{Find } E \in H^1(-h, h) \text{ such that}$$
$$B(E, \varphi) = R(\varphi), \quad \forall\, \varphi \in H^1(-h, h), \tag{B.5}$$

where

$$B(E,\varphi) := -i\gamma_{p,q}^{(u)} E(h)\overline{\varphi}(h) - i\gamma_{p,q}^{(w)} E(-h)\overline{\varphi}(-h) + \int_{-h}^{h} (\partial_z E)\overline{(\partial_z \varphi)}\, dz + \bar{\gamma}_{p,q}^2 \int_{-h}^{h} E\overline{\varphi}\, dz,$$

$$R(\varphi) := \int_{-h}^{h} -H\overline{\varphi}\, dz.$$

Next, we set the finite–dimensional function space for our approximate solution, V^M, to be P^M, the space of all complex–valued polynomials of degree less than or equal to M. With this, the Legendre–Galerkin approximation of (B.5) is

$$\text{Find } E^M \in V^M \text{ such that}$$

$$B(E^M,\varphi) = R(\varphi), \quad \forall\, \varphi \in V^M. \tag{B.6}$$

To prove our main result of this section we quote the following lemma from which defines a projection operator $\mathcal{P}^M$ from $H^1(-h,h)$ to V^M.

Lemma B.1. *There exists a projection operator $\mathcal{P}^M : H^1(-h,h) \to V^M$ satisfying $\mathcal{P}^M u(-h) = u(-h)$ and $\mathcal{P}^M u(h) = u(h)$ such that*

$$\left\| \partial_z (u - \mathcal{P}_M u) \right\|_{H^s} \lesssim M^{s-r} \left\| u \right\|_{H^r}$$

where $0 \le s \le 1 \le r$.

We can now establish the following result.

Theorem B.2. *Let E and E^M be the solutions of* (B.5) *and* (B.6)*, respectively. Then, for any $1 \le r \le M+1$,*

$$\left\| \partial_z (E - E^M) \right\|_{L^2} + \bar{\gamma}_{p,q} \left\| E - E^M \right\|_{L^2} \lesssim M^{1-r} (\left\| E \right\|_{H^r} + \bar{\gamma}_{p,q} \left\| E \right\|_{H^{r-1}}). \tag{B.7}$$

Proof. To begin we denote

$$e^M := E - E^M = (E - \mathcal{P}^M E) + (\mathcal{P}^M E - E^M) := \tilde{e}^M + \hat{e}^M.$$

Subtracting (B.5) from (B.6) we obtain the following error equation

$$\int_{-h}^{h} (\partial_z e^M)\overline{(\partial_z \varphi)}\, dz + \bar{\gamma}_{p,q}^2 \int_{-h}^{h} e^M \overline{\varphi}\, dz = i\gamma_{p,q}^{(u)} e^M(h)\overline{\varphi}(h) + i\gamma_{p,q}^{(w)} e^M(-h)\overline{\varphi}(-h), \quad \forall \varphi \in V^M.$$

Taking $\varphi = \hat{e}^M \in V^M$ in this equation, we obtain

$$\begin{aligned} \left\| \partial_z \hat{e}^M \right\|_{L^2}^2 + \bar{\gamma}_{p,q}^2 \left\| \hat{e}^M \right\|_{L^2}^2 &= -\int_{-h}^{h} (\partial_z \tilde{e}^M)\overline{(\partial_z \hat{e}^M)}\, dz - \bar{\gamma}_{p,q}^2 \int_{-h}^{h} \tilde{e}^M \overline{\hat{e}^M}\, dz \\ &\quad + i\gamma_{p,q}^{(u)} e^M(h)\overline{\hat{e}^M}(h) + i\gamma_{p,q}^{(w)} e^M(-h)\overline{\hat{e}^M}(-h) \\ &= -\int_{-h}^{h} (\partial_z \tilde{e}^M)\overline{(\partial_z \hat{e}^M)}\, dz - \bar{\gamma}_{p,q}^2 \int_{-h}^{h} \tilde{e}^M \overline{\hat{e}^M}\, dz \\ &\quad + i\gamma_{p,q}^{(u)} |\hat{e}^M(h)|^2 + i\gamma_{p,q}^{(w)} |\hat{e}^M(-h)|^2 \\ &\quad + i\gamma_{p,q}^{(u)} \tilde{e}^M(h)\overline{\hat{e}^M}(h) + i\gamma_{p,q}^{(w)} \tilde{e}^M(-h)\overline{\hat{e}^M}(-h). \end{aligned} \tag{B.8}$$

Now, we recall that

$$\gamma_{p,q}^{(m)} = \begin{cases} \sqrt{(k^{(m)})^2 - \alpha_p^2 - \beta_q^2}, & \alpha_p^2 + \beta_q^2 \le (k^{(m)})^2, \\ i\sqrt{\alpha_p^2 + \beta_q^2 - (k^{(m)})^2}, & \alpha_p^2 + \beta_q^2 > (k^{(m)})^2. \end{cases}$$

First we consider $\alpha_p^2 + \beta_q^2 \le (k^{(m)})^2$ so that $\gamma_{p,q}^{(m)}$ is real and nonegative. Taking the imaginary part of (B.8), we obtain

$$\begin{aligned} \gamma_{p,q}^{(u)} |\hat{e}^M(h)|^2 + \gamma_{p,q}^{(w)} |\hat{e}^M(-h)|^2 &= \mathrm{Im}\left\{(\partial_z \tilde{e}^M, \partial_z \hat{e}^M)\right\} + \bar{\gamma}_{p,q}^2 \mathrm{Im}\left\{(\tilde{e}^M, \hat{e}^M)\right\} \\ &\quad - \mathrm{Im}\left\{ i\gamma_{p,q}^{(u)} \tilde{e}^M(h) \overline{\hat{e}^M}(h) \right\} \\ &\quad - \mathrm{Im}\left\{ i\gamma_{p,q}^{(w)} \tilde{e}^M(-h) \overline{\hat{e}^M}(-h) \right\} \\ &=: I_1 + I_2. \end{aligned}$$

We estimate I_1 and I_2 as follows. Using the Cauchy-Schwarz inequality, for some $\delta_1, \delta_2 > 0$, we have

$$\begin{aligned} I_1 &= \mathrm{Im}\left\{(\partial_z \tilde{e}^M, \partial_z \hat{e}^M)\right\} + \bar{\gamma}_{p,q}^2 \mathrm{Im}\left\{(\tilde{e}^M, \hat{e}^M)\right\} \\ &\le \delta_1 \left\|\partial_z \hat{e}^M\right\|_{L^2}^2 + \frac{1}{4\delta_1} \left\|\partial_z \tilde{e}^M\right\|_{L^2}^2 + \delta_2 \bar{\gamma}_{p,q}^2 \left\|\hat{e}^M\right\|_{L^2}^2 + \frac{\bar{\gamma}_{p,q}^2}{4\delta_2} \left\|\tilde{e}^M\right\|_{L^2}^2, \\ I_2 &= -\mathrm{Im}\left\{ i\gamma_{p,q}^{(u)} \tilde{e}^M(h) \overline{\hat{e}^M}(h) \right\} - \mathrm{Im}\left\{ i\gamma_{p,q}^{(w)} \tilde{e}^M(-h) \overline{\hat{e}^M}(-h) \right\} \\ &\le \frac{\gamma_{p,q}^{(u)}}{2} \left|\hat{e}^M(h)\right|^2 + \frac{\gamma_{p,q}^{(u)}}{2} \left|\tilde{e}^M(h)\right|^2 + \frac{\gamma_{p,q}^{(w)}}{2} \left|\hat{e}^M(-h)\right|^2 + \frac{\gamma_{p,q}^{(w)}}{2} \left|\tilde{e}^M(-h)\right|^2. \end{aligned}$$

Combining the estimates of I_1 and I_2 we obtain

$$\begin{aligned} \gamma_{p,q}^{(u)} \left|\hat{e}^M(h)\right|^2 + \gamma_{p,q}^{(w)} \left|\hat{e}^M(-h)\right|^2 &\le 2\delta_1 \left\|\partial_z \hat{e}^M\right\|_{L^2}^2 + \frac{1}{2\delta_1} \left\|\partial_z \tilde{e}^M\right\|_{L^2}^2 \\ &\quad + 2\delta_2 \bar{\gamma}_{p,q}^2 \left\|\hat{e}^M\right\|_{L^2}^2 + \frac{\bar{\gamma}_{p,q}^2}{2\delta_2} \left\|\tilde{e}^M\right\|_{L^2}^2 \\ &\quad + \gamma_{p,q}^{(u)} \left|\tilde{e}^M(h)\right|^2 + \gamma_{p,q}^{(w)} \left|\tilde{e}^M(-h)\right|^2. \end{aligned} \tag{B.9}$$

In addition, taking the real part of (B.8), we obtain

$$\begin{aligned} \left\|\partial_z \hat{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^2 \left\|\hat{e}^M\right\|_{L^2}^2 &= \mathrm{Re}\left\{-(\partial_z \tilde{e}^M, \partial_z \hat{e}^M)\right\} + \bar{\gamma}_{p,q}^2 \mathrm{Re}\left\{-(\tilde{e}^M, \hat{e}^M)\right\} \\ &\quad + \mathrm{Re}\left\{ i\gamma_{p,q}^{(u)} \tilde{e}^M(h) \overline{\hat{e}^M}(h) \right\} + \mathrm{Re}\left\{ i\gamma_{p,q}^{(w)} \tilde{e}^M(-h) \overline{\hat{e}^M}(-h) \right\} \\ &=: I_3 + I_4. \end{aligned} \tag{B.10}$$

Using the Cauchy-Schwarz inequality, we estimate I_3, I_4 as follows.

$$\begin{aligned} I_3 &= \mathrm{Re}\left\{-(\partial_z \tilde{e}^M, \partial_z \hat{e}^M)\right\} + \bar{\gamma}_{p,q}^2 \mathrm{Re}\left\{-(\tilde{e}^M, \hat{e}^M)\right\} \\ &\le \frac{1}{4} \left\|\partial_z \hat{e}^M\right\|_{L^2}^2 + \left\|\partial_z \tilde{e}^M\right\|_{L^2}^2 + \frac{\bar{\gamma}_{p,q}^2}{4} \left\|\hat{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^2 \left\|\tilde{e}^M\right\|_{L^2}^2, \\ I_4 &= \mathrm{Re}\left\{ i\gamma_{p,q}^{(u)} \tilde{e}^M(h) \overline{\hat{e}^M}(h) \right\} + \mathrm{Re}\left\{ i\gamma_{p,q}^{(w)} \tilde{e}^M(-h) \overline{\hat{e}^M}(-h) \right\} \end{aligned}$$

$$\leq \gamma_{p,q}^{(u)} \left|\hat{e}^M(h)\right|^2 + \frac{\gamma_{p,q}^{(u)}}{4} \left|\tilde{e}^M(h)\right|^2 + \gamma_{p,q}^{(w)} \left|\hat{e}^M(-h)\right|^2 + \frac{\gamma_{p,q}^{(w)}}{4} \left|\tilde{e}^M(-h)\right|^2.$$

Substituting the estimates of I_3, I_4 into (B.10) and then using (B.9) we obtain

$$\begin{aligned}
\left\|\partial_z \hat{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^2 \left\|\hat{e}^M\right\|_{L^2}^2 &\leq \frac{4}{3}(\left\|\partial_z \tilde{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^2 \left\|\tilde{e}^M\right\|_{L^2}^2) \\
&\quad + \frac{\gamma_{p,q}^{(u)}}{3} \left|\tilde{e}^M(h)\right|^2 + \frac{\gamma_{p,q}^{(w)}}{3} \left|\tilde{e}^M(-h)\right|^2 \\
&\quad + \frac{4}{3}(\gamma_{p,q}^{(u)} \left|\hat{e}^M(h)\right|^2 + \gamma_{p,q}^{(w)} \left|\hat{e}^M(-h)\right|^2) \\
&\leq \frac{4}{3}(\left\|\partial_z \tilde{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^2 \left\|\tilde{e}^M\right\|_{L^2}^2) \\
&\quad + \frac{\gamma_{p,q}^{(u)}}{3} \left|\tilde{e}^M(h)\right|^2 + \frac{\gamma_{p,q}^{(w)}}{3} \left|\tilde{e}^M(-h)\right|^2 \\
&\quad + \frac{4}{3}(2\delta_1 \left\|\partial_z \hat{e}^M\right\|_{L^2}^2 + \frac{1}{2\delta_1} \left\|\partial_z \tilde{e}^M\right\|_{L^2}^2 \\
&\quad + 2\delta_2 \bar{\gamma}_{p,q}^2 \left\|\hat{e}^M\right\|_{L^2}^2 + \frac{\bar{\gamma}_{p,q}^2}{2\delta_2} \left\|\tilde{e}^M\right\|_{L^2}^2 \\
&\quad + \gamma_{p,q}^{(u)} \left|\tilde{e}^M(h)\right|^2 + \gamma_{p,q}^{(w)} \left|\tilde{e}^M(-h)\right|^2),
\end{aligned}$$

which implies that

$$\begin{aligned}
\left\|\partial_z \hat{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^2 \left\|\hat{e}^M\right\|_{L^2}^2 \lesssim \left\|\partial_z \tilde{e}^M\right\|_{L^2}^2 &+ \bar{\gamma}_{p,q}^2 \left\|\tilde{e}^M\right\|_{L^2}^2 \\
&+ \gamma_{p,q}^{(u)} \left|\tilde{e}^M(h)\right|^2 + \gamma_{p,q}^{(w)} \left|\tilde{e}^M(-h)\right|^2.
\end{aligned}$$

Finally, using the Gagliardo–Nirenberg interpolation inequality,

$$|\tilde{e}(\pm 1)|^2 \lesssim \|\tilde{e}\|_{L^2} \|\tilde{e}\|_{H^1} \lesssim \|\tilde{e}\|_{H^1}^2 \lesssim M^{2(1-r)} \|E\|_{H^r}^2$$

and using Lemma B.1, we obtain the desired estimate (B.7).

Next, if $\alpha_p^2 + \beta_q^2 > (k^{(m)})^2$ then $\gamma_{p,q}^{(m)}$ is purely complex and we define its imaginary part

$$\bar{\gamma}_{p,q}^{(m)} = \operatorname{Im}\left\{\sqrt{\alpha_p^2 + \beta_q^2 - (k^{(m)})^2}\right\}.$$

Since $i\gamma_{p,q}^{(m)} = -\bar{\gamma}_{p,q}^{(m)}$ we rewrite (B.8) as

$$\begin{aligned}
\text{(B.11)}\quad \left\|\partial_z \hat{e}^M\right\|_{L^2}^2 &+ \bar{\gamma}_{p,q}^2 \left\|\hat{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^{(u)} \left|\hat{e}^M(h)\right|^2 + \bar{\gamma}_{p,q}^{(w)} \left|\hat{e}^M(-h)\right|^2 \\
&= -\int_{-h}^{h} (\partial_z \tilde{e}^M)\overline{(\partial_z \hat{e}^M)}\, dz - \bar{\gamma}_{p,q}^2 \int_{-h}^{h} \tilde{e}^M \overline{\hat{e}^M}\, dz \\
&\quad - \bar{\gamma}_{p,q}^{(u)} \tilde{e}^M(h) \overline{\hat{e}^M}(h) - \bar{\gamma}_{p,q}^{(w)} \tilde{e}^M(-h) \overline{\hat{e}^M}(-h).
\end{aligned}$$

Now, applying the Cauchy-Schwarz inequality to the right hand side of (B.11) we obtain

$$\left\|\partial_z \hat{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^2 \left\|\hat{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^{(u)} \left|\hat{e}^M(h)\right|^2 + \bar{\gamma}_{p,q}^{(w)} \left|\hat{e}^M(-h)\right|^2$$

$$\leq \frac{1}{2}\left\|\partial_z \hat{e}^M\right\|_{L^2}^2 + \frac{1}{2}\left\|\partial_z \tilde{e}^M\right\|_{L^2}^2 + \frac{\bar{\gamma}_{p,q}^2}{2}\left\|\hat{e}\right\|_{L^2}^2 + \frac{\bar{\gamma}_{p,q}^2}{2}\left\|\tilde{e}^M\right\|_{L^2}^2$$
$$+ \frac{\bar{\gamma}_{p,q}^{(u)}}{2}\left|\hat{e}^M(h)\right|^2 + \frac{\bar{\gamma}_{p,q}^{(u)}}{2}\left|\tilde{e}^M(h)\right|^2 + \frac{\bar{\gamma}_{p,q}^{(w)}}{2}\left|\hat{e}^M(-h)\right|^2 + \frac{\bar{\gamma}_{p,q}^{(w)}}{2}\left|\tilde{e}^M(-h)\right|^2,$$

which implies that

$$\left\|\partial_z \hat{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^2\left\|\hat{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^{(u)}\left|\hat{e}^M(h)\right|^2 + \bar{\gamma}_{p,q}^{(w)}\left|\hat{e}^M(-h)\right|^2$$
$$\leq \left\|\partial_z \tilde{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^2\left\|\tilde{e}^M\right\|_{L^2}^2 + \bar{\gamma}_{p,q}^{(u)}\left|\tilde{e}^M(h)\right|^2 + \bar{\gamma}_{p,q}^{(w)}\left|\tilde{e}^M(-h)\right|^2.$$

The proof is completed by using the Gagliardo–Nirenberg interpolation inequality and Lemma B.1. □

REMARK B.1. *In exactly analogous fashion we obtain the estimate found in Theorem B.2 for $\hat{E}^y_{\ell,p,q}$. Furthermore, due to* (B.4) *the error estimate for $\hat{E}^z_{\ell,p,q}$ is achieved by the error estimates of $\hat{E}^{x,M}_{\ell,p,q}$ and $\hat{E}^{y,M}_{\ell,p,q}$. Namely, we have*

$$\left\|\hat{E}^z_{\ell,p,q} - \hat{E}^{z,M}_{\ell,p,q}\right\|_{L^2} \leq \frac{|\alpha_p|}{\bar{\gamma}_{p,q}^2}\left\|\partial_z(\hat{E}^x_{\ell,p,q} - \hat{E}^{x,M}_{\ell,p,q})\right\|_{L^2} + \frac{|\beta_q|}{\bar{\gamma}_{p,q}^2}\left\|\partial_z(\hat{E}^y_{\ell,p,q} - \hat{E}^{y,M}_{\ell,p,q})\right\|_{L^2},$$

which infers that

$$\begin{aligned}\left\|\hat{E}^z_{\ell,p,q} - \hat{E}^{z,M}_{\ell,p,q}\right\|_{L^2} &\lesssim \frac{\max\{|\alpha_p|,|\beta|_q\}}{\bar{\gamma}_{p,q}^2} M^{1-r}\left\{\left\|\hat{E}^x_{\ell,p,q}\right\|_{H^r} + \left\|\hat{E}^y_{\ell,p,q}\right\|_{H^r}\right.\\ &\quad \left. + \bar{\gamma}_{p,q}\left(\left\|\hat{E}^x_{\ell,p,q}\right\|_{H^{r-1}} + \left\|\hat{E}^y_{\ell,p,q}\right\|_{H^{r-1}}\right)\right\}\\ &\lesssim \frac{M^{1-r}}{\bar{\gamma}_{p,q}}\left\{\left\|\hat{E}^x_{\ell,p,q}\right\|_{H^r} + \left\|\hat{E}^y_{\ell,p,q}\right\|_{H^r}\right.\\ &\quad \left. + \bar{\gamma}_{p,q}\left(\left\|\hat{E}^x_{\ell,p,q}\right\|_{H^{r-1}} + \left\|\hat{E}^y_{\ell,p,q}\right\|_{H^{r-1}}\right)\right\}.\end{aligned}$$

Next, we recall the expansion

$$E(x,y,z) = \sum_{\ell=0}^{\infty} E_\ell(x,y,z)\delta^\ell, \tag{B.12}$$

and consider the following Fourier–Legendre approximation of the solution E_ℓ from (B.12)

$$E_\ell^{P,Q,M}(x,y,z) := \sum_{(p,q)\in\Gamma_{P,Q}} \hat{E}^{\ell,M}_{p,q}(z) e^{i\alpha_p x + i\beta_q y},$$

where

$$\Gamma_{P,Q} = \{(p,q) \in \mathbf{Z}^2 \mid |p| \leq P, |q| \leq Q\}.$$

Then, we have the following error estimate.

THEOREM B.3. *Let E_ℓ be the solution of the ℓ–th correction from the Maxwell system and let $r \geq 2$. Then*

$$\left\|E_\ell - E_\ell^{P,Q,M}\right\|_{L^2} \lesssim (P^{1-r} + Q^{1-r} + M^{1-r})\left\|E_\ell\right\|_{H^r}.$$

Proof. We have

$$E_\ell(x,y,z) - E_\ell^{P,M}(x,y,z) = \sum_{(p,q)\in\Gamma_{P,Q}} \left(\hat{E}_{\ell,p,q} - \hat{E}^M_{\ell,p,q}\right) e^{i\alpha_p x + i\beta_q y} + \sum_{(p,q)\in\mathbf{Z}^2\setminus\Gamma_{P,Q}} \hat{E}_{\ell,p,q} e^{i\alpha_p x + i\beta_q y}.$$

Therefore,

$$\left\|E_\ell - E_\ell^{P,Q,M}\right\|^2_{L^2} \lesssim \sum_{(p,q)\in\Gamma_{P,Q}} \left\|\hat{E}_{\ell,p,q} - \hat{E}^M_{\ell,p,q}\right\|^2_{L^2} + \sum_{n\in\mathbf{Z}^2\setminus\Gamma_P} \left\|\hat{E}_{\ell,p,q}\right\|^2_{L^2} =: X_1 + X_2.$$

From Theorem B.2 and Remark B.1 we obtain

$$\begin{aligned} X_1 &= \sum_{(p,q)\in\Gamma_{P,Q}} \left(\left\|\hat{E}^x_{\ell,p,q} - \hat{E}^{x,M}_{\ell,p,q}\right\|^2_{L^2} + \left\|\hat{E}^y_{ell,p,q} - \hat{E}^{y,M}_{\ell,p,q}\right\|^2_{L^2} + \left\|\hat{E}^z_{ell,p,q} - \hat{E}^{z,M}_{\ell,p,q}\right\|^2_{L^2} \right) \\ &\lesssim M^{2(1-r)} \sum_{(p,q)\in\Gamma_{P,Q}} \left(\left\|\hat{E}^x_{\ell,p,q}\right\|_{H^r} + \left\|\hat{E}^y_{\ell,p,q}\right\|_{H^r} \right). \end{aligned}$$

In addition,

$$\begin{aligned} X_2 &\lesssim (P^{2(1-r)} + Q^{2(1-r)}) \sum_{(p,q)\in\mathbf{Z}^2\setminus\Gamma_{P,Q}} \langle (p,q)\rangle^{2(r-1)} \left\|\hat{E}_{\ell,p,q}\right\|^2_{L^2} \\ &\lesssim (P^{2(1-r)} + Q^{2(1-r)}) \left\|E_\ell\right\|_{H^\cdot} \end{aligned}$$

The proof is complete by combining the estimates for X_1 and X_2. ☐

Finally, we consider the full HOPE–Fourier–Legendre–Galerkin approximation $E^{N,P,M}$ of the solution E of the full Maxwell system,

$$E^{L,P,Q,M}(x,y,z) := \sum_{\ell=0}^{L} E_\ell^{P,Q,M}(x,y,z)\delta^\ell.$$

We can estimate the full error by combining all the error estimates from Theorems B.2 and B.3 in the following result.

Theorem B.4. *Let E be the solution of the full Maxwell system and let $r \geq 2$, then*

$$\left\|E - E^{L,P,Q,M}\right\|_{L^2} \lesssim (B\delta)^{L+1} + (P^{1-r} + Q^{1-r} + M^{1-r}) \left\|\phi\right\|_{H^{r+1/2}},$$

for any constant

$$B > C_e k_0^2 \bar{\epsilon} \tilde{M} \left|\mathcal{E}\right|_{C^s},$$

giving convergence for $\delta < 1/B$.