On Analyticity of Variations of the Dirichlet–Neumann Operators, and Computational Concerns

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To my friends and family, who make my life thoroughly enjoyable.

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TABLE OF CONTENTS

CHAPTER			PAGE
1	INTRODUCTION		1
2	EQUATIONS OF MOTION		3
	2.1	Navier-Stokes and Euler Equations	3
	2.2	The Dirichlet–Neumann Operator	6
	2.3	Surface Formulation	7
3	SPECTRAL STABILITY OF WATER WAVES AND THE VARIATION OF THE DNO		10
	3.1	Dynamic Stability and the Functional Variation	10
	3.2	Spectral Stability Form	11
	3.3	Surface Formulation as Dynamical System	12
4	THE DIRICHLET–NEUMANN OPERATOR		15
	4.1	Perturbation Expansion Methods	16
	4.2	Operator Expansions	16
	4.3	Transformed Field Expansion	20
	4.4	Analyticity of DNO	24
5	VARIATION OF THE DNO		26
	5.1	Analyticity of the First variation of the DNO	28
	5.2	Higher Variations	38
	5.3	Analyticity of Higher Variations	43
6	NUMERICAL SIMULATION OF THE VARIATION OF THE DNO		61
	6.1	Spectral Collocation Method	61
	6.2	Exact Solution	66
	6.3	Sample Profiles	68
	6.4	Numerical Results	69
7	CONCLUSION		79
	7.1	Advantages	79
	7.2	Disadvantages	80
	7.3	Future Directions	81
	CITED LITERATURE		82

TABLE OF CONTENTS (Continued)

CHAPTER PAGE

VITA 85

LIST OF FIGURES

FIGURE		PAGE
1	Smooth Surface Deformation, Smooth Variation Direction	70
2	Smooth Surface Deformation, Rough Variation Direction	71
3	Smooth Surface Deformation, Lipschitz Variation Direction	72
4	Rough Surface Deformation, Smooth Variation Direction	73
5	Rough Surface Deformation, Rough Variation Direction	74
6	Rough Surface Deformation, Lipschitz Variation Direction	75
7	Lipschitz Surface Deformation, Smooth Variation Direction	76
8	Lipschitz Surface Deformation, Rough Variation Direction	77
9	Lipschitz Surface Deformation, Lipschitz Variation Direction	78

LIST OF ABBREVIATIONS

BVP	Boundary Value Problem
DFT	Discrete Fourier Transform
DNO	Dirichlet–Neumann Operator
FBP	Free Boundary Problem
FFT	Fast Fourier Transform
PDE	Partial Differential Equation

SUMMARY

One of the important open questions in the theory of free–surface ideal fluid flows is the dynamic stability of traveling wave solutions. In a spectral stability analysis, the first variation of the governing Euler equations is required which raises both theoretical and numerical issues. With Zakharov's 1968 and Craig & Sulem's 1993 formulation of the Euler equations in mind, we address the question of analyticity properties of first, and higher, variations of the Dirichlet–Neumann operator. This analysis will have consequences not only for theoretical investigations, but also for numerical simulations of the spectral stability of traveling water waves. We present the outcome of some computational experiments, as well as describe future applications of these results.

CHAPTER 1

INTRODUCTION

We begin by considering the famous governing equations for the evolution of an ideal fluid flow with a free air-fluid interface. The Navier-Stokes and Euler equations have been studied extensively since their initial formulation. Work by Zakharov (Zak68) demonstrated the Hamiltonian structure of the Euler equations, highlighting the importance of two surface quantities, the fluid interface shape and the velocity potential evaluated there. Craig & Sulem (CS93) then reformulated the problem into evolution equations posed entirely at the surface. Crucial in their reformulation is the appearance of the Dirichlet–Neumann Operator (DNO). We will view these surface formulation evolution equations, together with the DNO, as forming a dynamical system. Using some elementary dynamical systems theory, we find that the first functional variation of the Dirichlet–Neumann Operator is of significant interest. The DNO itself has been shown to posess a number of desirable characteristics, including analyticity with respect to the boundary shape, which justify a perturbation series expansion. These results are available through a number of approaches. Here, we apply techniques similar to those used by Nicholls & Reitich (NR03) to investigate the first variation of the DNO. We find that this operator possesses various desirable characteristics, and its series expansion is particularly amenable to numerical implementation. We execute several computational experiments in which current methodologies are tested against existing techniques. The results indicate that our new method should be utilized over alternative methods, particularly for extreme profiles which are very rough or of large amplitude. Our theoretical and computational work provides a possible new direction into the spectral stability analysis of the Euler equations.

CHAPTER 2

EQUATIONS OF MOTION

The following sections elucidate the background of our problem at hand. We describe the basic governing equations, as historically formulated. Common assumptions are made clear, as are certain motivations. We define the domain of interest.

2.1 Navier-Stokes and Euler Equations

The most famous equations in fluid mechanics are arguably the Navier-Stokes Equations. A detailed treatment and elementary analysis of these equations is available from many sources (Ach90), which we provide here for background. The equations are:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \frac{1}{\rho}\nabla p - \nu \Delta \mathbf{u} - \mathbf{g} = 0 \tag{2.1.1a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2.1.1b}$$

where $\mathbf{u}$ is the velocity field, ρ is the fluid density, p is the fluid pressure, ν is the *kinematic* fluid viscosity, and $\mathbf{g}$ is the gravitational force vector. It is often desirable to represent $\mathbf{g}$ as the gradient of a potential, $\mathbf{g} = -\nabla \chi$, where $\chi = \tilde{g}y$. Equation (2.1.1b), $\nabla \cdot \mathbf{u} = 0$, indicates that the fluid is incompressible. Incompressiblity is one of several assumptions that, when made together, define an *ideal fluid*. The other assumptions are that the fluid is inviscid ($\nu = 0$), irrotational ($\nabla \times \mathbf{u} = 0$), and of constant density ($\rho = 1$).

The assumption of irrotationality implies that the velocity field $\mathbf{u}$ can be written as the gradient of a scalar velocity potential: $\mathbf{u} = \nabla\phi$. By substituting $u = \nabla\phi$ into (2.1.1b), we see that this velocity potential will satisfy Laplace's equation $\Delta\phi = 0$.

Applying the vector identity $(\nabla \cdot F)F = (\nabla \times F) \times F + \nabla(\frac{1}{2}F^2)$ and using the fact that $\nabla \times \mathbf{u} = 0$, (2.1.1a) can be written as:

$$\nabla(\frac{\partial\phi}{\partial t} + p + \frac{1}{2}|\nabla\phi|^2 + \chi) = 0.$$

Integrating yields:

$$\frac{\partial\phi}{\partial t} + p + \frac{1}{2}|\nabla\phi|^2 + \chi = f(t),$$

where $f(t)$ is an arbitrary function of t, whose presence does not effect the velocity field $\mathbf{u}$. Because of this, we may choose it at our convenience. Relative to atmospheric thickness, water wave amplitudes are negligible, and we may assume that atmospheric pressure is constant and equal to, say, p_0 along the entire free surface. Specifying the air-fluid interface by $y = \eta(x,t)$, the water pressure at $y = \eta(x,t)$ is also constant and equal to the same value p_0. By choosing $f(t) = p_0$, we have:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \tilde{g}\eta = 0 \qquad \text{at } y = \eta(x,t).$$

This is commonly known as Bernoulli's condition at the free surface.

A second condition at the free surface is commonly known as the Kinematic Condition. This condition is often stated as requiring that fluid particles on the free surface stay at the free surface for all time. The mathematical representation is given by:

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial y} + \nabla_x \phi \cdot \nabla_x \eta = 0 \qquad \text{at } y = \eta(x,t), \tag{2.1.2}$$

The final boundary condition that must be specified is at the lower boundary of our domain. Here, we require only that our fluid does not penetrate the bottom. Because the fluid is inviscid, we do not enforce a "no-slip" condition that is often associated with phyical domain boundaries.

For the remainder of this work, we consider the problem posed on the domain $S_{h,\eta}$, defined below. This domain is characterized by periodicity in the lateral directions, with a flat bottom at depth $y = -h$ and a free surface $y = \eta(x,t)$ with mean zero:

$$S_{h,\eta} = \{(x,y) | (x,y) \in \mathbf{R}^{d-1} \times \mathbf{R}, -h \leq y \leq \eta(x,t)\}. \tag{2.1.3}$$

where $d = 2, 3$. The periodicity in the lateral x direction is with respect to a $(d-1)$-dimensional lattice Γ, so that, for instance, $\eta(x+\gamma, t) = \eta(x,t) \quad \forall \gamma \in \Gamma$. One of the primary difficulties associated with Free Boundary Problems (FBP) such as the Euler equations is that the domain on which the PDEs are posed evolves with time. Some details regarding this difficulty will be discussed later.

Let us summarize the situation so far, now that we have obtained a complete specification of the boundary conditions. The velocity potential ϕ together with the time-dependent free surface η must satisfy:

$$\Delta\phi = 0 \qquad \text{in } S_{h,\eta} \tag{2.1.4a}$$

$$\partial_y\phi = 0 \qquad \text{at } y = -h \tag{2.1.4b}$$

$$\partial_t\eta - \partial_y\phi + \nabla_x\eta \cdot \nabla_x\phi = 0 \qquad \text{at } y = \eta \tag{2.1.4c}$$

$$\partial_t\phi - \tilde{g}\eta + \frac{1}{2}\left|\nabla\phi\right|^2 = 0 \qquad \text{at } y = \eta, \tag{2.1.4d}$$

together with periodicity with respect to Γ. The final two equations are the kinematic and Bernoulli conditions, enforced at the free surface. They will henceforth be referred to as the evolution equations.

2.2 The Dirichlet–Neumann Operator

At this point, we introduce the Dirichlet–Neumann Operator to permit a restatement of (2.1.4) at the boundary. While the surface formulation of the problem due to Zakharov (Zak68) is implicit in nature, introduction of the DNO by Craig & Sulem (CS93) clarifies the issue. In general, the DNO is a boundary operator that takes Dirichlet data as input and returns Neumann data. Clearly, the explicit representation of this operator is highly dependent on the problem geometry. For concreteness, we define the DNO with respect to a generic elliptic problem which mirrors our present context of free-surface fluid flow:

$$\Delta v = 0 \qquad \text{in } S_{h,g} \tag{2.2.1a}$$

$$v(x, g(x)) = \xi(x) \tag{2.2.1b}$$

$$\partial_y v(x, -h) = 0 \tag{2.2.1c}$$

$$v(x+\gamma, y) = v(x, y) \qquad \forall\, \gamma \in \Gamma. \tag{2.2.1d}$$

So long as g is sufficiently smooth, (2.2.1) admits a unique solution v, whose normal derivative at the surface $y = g$ is easy to calculate. The DNO carries out this procedure by mapping the Dirichlet data, ξ, to the Neumann data:

$$G(g)[\xi] := [\nabla v]_{y=g} \cdot N = \partial_y v(x, g(x)) - \nabla_x g \cdot \nabla_x v(x, g(x)),$$

where $N = (-\nabla_x g, 1)^T$ is an exterior normal to $S_{h,g}$.

2.3 Surface Formulation

Zakharov's (Zak68) seminal formulation of the Euler equations as a Hamiltonian system is implicit in nature, thus it is desirable to find a more explicit surface formulation. Craig & Sulem (CS93) posed equations (2.1.4) in terms of only surface quantities, $\xi(x,t) = \phi(x, \eta(x,t), t)$, the velocity potential at the free surface, and $\eta(x,t)$. After some straightforward applications of the chain rule, $\partial_t \phi$ and $\nabla \phi$ are rewritten in terms of $\partial_t \xi$ and $\nabla \xi$. Inspecting the terms in

(2.1.4c), we see that the DNO makes its first explicit appearance. The evolution equations are now formulated solely at the surface:

$$\partial_t \eta = G(\eta)[\xi] \tag{2.3.1a}$$

$$\begin{aligned}\partial_t \xi = -\tilde{g}\eta - \frac{1}{2\left(1+|\nabla_x \eta|^2\right)} \Big[&|\nabla_x \xi|^2 - (G(\eta)[\xi])^2 \\ &-2\left(\nabla_x \xi \cdot \nabla_x \eta\right) G(\eta)[\xi] + |\nabla_x \xi|^2 |\nabla_x \eta|^2 - (\nabla_x \xi \cdot \nabla_x \eta)^2 \Big] .\end{aligned} \tag{2.3.1b}$$

Any analysis of the Euler equations, in particular the dynamic stability of traveling waves, can be performed equivalently on the surface equations (2.3.1).

Before proceeding, we recall that while the dependence of the DNO upon the Dirichlet data, ξ, is linear, the g dependence is genuinely nonlinear. In particular, this dependence is parametrically analytic (NR01) (see also (NR03; HN05; NT08)) which implies the strong convergence (see Theorem 4.4.2) of the following expansion:

$$G(g)[\xi] = G(\epsilon f)[\xi] = \sum_{n=0}^{\infty} G_n(f)[\xi]\epsilon^n \tag{2.3.2}$$

for $g(x) = \epsilon f(x)$ sufficiently small. Using this expansion the action of the DNO can be approximated by the truncated Taylor series:

$$G^N(g)[\xi] := \sum_{n=0}^{N} G_n(f)[\xi]\epsilon^n,$$

a method which has been used with great success in a number of numerical simulations (CS93; Sch97; Nic98; Nic01; CN02; GN05; GN07). Our purpose is to justify a similar expansion for the first variation of the DNO for use in spectral stability simulations.

CHAPTER 3

SPECTRAL STABILITY OF WATER WAVES AND THE VARIATION OF THE DNO

In this section, we motivate our study of variations of the DNO by interpreting the evolution equations (2.3.1) as a dynamical system. One of the fundamental questions in dynamical systems theory is the stability of solutions, for instance the evolution of a small perturbation of an equilibrium solution. Stability can be determined from the growth or decay of this perturbation.

3.1 Dynamic Stability and the Functional Variation

Consider the generic dynamical system:

$$\partial_t u = F(u), \tag{3.1.1}$$

which possesses an equilibrium solution $u(x,t) = \bar{u}(x)$. To decide upon the dynamic stability of $\bar{u}$, one adds to it a small perturbation:

$$u(x,t) = \bar{u}(x) + \delta\tilde{u}(x,t), \quad \delta \ll 1,$$

and studies the evolution of $\tilde{u}$. This form of u in (3.1.1) results in:

$$\partial_t \tilde{u} = \delta_u F(\bar{u})[\tilde{u}] + \mathcal{O}(\delta). \tag{3.1.2}$$

Here, the notation $\delta_u F(\bar{u})[\tilde{u}]$ denotes the first functional variation of the right hand side F of (3.1.1). For the purposes of this paper, we use Gateaux's definition of the variation of a functional F with respect to a function φ at φ_0 in the direction ψ as:

$$\delta_\varphi F(\varphi_0)\{\psi\} := \lim_{\tau \to 0} \frac{1}{\tau} \left[F(\varphi_0 + \tau\psi) - F(\varphi_0) \right]. \tag{3.1.3}$$

We will be using the following notation from this point onward:

$$\delta_\varphi F(\varphi_0)\{\psi\} := F^{(1)}(\varphi_0)\{\psi\}.$$

3.2 Spectral Stability Form

The evolution of the perturbation $\tilde{u}$ partially determines the stability results. Assuming a form:

$$\tilde{u}(x,t) = e^{\lambda t} w(x) \tag{3.2.1}$$

and ignoring the term $\mathcal{O}(\delta)$ in (3.1.2) comprises a linear spectral stability analysis:

$$\lambda w = \delta_u F(\bar{u})[w] =: \mathcal{A}(x)[w]. \tag{3.2.2}$$

The analysis is linear due to the neglect of the $\mathcal{O}(\delta)$ term, and is spectral due to analysis of the eigenvalues λ.

3.3 Surface Formulation as Dynamical System

The importance of computing the variation of the DNO becomes apparent when we view the surface formulation of the evolution equations (2.3.1) (modified slightly to incorporate a moving reference frame and solving for the time derivatives) as a dynamical system, with

$$u(x,t) = \begin{pmatrix} \eta(x,t) \\ \xi(x,t) \end{pmatrix},$$

and F representing the right-hand-side of (2.3.1). In this context the stationary equilibrium solution $\bar{u}(x)$ represents a traveling Stokes wave, which appears stationary in the moving reference frame. After perturbing the Stokes wave by $\tilde{u}(x,t)$, we require the first variation of F. Since this right-hand-side involves the DNO, a stability analysis will involve the first variation of the DNO.

It is of significant concern to specify which perturbations $w(x)$ in (3.2.1) will be considered. A natural first choice is the set of periodic functions with respect to the same lattice of periodicity Γ which controls $\bar{u}$. However, results for more general classes of perturbation are desirable. This problem is addressed by the "Generalized Principle of Reduced Instability" developed by Mielke (Mie97), which is essentially the Floquet theory of differential equations with periodic coefficients (DK06). This method distills the general setting of L^2 perturbations to the study of the "Bloch waves":

$$w(x) = e^{ip\cdot x} W(x),$$

where $W(x)$ is periodic with respect to Γ. The period lattice of the linear operator $\mathcal{A}(x)$ is the same as that of the original problem, since $\mathcal{A}$ inherits its properties from $\bar{u}$. The theory shows that the full L^2 spectral stability problem can be decided by simply considering Bloch waves with $p \in P(\Gamma')$, the fundamental cell of wavenumbers (e.g., if $\Gamma = (2\pi)\mathbf{Z}$, then $\Gamma' = \mathbf{Z}$, and $P(\Gamma') = [0,1]$). Thus we are left with the spectral problem (Mie97)

$$\mathcal{A}_p[W] = \lambda W,$$

c.f. (3.2.2), where $\mathcal{A}_p$ is the "Bloch operator"

$$\mathcal{A}_p[W] := e^{-ip\cdot x} \mathcal{A}[e^{ip\cdot x} W].$$

The crucial spectral identity (see (Mie97), Theorems 2.1 and A.4) is:

$$L^2\text{–spec}(\mathcal{A}) = L^2_{lu}\text{–spec}(\mathcal{A}) = \text{closure}\left(\bigcup_{p\in P(\Gamma')} \text{spec}(\mathcal{A}_p) \right), \tag{3.3.1}$$

where L^2_{lu} is the space of uniformly local L^2 functions. Thus, we can obtain information about stability with respect to *all* of these perturbations by simply considering periodic perturbations $W(x)$ and $\mathcal{A}_p$ with $p \in P(\Gamma')$ appearing as a parameter (Mie97).

For the current theoretical developments, this Bloch analysis is equivalent to considering the linear operator $\mathcal{A}$ acting on "Bloch periodic" (quasiperiodic) functions $w(x)$ which satisfy the "Bloch boundary conditions":

$$w(x+\gamma) = e^{ip\cdot\gamma}w(x), \quad \forall\, \gamma \in \Gamma.$$

Notice that if p is a rational number then these functions will be periodic with respect to the lattice Γ.

CHAPTER 4

THE DIRICHLET–NEUMANN OPERATOR

The purpose of this section is to familiarize the reader with the Dirichlet–Neumann Operator (DNO) and some of the techniques used to analyze it. Recall that the DNO is an operator which maps given Dirichlet data to the corresponding Neumann data on the boundary. In the setting of free-surface fluid flows, the DNO has representation:

$$G(g)[\xi] := [\nabla v]_{y=g} \cdot N = \partial_y v(x, g(x)) - \nabla_x g \cdot \nabla_x v(x, g(x)). \tag{4.0.1}$$

For certain geometries, this operator is simple to define. Namely, the case where $g(x) \equiv 0$ results in:

$$G(0)[\xi] = \partial_y v(x, 0). \tag{4.0.2}$$

The solution v to (2.2.1) is represented in the $g = 0$ case as:

$$v(x, y) = \sum_{k \in \Gamma'} \frac{\cosh(|k|(y + h))}{\cosh(|k|h)} \hat{\xi}_k e^{ik \cdot x}, \tag{4.0.3}$$

and therefore

$$G(0)[\xi] = \partial_y v(x, 0) = \sum_{k \in \Gamma'} |k| \tanh(|k|h) \hat{\xi}_k e^{ik \cdot x}. \tag{4.0.4}$$

We can use Fourier multiplier notation $D := -i\nabla_x$ to compactly express this operator as

$$G(0)[\xi] = |D| \tanh(|D|h)\xi. \tag{4.0.5}$$

The tanh term reflects the finite depth of our domain. In the limit $h \to \infty$, we have $G(0) \to |D|$

4.1 Perturbation Expansion Methods

The simplicity of the DNO in the case of separable geometry suggests a perturbative approach about the idle state $g(x) = 0$. We assume a small amplitude surface deformation, $g(x) = \epsilon f(x), \epsilon \ll 1$, and expand our operators and fields as:

$$\begin{aligned} v(x,y) &= \sum_{n=0}^{\infty} v_n(x,y)\epsilon^n \\ v^{(1)}(x,y) &= \sum_{n=0}^{\infty} v_n^{(1)}(x,y)\epsilon^n \\ G(\epsilon f)[\xi] &= \sum_{n=0}^{\infty} G_n(f)[\xi]\epsilon^n \\ G^{(1)}(\epsilon f)[\xi]\{w\} &= \sum_{n=0}^{\infty} G_n^{(1)}(f)[\xi]\epsilon^n \end{aligned}$$

4.2 Operator Expansions

The question now lies in the determination of terms in the expansions. Two common methods include Operator Expansions (OE) (Mil91) and Transformed Field Expansions (TFE) (NR01). The former method, which we describe here, involves expanding the definition of the

DNO as given in (4.0.1), without calculating the full field. For a given solution of (2.2.1a), (2.2.1c), and (2.2.1d),

$$v_k(x,y) = \cosh(|k|(y+h))e^{ik\cdot x}, \tag{4.2.1}$$

we have

$$G(\epsilon f)[\cosh(|k|(\epsilon f+h))e^{ik\cdot x}] = (\partial_y - \epsilon\nabla_x f\cdot\nabla_x)(\cosh(|k|(y+h))e^{ik\cdot x})|_{y=\epsilon f}. \tag{4.2.2}$$

Our goal is to use the linearity of G in the Dirichlet (square-bracket) argument to isolate its action on the basis function $e^{ik\cdot x}$. Once we have this, we sum over all $k \in \Gamma'$ to find the action on an arbitrary function $\xi(x)$. We expand the argument $\cosh(|k|(\epsilon f+h))e^{ik\cdot x}$ in a Taylor series, as well as G itself. We obtain expressions for the even and odd numbered terms of G. For clarity, we express the operators as Fourier multipliers. For $n = 0$ we have the familiar:

$$G_0(f) = |D_x|\tanh(|D_x|h). \tag{4.2.3}$$

For odd numbered terms $n = 2j-1 > 0$, we have:

$$\begin{aligned} G_{2j-1}(f) = {} & \frac{1}{(2j-1)!}D_x\cdot f^{2j-1}D_x|D_x|^{2(j-1)} \\ & -\sum_{s=0}^{j-1}\frac{1}{(2(j-s)-1)!}G_{2s}(f)[f^{2(j-s)-1}|D_x|^{2(j-s-1)}G_0] \\ & -\sum_{s=0}^{j-2}\frac{1}{(2(j-s-1))!}G_{2s+1}(f)[f^{2(j-s-1)}|D_x|^{2(j-s-1)}], \end{aligned} \tag{4.2.4a}$$

and for even numbered terms $n = 2j > 0$ we have:

$$\begin{aligned} G_{2j}(f) = & \frac{1}{(2j)!} D_x \cdot f^{2j} D_x |D_x|^{2(j-1)} G_0 \\ & - \sum_{s=0}^{j-1} \frac{1}{(2(j-s))!} G_{2s}(f) [f^{2(j-s)} |D_x|^{2(j-s)}] \\ & - \sum_{s=0}^{j-1} \frac{1}{(2(j-s)-1)!} G_{2s+1}(f) [f^{2(j-s)-1} |D_x|^{2(j-s-1)} G_0]. \end{aligned} \tag{4.2.5a}$$

Self adjointness properties of the DNO, for instance $G^* = G, G_n^* = G_n, |D|^* = |D|$, allow us to finally write the terms as:

$$G_0(f) = |D_x| \tanh(|D_x| h), \tag{4.2.6a}$$

$$\begin{aligned} G_{2j-1}(f) = & \frac{1}{(2j-1)!} |D_x|^{2(j-1)} D_x \cdot f^{2j-1} D_x \\ & - \sum_{s=0}^{j-1} \frac{1}{(2(j-s)-1)!} G_0 |D_x|^{2(j-s-1)} f^{2(j-s)-1} G_{2s}(f) \\ & - \sum_{s=0}^{j-2} \frac{1}{(2(j-s-1))!} |D_x|^{2(j-s-1)} f^{2(j-s-1)} G_{2s+1}(f), \end{aligned} \tag{4.2.6b}$$

$$\begin{aligned} G_{2j}(f) = & \frac{1}{(2j)!} G_0 |D_x|^{2(j-1)} D_x \cdot f^{2j} D_x \\ & - \sum_{s=0}^{j-1} \frac{1}{(2(j-s))!} |D_x|^{2(j-s)} f^{2(j-s)} G_{2s}(f) \\ & - \sum_{s=0}^{j-1} \frac{1}{(2(j-s)-1)!} G_0 |D_x|^{2(j-s-1)} f^{2(j-s)-1} G_{2s+1}(f). \end{aligned} \tag{4.2.6c}$$

To find $G^{(1)}$, we simply take the functional variation of each term in the expansion above, taking care to note the expansion order:

$$\delta_\eta G_0(f)\{w\} = 0, \tag{4.2.7a}$$

$$\begin{aligned}
\delta_\eta G_{2j-1}(f)\{w\} = {} & \frac{1}{(2(j-1))!}|D_x|^{2(j-1)} D_x \cdot f^{2(j-1)} w D_x \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s-1))!} G_0 |D_x|^{2(j-s-1)} f^{2(j-s-1)} w G_{2s}(f) \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s)-1)!} G_0 |D_x|^{2(j-s-1)} f^{2(j-s)-1} \delta_\eta G_{2s}(f) \\
& - \sum_{s=0}^{j-2} \frac{1}{(2(j-s-1)-1)!} |D_x|^{2(j-s-1)} f^{2(j-s-1)-1} w G_{2s+1}(f) \\
& - \sum_{s=0}^{j-2} \frac{1}{(2(j-s-1))!} |D_x|^{2(j-s-1)} f^{2(j-s-1)} \delta_\eta G_{2s+1}(f),
\end{aligned} \tag{4.2.7b}$$

$$\begin{aligned}
\delta_\eta G_{2j}(f)\{w\} = {} & \frac{1}{(2j-1)!} G_0 |D_x|^{2(j-1)} D_x \cdot f^{2j-1} w D_x \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s)-1)!} |D_x|^{2(j-s)} f^{2(j-s)-1} w G_{2s}(f) \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s))!} |D_x|^{2(j-s)} f^{2(j-s)} \delta_\eta G_{2s}(f) \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s-1))!} G_0 |D_x|^{2(j-s-1)} f^{2(j-s-1)} w G_{2s+1}(f) \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s)-1)!} G_0 |D_x|^{2(j-s-1)} f^{2(j-s)-1} \delta_\eta G_{2s+1}(f).
\end{aligned} \tag{4.2.7c}$$

While we note that $\delta_\eta G_0(f)\{w\} = 0$, it is not the case that $G_0^{(1)} = 0$. When using the superscript (1) to denote variation, we choose to let the subscript denote the power of ϵ in the expansion.

Since $\delta_\eta G_1(f)\{w\} = D_x w D_x \xi - G_0 w G_0 \xi$ is $\mathcal{O}(1)$, we see that the perturbation order in the variation is reduced by one, and we account for that in the subscript:

$$G_0^{(1)}(f)\{w\} = D_x w D_x - G_0 w G_0, \tag{4.2.8a}$$

$$\begin{aligned}
G_{2j-1}^{(1)}(f)\{w\} = {} & \frac{1}{(2j-1)!} G_0 |D_x|^{2(j-1)} D_x \cdot f^{2j-1} w D_x \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s)-1)!} |D_x|^{2(j-s)} f^{2(j-s)-1} w G_{2s}(f) \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s))!} |D_x|^{2(j-s)} f^{2(j-s)} G_{2s}^{(1)}(f) \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s-1))!} G_0 |D_x|^{2(j-s-1)} f^{2(j-s-1)} w G_{2s+1}(f) \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s)-1)!} G_0 |D_x|^{2(j-s-1)} f^{2(j-s)-1} G_{2s+1}^{(1)}(f),
\end{aligned} \tag{4.2.8b}$$

$$\begin{aligned}
G_{2j-2}^{(1)}(f)\{w\} = {} & \frac{1}{(2(j-1))!} |D_x|^{2(j-1)} D_x \cdot f^{2(j-1)} w D_x \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s-1))!} G_0 |D_x|^{2(j-s-1)} f^{2(j-s-1)} w G_{2s}(f) \\
& - \sum_{s=0}^{j-1} \frac{1}{(2(j-s)-1)!} G_0 |D_x|^{2(j-s-1)} f^{2(j-s)-1} G_{2s}^{(1)}(f) \\
& - \sum_{s=0}^{j-2} \frac{1}{(2(j-s-1)-1)!} |D_x|^{2(j-s-1)} f^{2(j-s-1)-1} w G_{2s+1}(f) \\
& - \sum_{s=0}^{j-2} \frac{1}{(2(j-s-1))!} |D_x|^{2(j-s-1)} f^{2(j-s-1)} G_{2s+1}^{(1)}(f).
\end{aligned} \tag{4.2.8c}$$

4.3 Transformed Field Expansion

Another approach to computing the DNO is to make a change of variables that alters the shape of the original problem domain. After the change of variables, the DNO is found via an expansion of the transformed field. This method is called Transformed Field Expansion (TFE), and has a number of theoretical and practical advantages over the OE approach outlined above. The change of variables that constitutes the first step in TFE is:

$$x' = x, \qquad y' = h\left(\frac{y - g(x)}{h + g(x)}\right), \tag{4.3.1}$$

which transforms the domain $S_{h,g}$ to $S_{h,0}$. The differential operators transform by:

$$\begin{aligned}
(h + g(x))\nabla_x &= (h + g(x'))\nabla_{x'} - (h + y')(\nabla_{x'} g(x'))\partial_{y'} \\
(h + g(x))\mathrm{div}_x &= (h + g(x'))\mathrm{div}_{x'} - (h + y')(\nabla_{x'} g(x')) \cdot \partial_{y'} \\
(h + g(x))\partial_y &= h\partial_{y'},
\end{aligned}$$

and the system (2.2.1) becomes a PDE and boundary conditions for the unknown transformed field:

$$u(x', y') := v\left(x', \frac{y'(h + g(x'))}{h} + g(x')\right).$$

These equations are, upon dropping primes,

$$\Delta u = F(x, y; g, u) \qquad \text{in } S_{h,0} \tag{4.3.2a}$$

$$u(x, 0) = \xi(x) \tag{4.3.2b}$$

$$\partial_y u(x, -h) = 0 \tag{4.3.2c}$$

$$u(x + \gamma, y) = u(x, y) \qquad \forall\, \gamma \in \Gamma, \tag{4.3.2d}$$

where

$$F = \mathrm{div}_x\,[F_x] + \partial_y F_y + F_h, \tag{4.3.2e}$$

and the x–derivative, y–derivative, and homogeneous parts of F are given by:

$$F_x = -\frac{2}{h} g \nabla_x u - \frac{1}{h^2} g^2 \nabla_x u + \frac{h+y}{h} \nabla_x g \partial_y u + \frac{h+y}{h^2} g \nabla_x g \partial_y u, \tag{4.3.2f}$$

$$F_y = \frac{h+y}{h} \nabla_x g \cdot \nabla_x u + \frac{h+y}{h^2} g \nabla_x g \cdot \nabla_x u - \frac{(h+y)^2}{h^2} \left|\nabla_x g\right|^2 \partial_y u, \tag{4.3.2g}$$

and

$$F_h = \frac{1}{h} \nabla_x g \cdot \nabla_x u + \frac{1}{h^2} g \nabla_x g \cdot \nabla_x u - \frac{h+y}{h^2} \left|\nabla_x g\right|^2 \partial_y u. \tag{4.3.2h}$$

Additionally, the DNO transforms to:

$$G(g)[\xi] = \partial_y u(x,0) + H(x; g, u), \tag{4.3.3a}$$

where

$$H = -\frac{1}{h} g G(g)[\xi] - \nabla_x g \cdot \nabla_x u(x,0) - \frac{1}{h} g \nabla_x g \cdot \nabla_x u(x,0) + \left|\nabla_x g\right|^2 \partial_y u(x,0). \tag{4.3.3b}$$

The reason for the particular gathering of terms in these equations is that both F and H are $\mathcal{O}(g)$.

Now that we have implemented the "transformation" in the TFE method, all that remains is to expand the field, u, and the DNO, G, in a power series in a parameter which measures the boundary deformation, e.g. ϵ in the relationship $g(x) = \epsilon f(x)$. Using this approach, several authors (see, e.g., (NR01; HN05)) have shown that if ϵ is small and f is smooth, then the expansion:

$$u(x, y, \epsilon) = \sum_{n=0}^{\infty} u_n(x, y)\epsilon^n \tag{4.3.4}$$

converges strongly in an appropriate function space, and each u_n satisfies:

$$\Delta u_n = F_n(x, y) \qquad \text{in } S_{h,0} \tag{4.3.5a}$$

$$u_n(x, 0) = \delta_{n,0}\ \xi(x) \tag{4.3.5b}$$

$$\partial_y u_n(x, -h) = 0 \tag{4.3.5c}$$

$$u_n(x + \gamma, y) = u_n(x, y) \qquad \forall\, \gamma \in \Gamma, \tag{4.3.5d}$$

where $\delta_{n,m}$ is the Kronecker delta,

$$F_n = \operatorname{div}_x [F_{x,n}] + \partial_y F_{y,n} + F_{h,n}, \tag{4.3.5e}$$

$$F_{x,n} = -\frac{2}{h} f \nabla_x u_{n-1} - \frac{1}{h^2} f^2 \nabla_x u_{n-2} + \frac{h+y}{h} \nabla_x f \partial_y u_{n-1} + \frac{h+y}{h^2} f \nabla_x f \partial_y u_{n-2}, \tag{4.3.5f}$$

$$F_{y,n} = \frac{h+y}{h} \nabla_x f \cdot \nabla_x u_{n-1} + \frac{h+y}{h^2} f \nabla_x f \cdot \nabla_x u_{n-2} - \frac{(h+y)^2}{h^2} \left|\nabla_x f\right|^2 \partial_y u_{n-2}, \tag{4.3.5g}$$

and

$$F_{h,n} = \frac{1}{h} \nabla_x f \cdot \nabla_x u_{n-1} + \frac{1}{h^2} f \nabla_x f \cdot \nabla_x u_{n-2} - \frac{h+y}{h^2} \left|\nabla_x f\right|^2 \partial_y u_{n-2}. \tag{4.3.5h}$$

In these formulas any function with a negative index is defined as zero. Under the same hypotheses (NR01; HN05) the expansion (2.3.2) can be shown to converge strongly, and the G_n can be computed via:

$$G_n(f)[\xi] = \partial_y u_n(x,0) + H_n(x), \tag{4.3.6a}$$

where

$$H_n = -\frac{1}{h} f G_{n-1}(f)[\xi] - \nabla_x f \cdot \nabla_x u_{n-1}(x,0) - \frac{1}{h} f \nabla_x f \cdot \nabla_x u_{n-2}(x,0) + |\nabla_x f|^2 \partial_y u_{n-2}(x,0). \tag{4.3.6b}$$

4.4 Analyticity of DNO

The recursions above can be used *directly* to establish the strong convergence of (4.3.4) and (2.3.2). The details are given in (NR01; NR03; HN05) but the results are summarized here for use in future sections.

Theorem 4.4.1. *Given an integer $s \geq 0$, if $f \in C^{s+2}$ and $\xi \in H^{s+3/2}$ then the series (4.3.4) converges strongly. In other words there exist constants $\tilde{C}_0$ and $\tilde{K}_0$ such that*

$$\|u_n\|_{H^{s+2}} \leq \tilde{K}_0 B_0^n$$

for any $B_0 > \tilde{C}_0 |f|_{C^{s+2}}$.

Theorem 4.4.2. *Given an integer $s \geq 0$, if $f \in C^{s+2}$ and $\xi \in H^{s+3/2}$ then the series* (2.3.2) *converges strongly as an operator from $H^{s+3/2}$ to $H^{s+1/2}$. In other words there exist constants C_0 and K_0 such that*

$$\|G_n(f)[\xi]\|_{H^{s+1/2}} \leq K_0 B_0^n$$

for any $B_0 > C_0 \, |f|_{C^{s+2}}$.

The function spaces in the theorems above are Sobolev spaces over our domain $S_{h,0}$:

$$H^s(S_{h,0}) = H^s\big([-h, 0] \times P(\Gamma)\big),$$

where $P(\Gamma)$ is the fundamental period cell of the lattice Γ.

CHAPTER 5

VARIATION OF THE DNO

From (4.3.2) and the definition (3.1.3), the first variation of the field, $u^{(1)}$, satisfies the following elliptic problem:

$$\Delta u^{(1)} = F^{(1)}(x,y) \qquad \text{in } S_{h,0} \tag{5.0.1a}$$

$$u^{(1)}(x,0) = 0 \tag{5.0.1b}$$

$$\partial_y u^{(1)}(x,-h) = 0 \tag{5.0.1c}$$

$$u^{(1)}(x+\gamma,y) = u^{(1)}(x,y) \qquad \forall\, \gamma \in \Gamma, \tag{5.0.1d}$$

where

$$F^{(1)} = \operatorname{div}_x \left[F_x^{(1)}\right] + \partial_y F_y^{(1)} + F_h^{(1)}, \tag{5.0.1e}$$

$$\begin{aligned} F_x^{(1)} = -\frac{2}{h} w \nabla_x u - \frac{2}{h} g \nabla_x u^{(1)} - \frac{2}{h^2} g w \nabla_x u - \frac{1}{h^2} g^2 \nabla_x u^{(1)} \\ + \frac{h+y}{h} \nabla_x w \partial_y u + \frac{h+y}{h} \nabla_x g \partial_y u^{(1)} \\ + \frac{h+y}{h^2} w \nabla_x g \partial_y u + \frac{h+y}{h^2} g \nabla_x w \partial_y u + \frac{h+y}{h^2} g \nabla_x g \partial_y u^{(1)}, \end{aligned} \tag{5.0.1f}$$

$$F_y^{(1)} = \frac{h+y}{h}\nabla_x w \cdot \nabla_x u + \frac{h+y}{h}\nabla_x g \cdot \nabla_x u^{(1)}$$
$$+ \frac{h+y}{h^2} w\nabla_x g \cdot \nabla_x u + \frac{h+y}{h^2} g\nabla_x w \cdot \nabla_x u + \frac{h+y}{h^2} g\nabla_x g \cdot \nabla_x u^{(1)}$$
$$- \frac{2(h+y)^2}{h^2}\nabla_x w \cdot \nabla_x g \partial_y u - \frac{(h+y)^2}{h^2} \left|\nabla_x g\right|^2 \partial_y u^{(1)}, \quad (5.0.1g)$$

and

$$F_h^{(1)} = \frac{1}{h}\nabla_x w \cdot \nabla_x u + \frac{1}{h}\nabla_x g \cdot \nabla_x u^{(1)}$$
$$+ \frac{1}{h^2} w\nabla_x g \cdot \nabla_x u + \frac{1}{h^2} g\nabla_x w \cdot \nabla_x u + \frac{1}{h^2} g\nabla_x g \cdot \nabla_x u^{(1)}$$
$$- \frac{2(h+y)}{h^2}\nabla_x w \cdot \nabla_x g \partial_y u - \frac{h+y}{h^2} \left|\nabla_x g\right|^2 \partial_y u^{(1)}. \quad (5.0.1h)$$

Next, the variation of the DNO satisfies the formula:

$$G^{(1)}(g)[\xi]\{w\} = \partial_y u^{(1)}(x,0) + H^{(1)}(x), \quad (5.0.2a)$$

where

$$H^{(1)} = -\frac{1}{h} wG(g)[\xi] - \frac{1}{h} gG^{(1)}(g)[\xi]\{w\} - \nabla_x w \cdot \nabla_x u(x,0) - \nabla_x g \cdot \nabla_x u^{(1)}(x,0)$$
$$- \frac{1}{h} w\nabla_x g \cdot \nabla_x u(x,0) - \frac{1}{h} g\nabla_x w \cdot \nabla_x u(x,0) - \frac{1}{h} g\nabla_x g \cdot \nabla_x u^{(1)}(x,0)$$
$$+ 2\nabla_x w \cdot \nabla_x g \partial_y u(x,0) + \left|\nabla_x g\right|^2 \partial_y u^{(1)}(x,0). \quad (5.0.2b)$$

5.1 Analyticity of the First variation of the DNO

As with the case of the field and DNO, we set $g(x) = \epsilon f(x)$ and the TFE methodology can be utilized to show that the expansions:

$$u^{(1)}(x,y,\epsilon)\{w\} = \sum_{n=0}^{\infty} u_n^{(1)}(x,y)\{w\}\ \epsilon^n, \quad G^{(1)}(g)[\xi]\{w\} = \sum_{n=0}^{\infty} G_n^{(1)}(f)[\xi]\{w\}\ \epsilon^n, \tag{5.1.1}$$

converge strongly; see Theorems 5.1.2 and 5.1.3. Given these expansions it is not difficult to see that the $u_n^{(1)}$ must satisfy:

$$\Delta u_n^{(1)} = F_n^{(1)}(x,y) \qquad \text{in } S_{h,0} \tag{5.1.2a}$$

$$u_n^{(1)}(x,0) = 0 \tag{5.1.2b}$$

$$\partial_y u_n^{(1)}(x,-h) = 0 \tag{5.1.2c}$$

$$u_n^{(1)}(x+\gamma,y) = u_n^{(1)}(x,y) \qquad \forall\, \gamma \in \Gamma, \tag{5.1.2d}$$

where

$$F_n^{(1)} = \operatorname{div}_x \left[F_{x,n}^{(1)} \right] + \partial_y F_{y,n}^{(1)} + F_{h,n}^{(1)}, \tag{5.1.2e}$$

and

$$F_{x,n}^{(1)} = -\frac{2}{h} w \nabla_x u_n - \frac{2}{h} f \nabla_x u_{n-1}^{(1)} - \frac{2}{h^2} w f \nabla_x u_{n-1} - \frac{1}{h^2} f^2 \nabla_x u_{n-2}^{(1)}$$
$$+ \frac{h+y}{h} \nabla_x w \partial_y u_n + \frac{h+y}{h} \nabla_x f \partial_y u_{n-1}^{(1)}$$
$$+ \frac{h+y}{h^2} w \nabla_x f \partial_y u_{n-1} + \frac{h+y}{h^2} f \nabla_x w \partial_y u_{n-1} + \frac{h+y}{h^2} f \nabla_x f \partial_y u_{n-2}^{(1)}, \quad (5.1.2f)$$

$$F_{y,n}^{(1)} = \frac{h+y}{h} \nabla_x w \cdot \nabla_x u_n + \frac{h+y}{h} \nabla_x f \cdot \nabla_x u_{n-1}^{(1)}$$
$$+ \frac{h+y}{h^2} w \nabla_x f \cdot \nabla_x u_{n-1} + \frac{h+y}{h^2} f \nabla_x w \cdot \nabla_x u_{n-1} + \frac{h+y}{h^2} f \nabla_x f \cdot \nabla_x u_{n-2}^{(1)}$$
$$- \frac{2(h+y)^2}{h^2} \nabla_x w \cdot \nabla_x f \partial_y u_{n-1} - \frac{(h+y)^2}{h^2} \left|\nabla_x f\right|^2 \partial_y u_{n-2}^{(1)}, \quad (5.1.2g)$$

and

$$F_{h,n}^{(1)} = \frac{1}{h} \nabla_x w \cdot \nabla_x u_n + \frac{1}{h} \nabla_x f \cdot \nabla_x u_{n-1}^{(1)}$$
$$+ \frac{1}{h^2} w \nabla_x f \cdot \nabla_x u_{n-1} + \frac{1}{h^2} f \nabla_x w \cdot \nabla_x u_{n-1} + \frac{1}{h^2} f \nabla_x f \cdot \nabla_x u_{n-2}^{(1)}$$
$$- \frac{2(h+y)}{h^2} \nabla_x w \cdot \nabla_x f \partial_y u_{n-1} - \frac{h+y}{h^2} \left|\nabla_x f\right|^2 \partial_y u_{n-2}^{(1)}. \quad (5.1.2h)$$

The $G_n^{(1)}$ can be computed via:

$$G_n^{(1)}(f)[\xi]\{w\} = \partial_y u_n^{(1)}(x,0) + H_n^{(1)}(x), \quad (5.1.3a)$$

where

$$
\begin{aligned}
H_n^{(1)} = -\frac{1}{h} w G_n(f)[\xi] - \frac{1}{h} f G_{n-1}^{(1)}(f)[\xi]\{w\} - \nabla_x w \cdot \nabla_x u_n(x,0) \\
- \nabla_x f \cdot \nabla_x u_{n-1}^{(1)}(x,0) - \frac{1}{h} w \nabla_x f \cdot \nabla_x u_{n-1}(x,0) \\
- \frac{1}{h} f \nabla_x w \cdot \nabla_x u_{n-1}(x,0) - \frac{1}{h} f \nabla_x f \cdot \nabla_x u_{n-2}^{(1)}(x,0) \\
+ 2 \nabla_x w \cdot \nabla_x f \partial_y u_{n-1}(x,0) + |\nabla_x f|^2 \partial_y u_{n-2}^{(1)}(x,0). \quad (5.1.3b)
\end{aligned}
$$

The primary result of this section is the parametric analyticity of the first variation of the DNO, $G^{(1)}$, with respect to the boundary variation $g = \epsilon f$. This can be shown directly from the next result on parametric analyticity of the first variation of the field, $u^{(1)}$. To make this precise we define the quantities D_1 and $\tilde{D}_1$ which help characterize the disk of convergence of the Taylor series of $G^{(1)}$ and $u^{(1)}$.

Definition 5.1.1. For any positive real number B_0 (see Theorems 4.4.1 & 4.4.2), and functions $f, w \in C^{s+2}$, let

$$
\begin{aligned}
D_1 &:= |f|_{C^{s+2}} + B_0 |w|_{C^{s+2}} \\
\tilde{D}_1 &:= |f|_{C^{s+2}}^2 + B_0 |f|_{C^{s+2}} |w|_{C^{s+2}}.
\end{aligned}
$$

Theorem 5.1.2. *Given an integer* $s \geq 0$, *if* $f \in C^{s+2}$, $\xi \in H^{s+3/2}$, *and* $w \in C^{s+2}$ *then the series for* $u^{(1)}$ *in* (5.1.1) *converges strongly. In other words there exist constants* $\tilde{C}_1$ *and* $\tilde{K}_1$ *such that*

$$\left\| u_n^{(1)} \right\|_{H^{s+2}} \leq \tilde{K}_1 B_1^n \tag{5.1.4}$$

for any $B_1 > \max\left\{B_0, 2C_e\tilde{C}_1 D_1, \sqrt{2C_e\tilde{C}_1\tilde{D}_1}\right\}$. C_e *is given in Lemma 5.1.5 and* B_0 *is given by Theorem 4.4.1 which holds with the hypotheses given above.*

The parametric analyticity of $G^{(1)}$ now follows.

Theorem 5.1.3. *Given an integer* $s \geq 0$, *if* $f \in C^{s+2}$, $\xi \in H^{s+3/2}$, *and* $w \in C^{s+2}$, *then the series for* $G^{(1)}$ *in* (5.1.1) *converges strongly as an operator from* $H^{s+3/2}$ *to* $H^{s+1/2}$. *In other words there exist constants* C_1 *and* K_1 *such that*

$$\left\| G_n^{(1)}(f)[\xi]\{w\} \right\|_{H^{s+1/2}} \leq K_1 B_1^n \tag{5.1.5}$$

for any $B_1 > \max\{B_0, C_1 D_1, C_1\sqrt{\tilde{D}_1}\}$.

A key element in the proof of these results is an "Algebra Property" of the function spaces H^s and C^s (Ada75; NR01).

Lemma 5.1.4. *For any integer $s \geq 0$ and any $\sigma > 0$, if $f \in C^s(P(\Gamma))$, $u \in H^s(P(\Gamma) \times [-h, 0])$, $g \in C^{s+1/2+\sigma}(P(\Gamma))$, and $\mu \in H^{s+1/2}(P(\Gamma))$, then*

$$\|fu\|_{H^s(P(\Gamma)\times[-h,0])} \leq M \, |f|_{C^s(P(\Gamma))} \, \|u\|_{H^s(P(\Gamma)\times[-h,0])}$$

$$\|g\mu\|_{H^{s+1/2}(P(\Gamma))} \leq M \, |g|_{C^{s+1/2+\sigma}(P(\Gamma))} \, \|\mu\|_{H^{s+1/2}(P(\Gamma))}$$

where M is a constant depending only on s and the dimension d.

Another invaluable tool in our analysis is the following well–known "Elliptic Estimate" (LU68; Eva98).

Lemma 5.1.5. *For any integer $s \geq 0$ there exists a constant C_e such that for any $F \in H^s$, $\xi \in H^{s+3/2}$, the solution $W \in H^{s+2}$ of*

$$\begin{aligned}
&\Delta W(x,y) = F(x,y) && \text{in } S_{h,0} \\
&W(x,0) = \xi(x) && \\
&\partial_y W(x,-h) = 0 && \\
&W(x+\gamma,y) = W(x,y) && \forall\, \gamma \in \Gamma
\end{aligned}$$

satisfies

$$\|W\|_{H^{s+2}} \leq C_e \left\{ \|F\|_{H^s} + \|\xi\|_{H^{s+3/2}} \right\}.$$

Our proof of Theorem 5.1.2 is inductive in nature relying upon the relation (5.1.2) for $u_n^{(1)}$; therefore a recursive estimate on the right–hand side $F_n^{(1)}$ is essential.

Lemma 5.1.6. *Let $s \geq 0$ be an integer and let $f, w \in C^{s+2}$. Assume*

$$\|u_n\|_{H^{s+2}} \leq \tilde{K}_0 B_0^n \qquad \forall n \tag{5.1.6a}$$

$$\left\|u_n^{(1)}\right\|_{H^{s+2}} \leq \tilde{K}_1 B_1^n \qquad n < N, \tag{5.1.6b}$$

and that the constants $\tilde{K}_0, \tilde{K}_1, B_0, B_1 > 0$. Then if $B_1 > B_0$, $\tilde{K}_1 > \tilde{K}_0$, there exists a constant $\tilde{C}_1$ such that

$$\left\|F_N^{(1)}\right\|_{H^s} \leq \tilde{C}_1 \tilde{K}_1 \left\{ D_1 B_1^{N-1} + \tilde{D}_1 B_1^{N-2} \right\}.$$

Proof. We recall that $F_N^{(1)} = \mathrm{div}_x \left[F_{x,N}^{(1)}\right] + \partial_y F_{y,N}^{(1)} + F_{h,N}^{(1)}$ and focus our attention upon $F_{x,N}^{(1)}$ as the other terms can be handled in a similar fashion. Using Lemma 5.1.4,

$$\begin{aligned}
\left\|\mathrm{div}_x \left[F_{x,N}^{(1)}\right]\right\|_{H^s} &\leq \left\|F_{x,N}^{(1)}\right\|_{H^{s+1}} \\
&\leq \frac{2M}{h} |w|_{C^{s+1}} \|u_N\|_{H^{s+2}} + \frac{2M}{h} |f|_{C^{s+1}} \left\|u_{N-1}^{(1)}\right\|_{H^{s+2}} \\
&+ \frac{2M^2}{h^2} |w|_{C^{s+1}} |f|_{C^{s+1}} \|u_{N-1}\|_{H^{s+2}} \\
&+ \frac{M^2}{h^2} |f|_{C^{s+1}}^2 \left\|u_{N-2}^{(1)}\right\|_{H^{s+2}} \\
&+ \frac{YM}{h} |w|_{C^{s+2}} \|u_N\|_{H^{s+2}} + \frac{YM}{h} |f|_{C^{s+2}} \left\|u_{N-1}^{(1)}\right\|_{H^{s+2}} \\
&+ \frac{YM^2}{h^2} |w|_{C^{s+1}} |f|_{C^{s+2}} \|u_{N-1}\|_{H^{s+2}} \\
&+ \frac{YM^2}{h^2} |f|_{C^{s+1}} |w|_{C^{s+2}} \|u_{N-1}\|_{H^{s+2}} \\
&+ \frac{YM^2}{h^2} |f|_{C^{s+1}} |f|_{C^{s+2}} \left\|u_{N-2}^{(1)}\right\|_{H^{s+2}},
\end{aligned}$$

where we have used

$$\|(h+y)v\|_{H^s} \leq Y\,\|v\|_{H^s}\,,$$

for some constant $Y = Y(s,d)$. By using $|f|_{C^{s+1}} \leq |f|_{C^{s+2}}$, $|w|_{C^{s+1}} \leq |w|_{C^{s+2}}$, and the inductive hypotheses (5.1.6), it is easy to show that

$$\begin{aligned}
\left\|\operatorname{div}_x\left[F^{(1)}_{x,N}\right]\right\|_{H^s} &\leq \frac{(2+Y)M}{h}\,|w|_{C^{s+2}}\,\tilde{K}_0 B_0^N \\
&+ \frac{2M^2(1+Y)}{h^2}\,|w|_{C^{s+2}}\,|f|_{C^{s+2}}\,\tilde{K}_0 B_0^{N-1} \\
&+ \frac{(2+Y)M}{h}\,|f|_{C^{s+2}}\,\tilde{K}_1 B_1^{N-1} \\
&+ \frac{M^2(1+Y)}{h^2}\,|f|^2_{C^{s+2}}\,\tilde{K}_1 B_1^{N-2} \\
&\leq \tilde{K}_1\left(\frac{(2+Y)M}{h}\,|w|_{C^{s+2}}\,B_0 B_1^{N-1}\right. \\
&+ \frac{2M^2(1+Y)}{h^2}\,|w|_{C^{s+2}}\,|f|_{C^{s+2}}\,B_0 B_1^{N-2} \\
&\left.+\frac{(2+Y)M}{h}\,|f|_{C^{s+2}}\,B_1^{N-1} + \frac{M^2(1+Y)}{h^2}\,|f|^2_{C^{s+2}}\,B_1^{N-2}\right) \\
&\leq \tilde{C}_1\tilde{K}_1\Big((|f|_{C^{s+2}} + |w|_{C^{s+2}}\,B_0)B_1^{N-1} \\
&+(|f|^2_{C^{s+2}} + |f|_{C^{s+2}}\,|w|_{C^{s+2}}\,B_0)B_1^{N-2}\Big)\,,
\end{aligned}$$

provided that $B_0 < B_1$, $\tilde{K}_0 < \tilde{K}_1$, and $\tilde{C}_1$ is chosen appropriately; the proof is now complete. $\square$

We are now in a position to prove the parametric analyticity of the first variation of the field, $u^{(1)}$.

Proof. (Theorem 5.1.2) We utilize an inductive method, therefore at order $n = 0$ we recall that we must solve (5.1.2) with

$$\begin{aligned} F_{x,0}^{(1)} &= -\frac{2}{h} w \nabla_x u_0 + \frac{h+y}{h} \nabla_x w \partial_y u_0, \\ F_{y,0}^{(1)} &= \frac{h+y}{h} \nabla_x w \cdot \nabla_x u_0, \\ F_{h,0}^{(1)} &= \frac{1}{h} \nabla_x w \cdot \nabla_x u_0. \end{aligned}$$

Using Lemma 5.1.5 we find that

$$\begin{aligned} \left\| u_0^{(1)} \right\|_{H^{s+2}} &\leq C_e \left\{ \frac{2M}{h} |w|_{C^{s+1}} \|u_0\|_{H^{s+2}} + \frac{YM}{h} |w|_{C^{s+2}} \|u_0\|_{H^{s+2}} \right. \\ &\quad \left. + \frac{YM}{h} |w|_{C^{s+2}} \|u_0\|_{H^{s+2}} + \frac{M}{h} |w|_{C^{s+1}} \|u_0\|_{H^{s+1}} \right\} \\ &\leq \frac{C_e M}{h} (3 + 2Y) |w|_{C^{s+2}} \tilde{K}_0 B_0. \end{aligned}$$

We set

$$\tilde{K}_1 = \max \left\{ \tilde{K}_0, \frac{C_e M}{h} (3 + 2Y) |w|_{C^{s+2}} \tilde{K}_0 B_0 \right\},$$

and the case $n = 0$ is established. We now assume (5.1.4) for all $n < N$ and use (5.1.2) and Lemma 5.1.5 to realize that:

$$\left\| u_N^{(1)} \right\|_{H^{s+2}} \leq C_e \left\| F_N^{(1)} \right\|_{H^s}.$$

Since the u_n satisfy the estimate of Theorem 4.4.1, we can use Lemma 5.1.6 to imply that

$$\left\| u_N^{(1)} \right\|_{H^{s+2}} \leq C_e \tilde{C}_1 \tilde{K}_1 \left\{ D_1 B_1^{N-1} + \tilde{D}_1 B_1^{N-2} \right\} \leq \tilde{K}_1 B_1^N,$$

if we choose

$$B_1 > \max \left\{ 2C_e \tilde{C}_1 D_1, \sqrt{2C_e \tilde{C}_1 \tilde{D}_1} \right\}.$$

$\square$

Finally, we show the parametric analyticity of $G^{(1)}$.

Proof. (Theorem 5.1.3) Again we work by induction and begin with $G_0^{(1)}$. An important realization to make is that our hypotheses guarantee that Theorem 4.4.2 holds together with its estimates on G_n. From (5.1.3a), we see at order zero that

$$G_0^{(1)}[\xi]\{w\} = \partial_y u_0^{(1)}(x,0) - \frac{1}{h} w G_0[\xi] - \nabla_x w \cdot \nabla_x u_0(x,0).$$

We now estimate

$$\begin{aligned} \left\| G_0^{(1)}[\xi]\{w\} \right\|_{H^{s+1/2}} &\leq \left\| u_0^{(1)}(x,0) \right\|_{H^{s+3/2}} + \frac{M}{h} \left| w \right|_{C^{s+1/2+\sigma}} \left\| G_0[\xi] \right\|_{H^{s+1/2}} \\ &\quad + M \left| w \right|_{C^{s+3/2+\sigma}} \left\| u_0(x,0) \right\|_{H^{s+3/2}} \\ &\leq \tilde{K}_1 + \frac{M}{h} \left| w \right|_{C^{s+2}} K_0 + M \left| w \right|_{C^{s+2}} \tilde{K}_0. \end{aligned}$$

If we set

$$K_1 = \tilde{K}_1 + \frac{M}{h} |w|_{C^{s+2}} K_0 + M |w|_{C^{s+2}} \tilde{K}_0,$$

then the case $n = 0$ is resolved. We now suppose that (5.1.5) holds for $n < N$ and examine $G_N^{(1)}$:

$$\begin{aligned}
\left\| G_N^{(1)}(f)[\xi]\{w\} \right\|_{H^{s+1/2}} \leq & \left\| u_N^{(1)}(x,0) \right\|_{H^{s+3/2}} \\
& + \frac{M}{h} |w|_{C^{s+1/2+\sigma}} \left\| G_N(f)[\xi] \right\|_{H^{s+1/2}} \\
& + \frac{M}{h} |f|_{C^{s+1/2+\sigma}} \left\| G_{N-1}^{(1)}(f)[\xi]\{w\} \right\|_{H^{s+1/2}} \\
& + M |w|_{C^{s+3/2+\sigma}} \left\| u_N(x,0) \right\|_{H^{s+3/2}} \\
& + M |f|_{C^{s+3/2+\sigma}} \left\| u_{N-1}^{(1)}(x,0) \right\|_{H^{s+3/2}} \\
& + \frac{M^2}{h} |w|_{C^{s+1/2+\sigma}} |f|_{C^{s+3/2+\sigma}} \left\| u_{N-1}(x,0) \right\|_{H^{s+3/2}} \\
& + \frac{M^2}{h} |f|_{C^{s+1/2+\sigma}} |w|_{C^{s+3/2+\sigma}} \left\| u_{N-1}(x,0) \right\|_{H^{s+3/2}} \\
& + \frac{M^2}{h} |f|_{C^{s+1/2+\sigma}} |f|_{C^{s+3/2+\sigma}} \left\| u_{N-2}^{(1)}(x,0) \right\|_{H^{s+3/2}} \\
& + 2M^2 |w|_{C^{s+3/2+\sigma}} |f|_{C^{s+3/2+\sigma}} \left\| u_{N-1}(x,0) \right\|_{H^{s+3/2}} \\
& + M^2 |f|^2_{C^{s+3/2+\sigma}} \left\| u_{N-2}^{(1)}(x,0) \right\|_{H^{s+3/2}} .
\end{aligned}$$

Using the fact that $B_1 > B_0$,

$$
\begin{aligned}
\left\| G_N^{(1)} \right\|_{H^{s+1/2}} \leq & \tilde{K}_1 B_1^N + M \left(\frac{K_1}{h} + \tilde{K}_1 \right) |f|_{C^{s+2}} B_1^{N-1} \\
& + M^2 \tilde{K}_1 \left(\frac{1}{h^2} + 1 \right) |f|_{C^{s+2}}^2 B_1^{N-2} \\
& + M \left(\frac{K_0}{h} + \tilde{K}_0 \right) |w|_{C^{s+2}} B_0^N \\
& + 2M^2 \tilde{K}_0 \left(\frac{1}{h} + 1 \right) |f|_{C^{s+2}} |w|_{C^{s+2}} B_0^{N-1} \\
\leq & \tilde{K}_1 B_1^N + M \left[\left(\frac{K_1}{h} + \tilde{K}_1 \right) |f|_{C^{s+2}} \right. \\
& \left. + \left(\frac{K_0}{h} + \tilde{K}_0 \right) \left(B_0 |w|_{C^{s+2}} \right) \right] B_1^{N-1} \\
& + M^2 \left[\tilde{K}_1 \left(\frac{1}{h^2} + 1 \right) |f|_{C^{s+2}}^2 \right. \\
& \left. + 2\tilde{K}_0 \left(\frac{1}{h} + 1 \right) |f|_{C^{s+2}} \left(B_0 |w|_{C^{s+2}} \right) \right] B_1^{N-2}.
\end{aligned}
$$

By the bound $2C_e\tilde{C}_1 D_1 < B_1$ we are done provided that K_1 is chosen sufficiently large. □

5.2 Higher Variations

Though the impact of higher variations of the DNO on a spectral stability analysis is not immediately apparent, we establish in this section parametric analyticity results for these higher derivatives. However, we do restrict ourselves to the case of periodic perturbations as products of these functions appear in the relevant formulas, but the space of Bloch periodic functions is *not* closed under multiplication. At this point the key role that the transformation (4.3.1) plays is particularly evident as the proof of the relevant analyticity theorem is no more difficult than that of the first variation case.

To begin, we record a helpful Proposition regarding variations of products which is easily established using induction.

Proposition 5.2.1. *Suppose that A and B are linear operators and U is a nonlinear function of g, then if*

$$R(g) = A[g]U(g), \quad S(g) = A[g]B[g]U(g),$$

and $U^{(k)}$, $R^{(k)}$, and $S^{(k)}$ denote the k–th variations of U, R, and S, respectively, then

$$R^{(m)}\{w\} = A[g]U^{(m)}\{w\} + \sum_{j=1}^{m} A[w_j]U^{(m-1)}\{\tilde{w}_j\} \tag{5.2.1a}$$

$$\begin{aligned} S^{(m)}[w] = {} & A[g]B[g]U^{(m)}\{w\} + A[g]\sum_{j=1}^{m} B[w_j]U^{(m-1)}\{\tilde{w}_j\} \\ & + B[g]\sum_{j=1}^{m} A[w_j]U^{(m-1)}\{\tilde{w}_j\} \\ & + \sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} A[w_j]B[w_k]U^{(m-2)}\{\tilde{w}_{j,k}\}, \end{aligned} \tag{5.2.1b}$$

where

$$w = (w_1, \ldots, w_m)$$

$$\tilde{w}_j = (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_m)$$

$$\tilde{w}_{j,k} = (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_{k-1}, w_{k+1}, \ldots, w_m).$$

Gateaux's definition (LL01) of the m–th variation of a functional F with respect to a function φ at φ_0 in the direction $\psi = (\psi_1, \ldots, \psi_m)$ is

$$\delta_\varphi^m F(\varphi_0)\{\psi\} := \lim_{\tau_m \to 0} \frac{1}{\tau_m} \left[\delta_\varphi^{m-1} F(\varphi_0 + \tau_m \psi_m)\{\psi_1, \ldots \psi_{m-1}\} \right.$$
$$\left. - \delta_\varphi^{m-1} F(\varphi_0)\{\psi_1, \ldots, \psi_{m-1}\} \right].$$

As the DNO and its underlying elliptic BVP (in transformed coordinates) are given in (4.3.3) and (4.3.2), it is easy to derive equations for their m–th variations,

$$u^{(m)}(x, y; g)\{w\} := \delta_g^m u(x, y; g)\{w\}, \qquad G^{(m)}(g)[\xi]\{w\} := \delta_g^m G(g)[\xi]\{w\}.$$

First, for the m–th variation of the field, $u^{(m)}$ satisfies the following elliptic problem:

$$\Delta u^{(m)} = F^{(m)}(x, y) \qquad \text{in } S_{h,0} \tag{5.2.2a}$$

$$u^{(m)}(x, 0) = 0 \tag{5.2.2b}$$

$$\partial_y u^{(m)}(x, -h) = 0 \tag{5.2.2c}$$

$$u^{(m)}(x + \gamma, y) = u^{(m)}(x, y) \qquad \forall\, \gamma \in \Gamma, \tag{5.2.2d}$$

where

$$F^{(m)} = \operatorname{div}_x \left[F_x^{(m)} \right] + \partial_y F_y^{(m)} + F_h^{(m)}. \tag{5.2.2e}$$

To derive the forms of $F_x^{(m)}$, $F_y^{(m)}$, and $F_h^{(m)}$ we use Proposition 5.2.1 repeatedly. For instance, the first term in the expression for F_x is

$$R(u) = -\frac{2}{h} g \nabla_x u = -\frac{2}{h} A[g] U(g),$$

where $A = I$ is the identity and $U(g) = \nabla_x u(g)$. By Proposition 5.2.1,

$$\begin{aligned} R^{(m)}[w] &= -\frac{2}{h}\left(A[g]U^{(m)}[w] + \sum_{j=1}^{m} A[w_j]U^{(m-1)}[\tilde{w}_j] \right) \\ &= -\frac{2}{h}\left(g\nabla_x u^{(m)}\{w\} + \sum_{j=1}^{m} w_j \nabla_x u^{(m-1)}\{\tilde{w}_j\} \right). \end{aligned}$$

Proceeding in this way we can derive the following expressions:

$$\begin{aligned} F_x^{(m)} = -\frac{2}{h} & \left(g\nabla_x u^{(m)}\{w\} + \sum_{j=1}^{m} w_j \nabla_x u^{(m-1)}\{\tilde{w}_j\} \right) \\ & - \frac{1}{h^2}\left(g^2 \nabla_x u^{(m)}\{w\} + 2g \sum_{j=1}^{m} w_j \nabla_x u^{(m-1)}\{\tilde{w}_j\} \right. \\ & \left. + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j w_k \nabla_x u^{(m-2)}\{\tilde{w}_{j,k}\} \right) \\ & + \frac{h+y}{h}\left(\nabla_x g \partial_y u^{(m)}\{w\} + \sum_{j=1}^{m} \nabla_x w_j \partial_y u^{(m-1)}\{\tilde{w}_j\} \right) \\ & + \frac{h+y}{h^2}\left(g\nabla_x g \partial_y u^{(m)}\{w\} + g\sum_{j=1}^{m} \nabla_x w_j \partial_y u^{(m-1)}\{\tilde{w}_j\} \right. \\ & \left. + \nabla_x g \sum_{j=1}^{m} w_j \partial_y u^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \partial_y u^{(m-2)}\{\tilde{w}_{j,k}\} \right), \end{aligned} \tag{5.2.2f}$$

$$F_y^{(m)} = \frac{h+y}{h}\left(\nabla_x g \cdot \nabla_x u^{(m)}\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}\{\tilde{w}_j\}\right)$$
$$+ \frac{h+y}{h^2}\left(g\nabla_x g \cdot \nabla_x u^{(m)}\{w\} + g\sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}\{\tilde{w}_j\}\right.$$
$$\left. +\nabla_x g \cdot \sum_{j=1}^{m} w_j \nabla_x u^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u^{(m-2)}\{\tilde{w}_{j,k}\}\right)$$
$$- \frac{(h+y)^2}{h^2}\left(|\nabla_x g|^2 \partial_y u^{(m)}\{w\} + 2\nabla_x g \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u^{(m-1)}\{\tilde{w}_j\}\right.$$
$$\left. +\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u^{(m-2)}\{\tilde{w}_{j,k}\}\right), \quad (5.2.2g)$$

and

$$F_h^{(m)} = \frac{1}{h}\left(\nabla_x g \cdot \nabla_x u^{(m)}\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}\{\tilde{w}_j\}\right)$$
$$+ \frac{1}{h^2}\left(g\nabla_x g \cdot \nabla_x u^{(m)}\{w\} + g\sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}\{\tilde{w}_j\}\right.$$
$$\left. +\nabla_x g \cdot \sum_{j=1}^{m} w_j \cdot \nabla_x u^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u^{(m-2)}\{\tilde{w}_{j,k}\}\right)$$
$$- \frac{h+y}{h^2}\left(|\nabla_x g|^2 \partial_y u^{(m)}\{w\} + 2\nabla_x g \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u^{(m-1)}\{\tilde{w}_j\}\right.$$
$$\left. +\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u^{(m-2)}\{\tilde{w}_{j,k}\}\right). \quad (5.2.2h)$$

Now, the variation of the DNO satisfies:

$$G^{(m)}(g)[\xi]\{w\} = \partial_y u^{(m)}(x,0)\{w\} + H^{(m)}(x)\{w\}, \quad (5.2.3a)$$

where

$$
\begin{aligned}
H^{(m)} = -\frac{1}{h}&\left(gG^{(m)}(g)[\xi]\{w\} + \sum_{j=1}^{m} w_j G^{(m-1)}(g)[\xi]\{\tilde{w}_j\}\right) \\
&- \left(\nabla_x g \cdot \nabla_x u^{(m)}(x,0)\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}(x,0)\{\tilde{w}_j\}\right) \\
&- \frac{1}{h}\left(g\nabla_x g \cdot \nabla_x u^{(m)}(x,0)\{w\} + g\sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u^{(m-1)}(x,0)\{\tilde{w}_j\}\right. \\
&\qquad + \nabla_x g \cdot \sum_{j=1}^{m} w_j \nabla_x u^{(m-1)}(x,0)\{\tilde{w}_j\} \\
&\qquad \left. + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u^{(m-2)}(x,0)\{\tilde{w}_{j,k}\}\right) \\
&+ \left(|\nabla_x g|^2 \partial_y u^{(m)}(x,0)\{w\} + 2\nabla_x g \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u^{(m-1)}(x,0)\{\tilde{w}_j\}\right. \\
&\qquad \left. + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u^{(m-2)}(x,0)\{\tilde{w}_{j,k}\}\right). \qquad (5.2.3b)
\end{aligned}
$$

5.3 Analyticity of Higher Variations

Following the development of § 5.1 we can now establish the analyticity of the m–th variations of the field and the DNO. Again, if $g = \epsilon f$ is sufficiently smooth then both

$$u^{(m)}(x,y,\epsilon)\{w\} = \sum_{n=0}^{\infty} u_n^{(m)}(x,y)\{w\}\ \epsilon^n, \qquad (5.3.1a)$$

$$G^{(m)}(\eta)[\xi]\{w\} = \sum_{n=0}^{\infty} G_n^{(m)}[\xi]\{w\}\ \epsilon^n, \qquad (5.3.1b)$$

will converge strongly; see Theorems 5.3.2 and 5.3.3. Given these expansions it is not difficult to see that the $u_n^{(m)}$ must satisfy

$$\Delta u_n^{(m)} = F_n^{(m)}(x,y) \qquad \text{in } S_{h,0} \tag{5.3.2a}$$

$$u_n^{(m)}(x,0) = 0 \tag{5.3.2b}$$

$$\partial_y u_n^{(m)}(x,-h) = 0 \tag{5.3.2c}$$

$$u_n^{(m)}(x+\gamma,y) = u_n^{(m)}(x,y) \qquad \forall\, \gamma \in \Gamma, \tag{5.3.2d}$$

where

$$F_n^{(m)} = \operatorname{div}_x\left[F_{x,n}^{(m)}\right] + \partial_y F_{y,n}^{(m)} + F_{h,n}^{(m)}, \tag{5.3.2e}$$

and

$$
\begin{aligned}
F_{x,n}^{(m)} = -\frac{2}{h} & \left(f \nabla_x u_{n-1}^{(m)}\{w\} + \sum_{j=1}^{m} w_j \nabla_x u_n^{(m-1)}\{\tilde{w}_j\} \right) \\
& - \frac{1}{h^2} \left(f^2 \nabla_x u_{n-2}^{(m)}\{w\} + 2f \sum_{j=1}^{m} w_j \nabla_x u_{n-1}^{(m-1)}\{\tilde{w}_j\} \right. \\
& \left. + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j w_k \nabla_x u_n^{(m-2)}\{\tilde{w}_{j,k}\} \right) \\
& + \frac{h+y}{h} \left(\nabla_x f \partial_y u_{n-1}^{(m)}\{w\} + \sum_{j=1}^{m} \nabla_x w_j \partial_y u_n^{(m-1)}\{\tilde{w}_j\} \right) \\
& + \frac{h+y}{h^2} \left(f \nabla_x f \partial_y u_{n-2}^{(m)}\{w\} + f \sum_{j=1}^{m} \nabla_x w_j \partial_y u_{n-1}^{(m-1)}\{\tilde{w}_j\} \right. \\
& \left. + \nabla_x f \sum_{j=1}^{m} w_j \partial_y u_{n-1}^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \partial_y u_n^{(m-2)}\{\tilde{w}_{j,k}\} \right), \qquad (5.3.2f)
\end{aligned}
$$

$$
\begin{aligned}
F_{y,n}^{(m)} = \frac{h+y}{h} & \left(\nabla_x f \cdot \nabla_x u_{n-1}^{(m)}\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_n^{(m-1)}\{\tilde{w}_j\} \right) \\
& + \frac{h+y}{h^2} \left(f \nabla_x f \cdot \nabla_x u_{n-2}^{(m)}\{w\} + f \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_{n-1}^{(m-1)}\{\tilde{w}_j\} \right. \\
& \left. + \nabla_x f \cdot \sum_{j=1}^{m} w_j \nabla_x u_{n-1}^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_n^{(m-2)}\{\tilde{w}_{j,k}\} \right) \\
& - \frac{(h+y)^2}{h^2} \left(|\nabla_x f|^2 \partial_y u_{n-2}^{(m)}\{w\} + 2\nabla_x f \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u_{n-1}^{(m-1)}\{\tilde{w}_j\} \right. \\
& \left. + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_n^{(m-2)}\{\tilde{w}_{j,k}\} \right), \qquad (5.3.2g)
\end{aligned}
$$

and

$$
\begin{aligned}
F_{h,n}^{(m)} = \frac{1}{h}\left(\nabla_x f \cdot \nabla_x u_{n-1}^{(m)}\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_n^{(m-1)}\{\tilde{w}_j\}\right) \\
+ \frac{1}{h^2}\left(f \nabla_x f \cdot \nabla_x u_{n-2}^{(m)}\{w\} + f\sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_{n-1}^{(m-1)}\{\tilde{w}_j\}\right. \\
\left. + \nabla_x f \cdot \sum_{j=1}^{m} w_j \cdot \nabla_x u_{n-1}^{(m-1)}\{\tilde{w}_j\} + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_n^{(m-2)}\{\tilde{w}_{j,k}\}\right) \\
- \frac{h+y}{h^2}\left(|\nabla_x f|^2\, \partial_y u_{n-2}^{(m)}\{w\} + 2\nabla_x f \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u_{n-1}^{(m-1)}\{\tilde{w}_j\}\right. \\
\left. + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_n^{(m-2)}\{\tilde{w}_{j,k}\}\right). \qquad (5.3.2h)
\end{aligned}
$$

The $G_n^{(m)}$ can be computed via

$$
G_n^{(m)}(f)[\xi]\{w\} = \partial_y u_n^{(m)}(x,0) + H_n^{(m)}(x), \qquad (5.3.3a)
$$

where

$$
\begin{aligned}
H_n^{(m)} = -\frac{1}{h}\Bigg(& fG_{n-1}^{(m)}(f)[\xi]\{w\} + \sum_{j=1}^{m} w_j G_n^{(m-1)}(f)[\xi]\{\tilde{w}_j\}\Bigg) \\
& - \Bigg(\nabla_x f \cdot \nabla_x u_{n-1}^{(m)}(x,0)\{w\} + \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_n^{(m-1)}(x,0)\{\tilde{w}_j\}\Bigg) \\
& - \frac{1}{h}\Bigg(f\nabla_x f \cdot \nabla_x u_{n-2}^{(m)}(x,0)\{w\} + f\sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_{n-1}^{(m-1)}(x,0)\{\tilde{w}_j\} \\
& \qquad + \nabla_x f \cdot \sum_{j=1}^{m} w_j \nabla_x u_{n-1}^{(m-1)}(x,0)\{\tilde{w}_j\} \\
& \qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_n^{(m-2)}(x,0)\{\tilde{w}_{j,k}\}\Bigg) \\
& + \Bigg(|\nabla_x f|^2 \partial_y u_{n-2}^{(m)}(x,0)\{w\} + 2\nabla_x f \cdot \sum_{j=1}^{m} \nabla_x w_j \partial_y u_{n-1}^{(m-1)}(x,0)\{\tilde{w}_j\} \\
& \qquad + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_n^{(m-2)}(x,0)\{\tilde{w}_{j,k}\}\Bigg). \qquad (5.3.3\text{b})
\end{aligned}
$$

We are now in a position to prove our final results, the parametric analyticity of the m–th variation of the field and DNO with respect to ϵ. Again, for precision, we define the quantities D_m and $\tilde{D}_m$ which quantify the radius of convergence of the Taylor series, (5.3.1).

Definition 5.3.1. For any integer $m \geq 2$, positive real numbers B_{m-1} and B_{m-2}, and functions $f, w_1, \ldots, w_m \in C^{s+2}$, let

$$D_m := |f|_{C^{s+2}} + B_{m-1} \sum_{j=1}^{m} |w_j|_{C^{s+2}}$$
$$\tilde{D}_m := |f|^2_{C^{s+2}} + 2B_{m-1} |f|_{C^{s+2}} \sum_{j=1}^{m} |w_j|_{C^{s+2}} + B^2_{m-2} \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} |w_j|_{C^{s+2}} |w_k|_{C^{s+2}} .$$

Theorem 5.3.2. *Given an integer $s \geq 0$, if*

$$f \in C^{s+2}, \quad \xi \in H^{s+3/2}, \quad w_1, \ldots, w_m \in C^{s+2},$$

and the series for $u^{(p)}$ $(0 \leq p \leq m-1)$ in (5.3.1) *are strongly convergent, then the series for $u^{(m)}$ in* (5.3.1) *converges strongly. In other words there exist constants $\tilde{C}_m$ and $\tilde{K}_m$ such that*

$$\left\| u_n^{(m)} \right\|_{H^{s+2}} \leq \tilde{K}_m B_m^n \tag{5.3.4}$$

for any

$$B_m > \max \left\{ B_0, \ldots, B_{m-1}, 2C_e \tilde{C}_m D_m, \sqrt{2C_e \tilde{C}_m \tilde{D}_m} \right\},$$

where $B_0, B_1, \ldots, B_{m-1}$ are obtained by the analyticity estimates of u, $u^{(1)}$, $\ldots$, $u^{(m-1)}$.

The parametric analyticity of $G^{(m)}$ now follows.

Theorem 5.3.3. *Given an integer $s \geq 0$, if*

$$f \in C^{s+2}, \quad \xi \in H^{s+3/2}, \quad w_1, \ldots, w_m \in C^{s+2},$$

and the series for $G^{(p)}$ $(0 \leq p \leq m-1)$ in (5.3.1) are strongly convergent, then the series for $G^{(m)}$ in (5.3.1) converges strongly as an operator from $H^{s+3/2}$ to $H^{s+1/2}$. In other words there exist constants C_m and K_m such that

$$\left\| G_n^{(m)}(f)[\xi]\{w\} \right\|_{H^{s+1/2}} \leq K_m B_m^n \tag{5.3.5}$$

for any

$$B_m > \max\left\{ B_0, \ldots, B_{m-1}, C_m D_m, C_m \sqrt{D_m} \right\}.$$

Remark 5.3.4. These results would easily lead to an inductive proof for the parametric analyticity of *all* variations of the field and DNO provided one has control over the growth of the B_m as $m \to \infty$. At present it is not clear whether such a bound can be found so we make no such claim.

Our inductive proof again requires a recursive estimate.

Lemma 5.3.5. *Let $s \geq 0$ be an integer and let $f, w_1, \dots w_m \in C^{s+2}$. Assume*

$$\|u_n\|_{H^{s+2}} \leq \tilde{K}_0 B_0^n \qquad \forall n$$

$$\left\|u_n^{(p)}\right\|_{H^{s+2}} \leq \tilde{K}_p B_p^n \qquad 0 < p < m, \forall n$$

$$\left\|u_n^{(m)}\right\|_{H^{s+2}} \leq \tilde{K}_m B_m^n \qquad n < N,$$

and constants $\tilde{K}_0, \dots \tilde{K}_m, B_0, \dots, B_m > 0$. Then if

$$B_m > \max\{B_0, \dots, B_{m-1}\}, \quad \tilde{K}_m > \max\{\tilde{K}_0, \dots, \tilde{K}_{m-1}\},$$

there exists a constant $\tilde{C}_m$ such that

$$\left\|F_N^{(m)}\right\|_{H^s} \leq \tilde{C}_m \tilde{K}_m \left\{D_m B_m^{N-1} + \tilde{D}_m B_m^{N-2}\right\}.$$

Proof. Again, we focus our attention on one term in $F_N^{(m)}$ as the others can be handled in a similar fashion; consider $F_{y,N}^{(m)}$ and recall that since it is $\partial_y F_{y,N}^{(m)}$ which appears in $F_N^{(m)}$ we measure in the H^{s+1} norm.

$$
\begin{aligned}
\left\| F_{y,N}^{(m)} \right\|_{H^{s+1}} \leq \frac{Y}{h} \Bigg(& M \left| f \right|_{C^{s+2}} \left\| u_{N-1}^{(m)} \right\|_{H^{s+2}} \\
& + \sum_{j=1}^{m} M \left| w_j \right|_{C^{s+2}} \left\| u_N^{(m-1)} \right\|_{H^{s+2}} \Bigg) \\
& + \frac{Y}{h^2} \Bigg(M^2 \left| f \right|_{C^{s+1}} \left| f \right|_{C^{s+2}} \left\| u_{N-2}^{(m)} \right\|_{H^{s+2}} \\
& + M \left| f \right|_{C^{s+1}} \sum_{j=1}^{m} M \left| w_j \right|_{C^{s+2}} \left\| u_{N-1}^{(m-1)} \right\|_{H^{s+2}} \\
& + M \left| f \right|_{C^{s+2}} \sum_{j=1}^{m} M \left| w_j \right|_{C^{s+1}} \left\| u_{N-1}^{(m-1)} \right\|_{H^{s+2}} \\
& + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} M^2 \left| w_j \right|_{C^{s+1}} \left| w_k \right|_{C^{s+2}} \left\| u_N^{(m-2)} \right\|_{H^{s+2}} \Bigg) \\
+ \frac{Y^2}{h^2} \Bigg(& M^2 \left| f \right|_{C^{s+2}}^2 \left\| u_{N-2}^{(m)} \right\|_{H^{s+2}} + 2M \left| f \right|_{C^{s+2}} \sum_{j=1}^{m} M \left| w_j \right|_{C^{s+2}} \left\| u_{N-1}^{(m-1)} \right\|_{H^{s+2}} \\
& + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} M^2 \left| w_j \right|_{C^{s+2}} \left| w_k \right|_{C^{s+2}} \left\| u_N^{(m-2)} \right\|_{H^{s+2}} \Bigg) .
\end{aligned}
$$

Using the inductive bounds we now conclude the following:

$$
\begin{aligned}
\left\| F^{(m)}_{y,N} \right\|_{H^{s+1}} \leq \frac{MY}{h} & \left(|f|_{C^{s+2}} \tilde{K}_m B_m^{N-1} + \sum_{j=1}^{m} |w_j|_{C^{s+2}} \tilde{K}_{m-1} B_{m-1}^{N} \right) \\
& + \frac{M^2 Y}{h^2} \left(|f|^2_{C^{s+2}} \tilde{K}_m B_m^{N-2} + 2\,|f|_{C^{s+2}} \sum_{j=1}^{m} |w_j|_{C^{s+2}} \tilde{K}_{m-1} B_{m-1}^{N-1} \right. \\
& \qquad \left. + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+2}} \, |w_k|_{C^{s+2}} \tilde{K}_{m-2} B_{m-2}^{N} \right) \\
& + \frac{M^2 Y^2}{h^2} \left(|f|^2_{C^{s+2}} \tilde{K}_m B_m^{N-2} + 2\,|f|_{C^{s+2}} \sum_{j=1}^{m} |w_j|_{C^{s+2}} \tilde{K}_{m-1} B_{m-1}^{N-1} \right. \\
& \qquad \qquad \left. + \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+2}} \, |w_k|_{C^{s+2}} \tilde{K}_{m-2} B_{m-2}^{N} \right).
\end{aligned}
$$

By rearranging and using

$$
B_m > \max\{B_0, \ldots, B_{m-1}\}, \quad \tilde{K}_m > \max\{\tilde{K}_0, \ldots, \tilde{K}_{m-1}\},
$$

we obtain:

$$
\begin{aligned}
\left\| F_{y,N}^{(m)} \right\|_{H^{s+1}} \leq \frac{MY}{h} \tilde{K}_m & \left(|f|_{C^{s+2}} + \sum_{j=1}^{m} |w_j|_{C^{s+2}} B_{m-1} \right) B_m^{N-1} \\
& + \frac{M^2 Y}{h^2} \tilde{K}_m \left(|f|_{C^{s+2}}^2 + 2 |f|_{C^{s+2}} \sum_{j=1}^{m} |w_j|_{C^{s+2}} B_{m-1} \right. \\
& \left. + \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} |w_j|_{C^{s+2}} |w_k|_{C^{s+2}} B_{m-2}^2 \right) B_m^{N-2} \\
& + \frac{M^2 Y^2}{h^2} \tilde{K}_m \left(|f|_{C^{s+2}}^2 + 2 |f|_{C^{s+2}} \sum_{j=1}^{m} |w_j|_{C^{s+2}} B_{m-1} \right. \\
& \left. + \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} |w_j|_{C^{s+2}} |w_k|_{C^{s+2}} B_{m-2}^2 \right) B_m^{N-2},
\end{aligned}
$$

and we are done if $\tilde{C}_m$ is chosen appropriately. □

We are now in a position to prove the parametric analyticity of the m–th variation of the field, $u^{(m)}$.

Proof. (Theorem 5.3.2) We utilize an induction in n; at order $n = 0$ we recall that we must solve (5.3.2) with

$$
\begin{aligned}
F_{x,0}^{(m)} = & -\frac{2}{h} \sum_{j=1}^{m} w_j \nabla_x u_0^{(m-1)} \{\tilde{w}_j\} - \frac{1}{h^2} \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} w_j w_k \nabla_x u_0^{(m-2)} \{\tilde{w}_{j,k}\} \\
& + \frac{h+y}{h} \sum_{j=1}^{m} \nabla_x w_j \partial_y u_0^{(m-1)} \{\tilde{w}_j\} \\
& + \frac{h+y}{h^2} \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} w_j \nabla_x w_k \partial_y u_0^{(m-2)} \{\tilde{w}_{j,k}\},
\end{aligned}
$$

$$
\begin{aligned}
F_{y,0}^{(m)} &= \frac{h+y}{h} \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_0^{(m-1)}\{\tilde{w}_j\} \\
&+ \frac{h+y}{h^2} \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_0^{(m-2)}\{\tilde{w}_{j,k}\} \\
&- \frac{(h+y)^2}{h^2} \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_0^{(m-2)}\{\tilde{w}_{j,k}\},
\end{aligned}
$$

$$
\begin{aligned}
F_{h,0}^{(m)} &= \frac{1}{h} \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_0^{(m-1)}\{\tilde{w}_j\} \\
&+ \frac{1}{h^2} \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_0^{(m-2)}\{\tilde{w}_{j,k}\} \\
&- \frac{h+y}{h^2} \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_0^{(m-2)}\{\tilde{w}_{j,k}\}.
\end{aligned}
$$

Using Lemmas 5.1.4 & 5.1.5 we find that

$$\begin{aligned}\left\|u_0^{(1)}\right\|_{H^{s+2}} \leq C_e \Bigg\{ &\frac{2M}{h}\sum_{j=1}^{m}|w_j|_{C^{s+1}}\left\|u_0^{(m-1)}\right\|_{H^{s+2}} \\ &+ \frac{M^2}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+1}}|w_k|_{C^{s+1}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}} \\ &+ 2\frac{MY}{h}\sum_{j=1}^{m}|w_j|_{C^{s+2}}\left\|u_0^{(m-1)}\right\|_{H^{s+2}} \\ &+ 2\frac{M^2Y}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+1}}|w_k|_{C^{s+2}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}} \\ &+ \frac{M^2Y^2}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+2}}|w_k|_{C^{s+2}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}} \\ &+ \frac{M}{h}\sum_{j=1}^{m}|w_j|_{C^{s+2}}\left\|u_0^{(m-1)}\right\|_{H^{s+2}} \\ &+ \frac{M^2}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+1}}|w_k|_{C^{s+2}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}} \\ &+ \frac{M^2Y}{h^2}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+2}}|w_k|_{C^{s+2}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}}\Bigg\}.\end{aligned}$$

We set

$$\begin{aligned}\tilde{K}_m = (3+2Y)\frac{M}{h}\sum_{j=1}^{m}|w_j|_{C^{s+2}}\left\|u_0^{(m-1)}\right\|_{H^{s+2}} \\ + (Y^2+3Y+2)\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m}|w_j|_{C^{s+2}}|w_k|_{C^{s+2}}\left\|u_0^{(m-2)}\right\|_{H^{s+2}}\end{aligned}$$

and the case $n = 0$ is established. We now assume (5.3.4) for all $n < N$ and use (5.3.2) and Lemma 5.1.5 to realize

$$\left\| u_N^{(m)} \right\|_{H^{s+2}} \leq C_e \left\| F_N^{(m)} \right\|_{H^s}.$$

By our hypotheses on the analyticity of u, $u^{(1)}$, $\dots u^{(m-1)}$, Lemma 5.3.5 holds which we now use to imply that

$$\begin{aligned}
\left\| u_N^{(m)} \right\|_{H^{s+2}} &\leq C_e \tilde{C}_m \tilde{K}_m \left\{ D_m B_m^{N-1} + \tilde{D}_m B_m^{N-2} \right\} \\
&\leq \tilde{K}_m B_m^N,
\end{aligned}$$

provided we choose

$$B_m > \max \left\{ 2C_e \tilde{C}_m D_m, \sqrt{2C_e \tilde{C}_m \tilde{D}_m} \right\}.$$

□

Finally, we show the parametric analyticity of $G^{(m)}$.

Proof. (Theorem 5.3.3) By our hypotheses of the analyticity of G, $G^{(1)}$, ..., $G^{(m-1)}$ we have estimates on the terms G_n, $G_n^{(1)}$, ..., $G_n^{(m-1)}$, which are used later in this proof. We proceed inductively in n and from (5.3.3a), we see at order zero that

$$\begin{aligned}
G_0^{(m)}[\xi]\{w\} = \partial_y u_0^{(m)}(x,0)\{w\} &- \frac{1}{h}\sum_{j=1}^{m} w_j G_0^{(m-1)}(f)[\xi]\{\tilde{w}_j\} \\
&- \sum_{j=1}^{m} \nabla_x w_j \cdot \nabla_x u_0^{(m-1)}(x,0)\{\tilde{w}_j\} \\
&- \frac{1}{h}\sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} w_j \nabla_x w_k \cdot \nabla_x u_0^{(m-2)}(x,0)\{\tilde{w}_{j,k}\} \\
&+ \sum_{j=1}^{m}\sum_{k=1,k\neq j}^{m} \nabla_x w_j \cdot \nabla_x w_k \partial_y u_0^{(m-2)}(x,0)\{\tilde{w}_{j,k}\}.
\end{aligned}$$

We now estimate

$$
\begin{aligned}
\left\| G_0^{(m)}[\xi]\{w\} \right\|_{H^{s+1/2}} \leq & \left\| u_0^{(m)}(x,0) \right\|_{H^{s+3/2}} \\
& + \frac{M}{h} \sum_{j=1}^{m} |w_j|_{C^{s+1/2+\sigma}} \left\| G_0^{(m-1)}(f)[\xi]\{\tilde{w}_j\} \right\|_{H^{s+1/2}} \\
& + M \sum_{j=1}^{m} |w_j|_{C^{s+3/2+\sigma}} \left\| u_0^{(m-1)} \right\|_{H^{s+3/2}} \\
& + \frac{M^2}{h} \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+1/2+\sigma}} |w_k|_{C^{s+3/2+\sigma}} \\
& \quad \times \left\| u_0^{(m-2)} \right\|_{H^{s+3/2}} \\
& + M^2 \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|^2_{C^{s+3/2+\sigma}} \left\| u_0^{(m-2)} \right\|_{H^{s+3/2}} \\
\leq & \tilde{K}_m + \frac{M}{h} \sum_{j=1}^{m} |w_j|_{C^{s+2}} K_{m-1} + M \sum_{j=1}^{m} |w_j|_{C^{s+2}} \tilde{K}_{m-1} \\
& + \frac{M^2}{h} \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+2}} |w_k|_{C^{s+2}} \tilde{K}_{m-2} \\
& + M^2 \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+2}} |w_k|_{C^{s+2}} \tilde{K}_{m-2}.
\end{aligned}
$$

If we set

$$
\begin{aligned}
K_m = & \tilde{K}_m + \frac{M}{h} \sum_{j=1}^{m} |w_j|_{C^{s+2}} K_{m-1} + M \sum_{j=1}^{m} |w_j|_{C^{s+2}} \tilde{K}_{m-1} \\
& + \frac{M^2}{h} \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+2}} |w_k|_{C^{s+2}} \tilde{K}_{m-2} \\
& + M^2 \sum_{j=1}^{m} \sum_{k=1,k\neq j}^{m} |w_j|_{C^{s+2}} |w_k|_{C^{s+2}} \tilde{K}_{m-2},
\end{aligned}
$$

then the case $n = 0$ is resolved. We now suppose that (5.3.5) holds for $n < N$ and examine $G_N^{(m)}$ in $H^{s+1/2}$:

$$
\begin{aligned}
\left\| G_N^{(m)} \right\|_{H^{s+1/2}} \leq & \left\| u_N^{(m)}(x,0) \right\|_{H^{s+3/2}} \\
& + \frac{M}{h} \left(|f|_{C^{s+1/2+\sigma}} \left\| G_{N-1}^{(m)} \right\|_{H^{s+1/2}} + \sum_{j=1}^{m} |w_j|_{C^{s+1/2+\sigma}} \left\| G_N^{(m-1)} \right\|_{H^{s+1/2}} \right) \\
& \qquad + M \left(|f|_{C^{s+3/2+\sigma}} \left\| u_{N-1}^{(m)}(x,0) \right\|_{H^{s+3/2}} \right. \\
& \qquad \left. + \sum_{j=1}^{m} |w_j|_{C^{s+3/2+\sigma}} \left\| u_N^{(m-1)}(x,0) \right\|_{H^{s+3/2}} \right) \\
& \quad + \frac{M^2}{h} \left(|f|_{C^{s+1/2+\sigma}} |f|_{C^{s+3/2+\sigma}} \left\| u_{N-2}^{(m)}(x,0) \right\|_{H^{s+3/2}} \right. \\
& \quad + |f|_{C^{s+1/2+\sigma}} \sum_{j=1}^{m} |w_j|_{C^{s+3/2+\sigma}} \left\| u_{N-1}^{(m-1)}(x,0) \right\|_{H^{s+3/2}} \\
& \quad + |f|_{C^{s+3/2+\sigma}} \sum_{j=1}^{m} |w_j|_{C^{s+1/2+\sigma}} \left\| u_{N-1}^{(m-1)}(x,0) \right\|_{H^{s+3/2}} \\
& \left. + \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} |w_j|_{C^{s+1/2+\sigma}} |w_k|_{C^{s+3/2+\sigma}} \left\| u_N^{(m-2)}(x,0) \right\|_{H^{s+3/2}} \right) \\
& \qquad + M^2 \left(|f|_{C^{s+3/2+\sigma}}^2 \left\| u_{N-2}^{(m)}(x,0) \right\|_{H^{s+3/2}} \right. \\
& \quad + 2 |f|_{C^{s+3/2+\sigma}} \sum_{j=1}^{m} |w_j|_{C^{s+3/2+\sigma}} \left\| u_{N-1}^{(m-1)}(x,0) \right\|_{H^{s+3/2}} \\
& \qquad \left. + \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} |w_j|_{C^{s+3/2+\sigma}} |w_k|_{C^{s+3/2+\sigma}} \left\| u_N^{(m-2)}(x,0) \right\|_{H^{s+3/2}} \right).
\end{aligned}
$$

Now,

$$
\begin{aligned}
\left\| G_N^{(m)} \right\|_{H^{s+1/2}} \leq \tilde{K}_m B_m^N &+ \frac{M}{h} \Bigg(|f|_{C^{s+2}} K_m B_m^{N-1} \\
&\qquad + \sum_{j=1}^{m} |w_j|_{C^{s+2}} K_{m-1} B_{m-1}^N \Bigg) \\
&+ M \left(|f|_{C^{s+2}} \tilde{K}_m B_m^{N-1} + \sum_{j=1}^{m} |w_j|_{C^{s+2}} \tilde{K}_{m-1} B_{m-1}^N \right) \\
&+ \frac{M^2}{h} \Bigg(|f|_{C^{s+2}}^2 \tilde{K}_m B_m^{N-2} + 2 |f|_{C^{s+2}} \sum_{j=1}^{m} |w_j|_{C^{s+2}} \tilde{K}_{m-1} B_{m-1}^{N-1} \\
&\qquad + \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} |w_j|_{C^{s+2}} |w_k|_{C^{s+2}} \tilde{K}_{m-2} B_{m-2}^N \Bigg) \\
&+ M^2 \Bigg(|f|_{C^{s+2}}^2 \tilde{K}_m B_m^{N-2} + 2 |f|_{C^{s+2}} \sum_{j=1}^{m} |w_j|_{C^{s+2}} \tilde{K}_{m-1} B_{m-1}^{N-1} \\
&\qquad + \sum_{j=1}^{m} \sum_{k=1, k \neq j}^{m} |w_j|_{C^{s+2}} |w_k|_{C^{s+2}} \tilde{K}_{m-2} B_{m-2}^N \Bigg),
\end{aligned}
$$

which can be bounded above by $K_m B_m^N$ provided that:

$$B_m > \max\{B_0, \ldots, B_{m-1}\},$$

and K_m is chosen sufficiently large (see the proof of Theorem 5.1.3). $\square$

CHAPTER 6

NUMERICAL SIMULATION OF THE VARIATION OF THE DNO

In this section, we outline our numerical implementation of the OE and TFE methods for simulating the first variation of the DNO. For simplicity, we implement our schemes in the $d = 2$ case. Only a comparison of the first variation is made, as this is the operator directly connected with questions about stability.

6.1 Spectral Collocation Method

When computing an approximation to the field $u(x, y)$ or its variation $u^{(1)}(x, y)$, we begin by choosing a set of collocation points in our domain. We are required to compute u and its variation for the TFE method, since these quantities appear in the expansions of the DNO and its variation. Since the problem is x-periodic, we will expand using a Fourier spectral method in that direction, and we choose an evenly spaced grid of N_x-many x-values. Since the problem is *not* y-periodic, we will expand using a Chebyshev spectral method in that direction, and we choose an unevenly spaced grid of Chebyshev node y-values, $y_n = \frac{h}{2}\cos\left(\frac{n\pi}{N_y}\right) - \frac{h}{2} \quad 0 \le n \le N_y$, where N_y is the number of collocation points in the y-direction. The approximations are:

$$u_n(x,y) \approx u_{n,N_x,N_y}(x,y) := \sum_{k=-N_x/2}^{N_x/2-1} \sum_{l=0}^{N_y} \hat{u}_n(k,l) T_l(\frac{2y+h}{h}) e^{ikx} \tag{6.1.1}$$

$$u_n^{(1)}(x,y) \approx u_{n,N_x,N_y}^{(1)}(x,y) := \sum_{k=-N_x/2}^{N_x/2-1} \sum_{l=0}^{N_y} \hat{u}_n^{(1)}(k,l) T_l(\frac{2y+h}{h}) e^{ikx} \tag{6.1.2}$$

$$G_n(x) \approx G_{n,N_x}(x) := \sum_{k=-N_x/2}^{N_x/2-1} \hat{G}_n(k) e^{ikx} \tag{6.1.3}$$

$$G_n^{(1)}(x) \approx G_{n,N_x}^{(1)}(x) := \sum_{k=-N_x/2}^{N_x/2-1} \hat{G}_n^{(1)}(k) e^{ikx}, \tag{6.1.4}$$

where T_l is the l-th Chebyshev polynomial. Our choice for these two different, yet related, spectral methods is motivated by Boyd's "Moral Principle 1": *When in doubt, use Chebyshev polynomials unless the solution is spatially periodic, in which case an ordinary Fourier series is better* (Boy01).

The first two Chebyshev polynomials are given by

$$T_0(y) = 1$$

$$T_1(y) = y,$$

and further polynomials are defined by the recursion

$$T_n(y) = 2yT_{n-1}(y) - T_{n-2}(y) \qquad \forall n \ge 2.$$

Note that each Chebyshev polynomial is of degree n. These functions form a basis for the space of continuous functions on the interval $y \in [-1, 1]$, which can be easily mapped to any finite interval (i.e. $y \in [-h, 0]$).

From a practical standpoint we must be able to transform function values to Fourier and Chebyshev coefficients. The Discrete Fourier Transform (DFT) accomplishes this for the Fourier coefficients, while for the Chebyshev coefficients, we create an $N_y \times N_y$ square matrix $\mathcal{T}$, whose columns consist of the first N_y Chebyshev polynomials evaluated on the Chebyshev grid $y_n = \cos(n\pi/(N_y - 1))$. Thus, given a vector of Chebyshev coefficients $\bar{a} = \{a_n\}$, the values of their linear combination $\sum a_n T_n$ are given by:

$$\bar{f} = \mathcal{T}\bar{a}.$$

Similarly, given a set of function values $\bar{f}$ at the Chebyshev grid points, the coefficients $\bar{a}$ of the T_n that represent f are given by:

$$\bar{a} = \mathcal{T}^{-1}\bar{f}.$$

Interestingly, these operations do not require any change of variables to account for the differing working intervals $[-h, 0]$ vs. $[-1, 1]$. Moving between those intervals involves only a translation and dialation, and the transformation matrix is unaffected. Since the DFT (accelerated by the FFT algorithm) algorithm is commonly used and is available in any major scientific computing platform, we do not discuss the details here.

Derivatives are computed on the Fourier and Chebyshev spectral sides, respectively, while multiplcations of functions are computed on the physical side. We move between the spectral and physical sides using the FFT algorithm in the x-direction, and the Chebyshev transform algorithm listed above, in the y-direction. Function multiplications are pointwise for the vector containing the function values. Differentiation on the Fourier side is performed by simply multiplying by ik pointwise. The Chebyshev differentiation routine is somewhat more involved. One creates an $(N_y - 1) \times (N_y - 1)$ matrix $\mathcal{D}$, derived from recursions between the Chebyshev polynomials and their derivatives. This matrix is smaller in dimension than the transformation matrix because some information is lost when differentiating a finite linear combination of Chebyshev polynomials, as the derivative $T_0' = 0$ and there is no $N_y + 1^{st}$ polynomial to contribute a data point. Furthermore, the matrix $\mathcal{D}$ is created specifically to operate on the interval $[-1, 1]$, and so a scale factor of $2/h$ is introduced to account for the dilation.

As an example of how these techniques are applied when computing a particular expansion order, consider the following term occurring on the right-hand-side of (5.1.2), for $n = 1$:

$$\operatorname{div}_x \left[\frac{h+y}{h} \nabla_x f \partial_y u_0^{(1)} \right].$$

To compute this quantity, we transform f to the Fourier side, multiply by ik at each wave number to differentiate, and invert the transform to obtain $\nabla_x f$ (which is merely $\partial_x f$ when $d = 1$). Similarly, we transform $u_0^{(1)}$ to the Chebyshev side and differentiate algebraically using recurrence for the $T_n(y)$, then invert to obtain $\partial_y u_0^{(1)}$. These two quantities are multiplied together componentwise for $d = 2$ and then by $\frac{h+y}{h}$ in turn. At this point, we transform this quantity to the Fourier side for another differentiation in x. The process is repeated in calculations of terms of the DNO and its variation using the OE and TFE methods.

To find u_n and $u_n^{(1)}$ in the TFE method, we must solve the PDE governing each expansion order. After first Fourier transforming the PDE in the x-direction, we are left with a 2–point nonhomogeneous boundary value problem in the y-direction for each wave number. In order to use Chebyshev methods, we consider a generic 2–point BVP motivated by that obtained through the TFE method, having the form:

$$\begin{aligned}
\partial_y^2 \hat{u}_k(y) - k^2 \hat{u}_k(y) &= f(y) \qquad\qquad \text{for } y \in [-1, 1] \\
\hat{u}_k(-1) &= A \\
\hat{u}_k(1) &= B,
\end{aligned}$$

with constants A, B typically equal to zero except at the zeroth order. The Chebyshev coefficients $\{a_n\}$ are found by solving a nearly tridiagonal system (GO77) defined by the following equations:

$$\begin{aligned}
&\sum_{n=0}^{N_y-1}(-1)^n a_n = A \\
&\sum_{n=0}^{N_y-1} a_n = B \\
&\frac{k^2 c_{n-2}}{4n(n-1)}a_{n-2} - (1+\frac{k^2 e_{n+2}}{2(n^2-1)})a_n + \frac{k^2 e_{n+4}}{4n(n+1)}a_{n+2} = \\
&\qquad -\frac{c_{n-2}f_{n-2}}{4n(n-1)} + \frac{e_{n+2}f_n}{2(n^2-1)} - \frac{e_{n+4}f_{n+2}}{4n(n+1)} \qquad \text{for } 2 \le n \le N_y - 1,
\end{aligned}$$

where $\{f_n\}$ are the Chebyshev coefficients of the nonhomogeneous term, $c_0 = 2$, $c_n = 1$ for $n \ge 1$, $e_n = 1$ for $n \le N_y$, and $e_n = 0$ for $n \ge N_y$.

6.2 Exact Solution

We proceed with our analysis by considering certain conditions under which an exact solution is 'easy' to find. We evaluate this exact solution on our collocation grid, and compare it against the values of the perturbation expansions. As described in previous work, for a given $g(x)$, the system (4.3.2) has exact solution given by

$$u_k(x,y) = \cosh\left(|k|\left(\frac{h+g(x)}{h}y + g(x) + h\right)\right)e^{ikx} \qquad k \in \Gamma',$$

for the particular choice of Dirichlet data

$$\xi_k(x) = \cosh(|k|(g(x) + h)e^{ikx}.$$

From this Dirichlet data, we can contruct exact Neumann data, i.e. the DNO:

$$\nu_k(x) = [|k| \sinh(|k|(g(x) + h) - ik \cdot \nabla_x g(x) \cosh(|k|(g(x) + h)]\, e^{ikx}.$$

What is left is to take the variation of each of these objects, in the direction $w(x)$:

$$u_k^{(1)}(x, y)\{w\} = |k| \left(\frac{h + y}{h}\right) w(x) \sinh\left(|k|(\frac{h + g(x)}{h} y + g(x) + h)\right) e^{ikx},$$

$$\xi_k^{(1)}(x)\{w\} = |k| w(x) \sinh(|k|(g(x) + h)e^{ikx},$$

and

$$\begin{aligned}\nu_k^{(1)}(x)\{w\} = \big[\big(|k|^2 w(x) - ik \cdot \nabla_x w(x)\big) \cosh(|k|(g(x) + h) \\ -ik \cdot \nabla_x g(x) |k| w(x) \sinh(|k|(g(x) + h)]\, e^{ikx}. \quad (6.2.1)\end{aligned}$$

These are the exact variation of field, variation of Dirichlet data, and variation of DNO. For our numerical investigations, we perform all computations with $k = 3$. It is important to obtain the variation of the Dirichlet data $\xi_k^{(1)}$ because of the following fact:

$$\delta_g \left\{ G(g)[\xi_k] \right\} = G^{(1)}(g)[\xi_k]\{w\} + G(g)[\xi_k^{(1)}\{w\}].$$

Since the Dirichlet data for our exact solution is explicitly g–dependent we account for it using the above "product rule", and obtain:

$$G^{(1)}(g)[\xi_k]\{w\} = \nu_k^{(1)} - G(g)[\xi_k^{(1)}\{w\}].$$

6.3 Sample Profiles

We investigate numerical results under a variety of conditions regarding the smoothness of the surface shape $g(x) = \epsilon f(x)$ and the direction of variation, $w(x)$. For each, we choose generic functions from the classes 'smooth', 'rough' (C^4, C^3), and Lipschitz. The sample surface deformations are

$$f_s(x) = \sin(x)$$

$$f_r(x) = Ax^4(2\pi - x)^4 + B$$

$$f_L(x) = \begin{cases} -(\frac{2}{\pi})x + 1 & x \in [0, \pi) \\ (\frac{2}{\pi})x - 3 & x \in [\pi, 2\pi) \end{cases},$$

and the directions of variation are

$$
\begin{aligned}
w_s(x) &= -\sin(2x) \\
w_r(x) &= A'x^3(2\pi - x)^3 + B' \\
w_L(x) &= \frac{3}{2}\begin{cases} -(\frac{2}{\pi})x + \frac{1}{2} & x \in [0, \pi/2) \\ (\frac{2}{\pi})x - \frac{3}{2} & x \in [\pi/2, \pi) \\ -(\frac{2}{\pi})x + \frac{5}{2} & x \in [\pi, 3\pi/2) \\ (\frac{2}{\pi})x - \frac{7}{2} & x \in [3\pi/2, 2\pi) \end{cases}.
\end{aligned}
$$

The constants A, A', B, B' are chosen so that the functions are zero-mean and have maximum aplitude and slope $\mathcal{O}(1)$. In order to demonstrate effectiveness of the TFE method under the broadest circumstances, all nine possible combinations of surface deformation and variation directions smoothness are tested.

6.4 Numerical Results

Here we show error results of the various computations. Both OE and TFE schemes show improvement in accuracy as the number of Taylor orders is increased, until about the 7^{th} order. At this point, the OE scheme experiences exponentially growing errors, while the TFE scheme errors steadily approach machine–zero, 10^{-16}. This tendency is found in all combinations of smoothness of surface deformation and variation direction. However, we do notice that the results are moderately worse for OE with the Lipschitz and "rough" profiles – namely, divergence occurs for smaller n. The results are from computations with a collocation grid composed of $N_x = 256$ points in the x-direction, $N_y = 64$ points in the y-direction, a deformation paramter

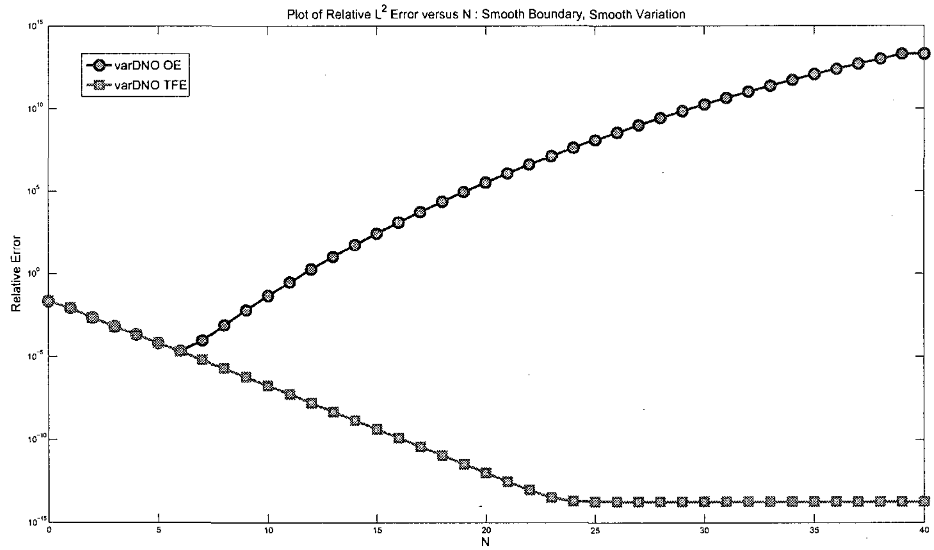

Figure 1. Smooth Surface Deformation, Smooth Variation Direction

value $\epsilon = 0.3$, and a maximum Taylor order of $N = 40$. In order to minimize the effects of aliasing, the infinite Fourier series representation of the functions $f_s(x), f_L(x), w_s(x), w_L(x)$ are truncated to include only 40 wave numbers: $k \in \{-20, -19, \ldots, 18, 19\}$.

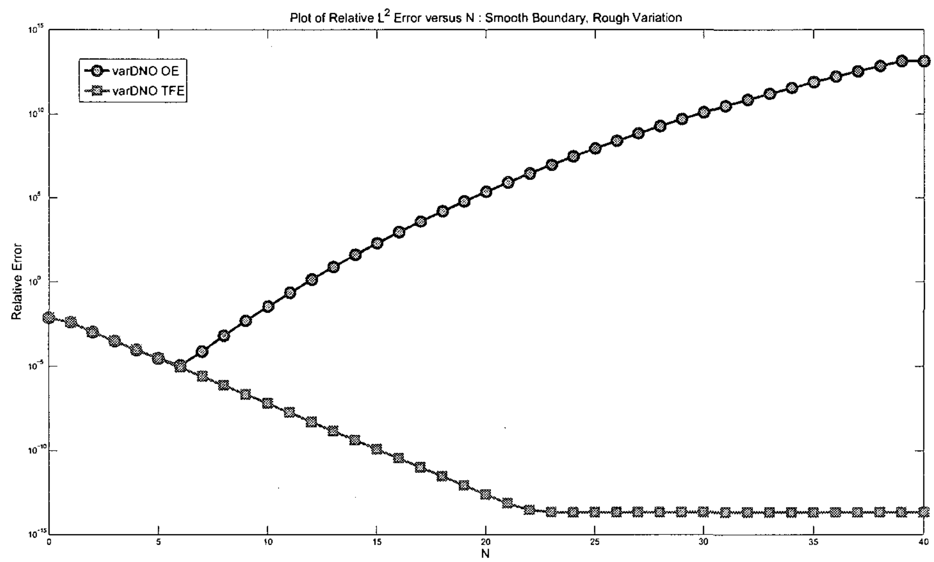

Figure 2. Smooth Surface Deformation, Rough Variation Direction

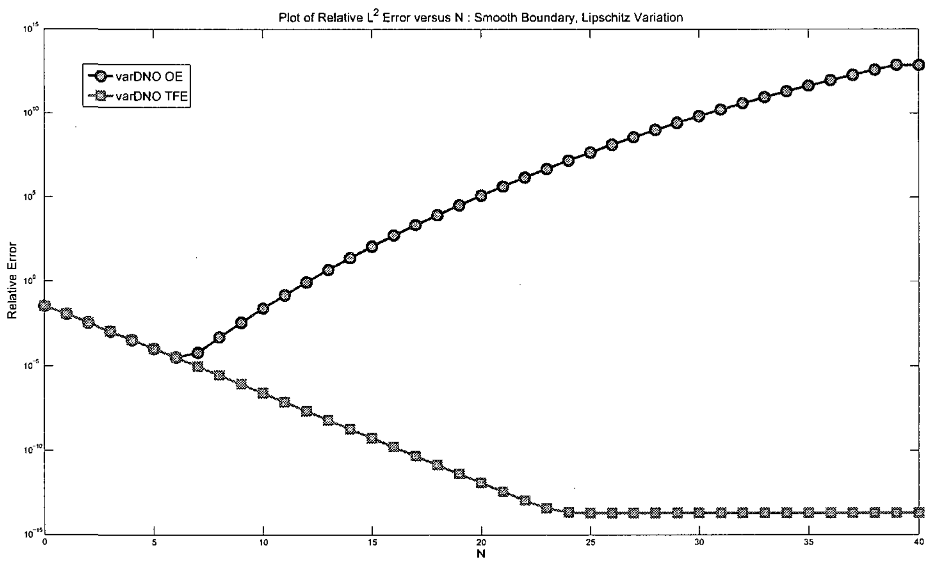

Figure 3. Smooth Surface Deformation, Lipschitz Variation Direction

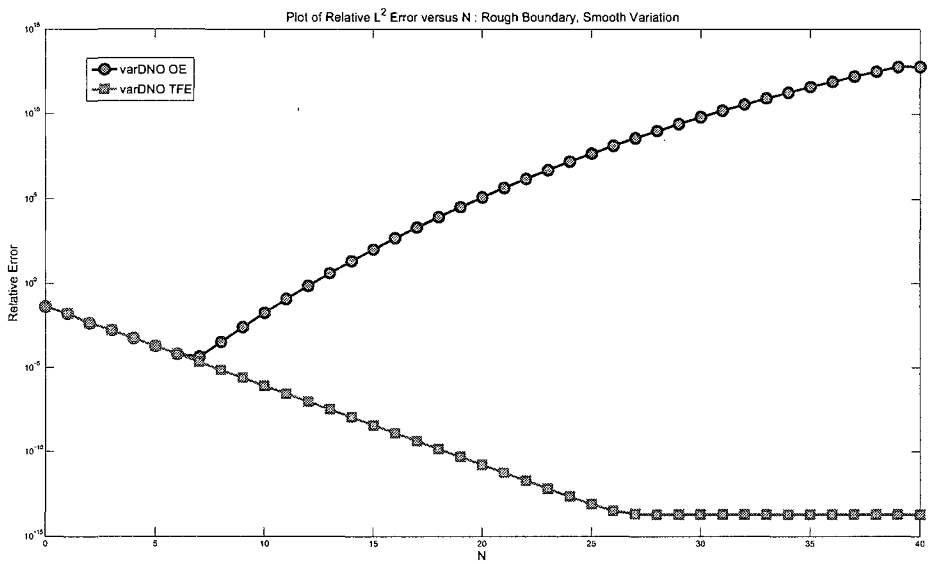

Figure 4. Rough Surface Deformation, Smooth Variation Direction

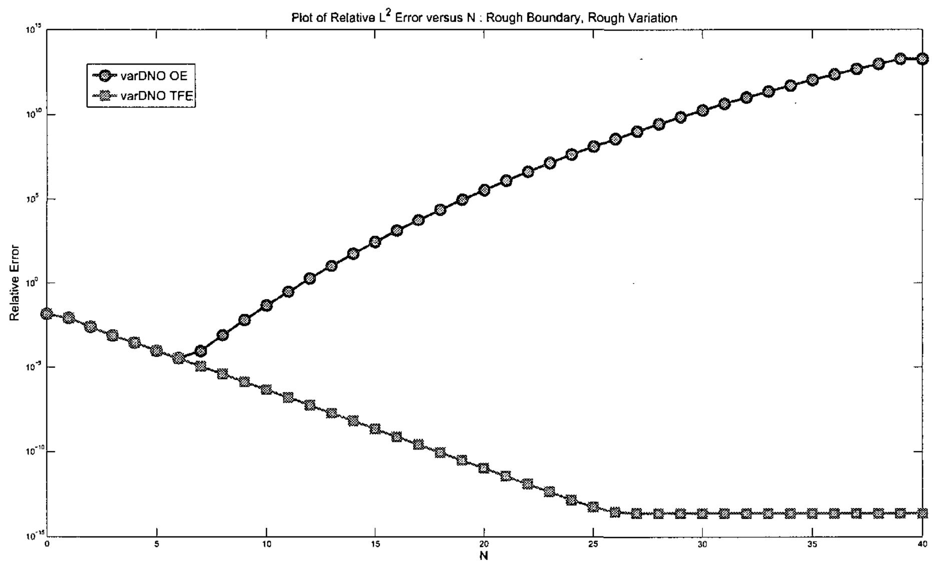

Figure 5. Rough Surface Deformation, Rough Variation Direction

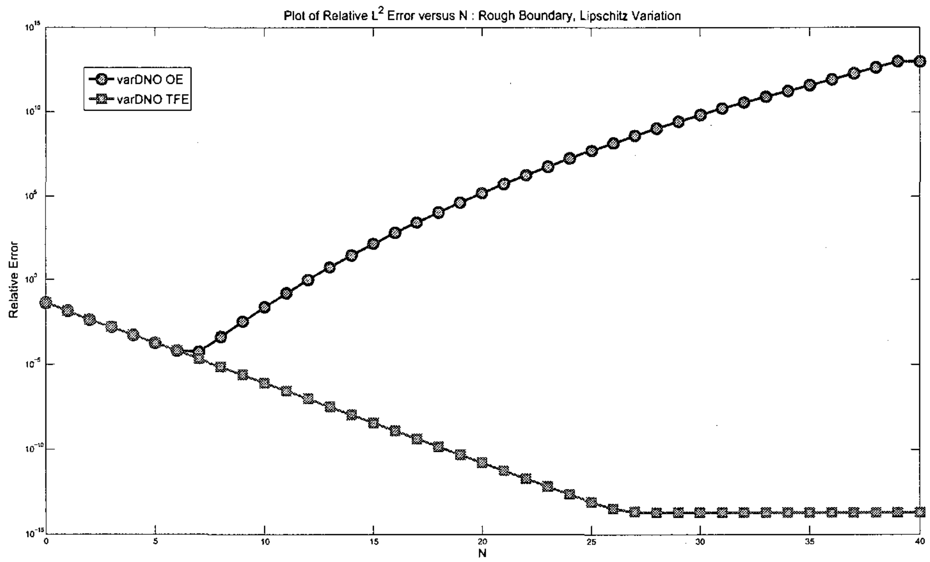

Figure 6. Rough Surface Deformation, Lipschitz Variation Direction

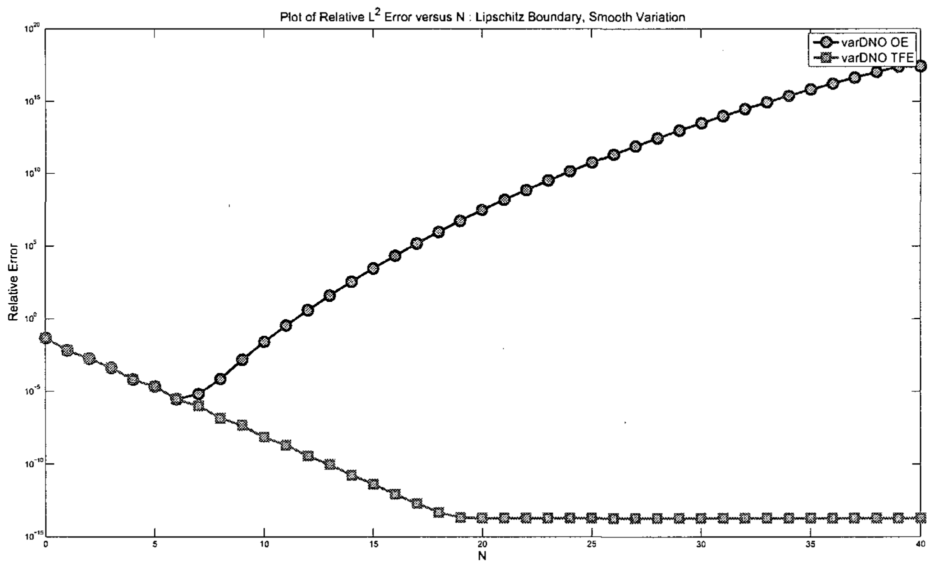

Figure 7. Lipschitz Surface Deformation, Smooth Variation Direction

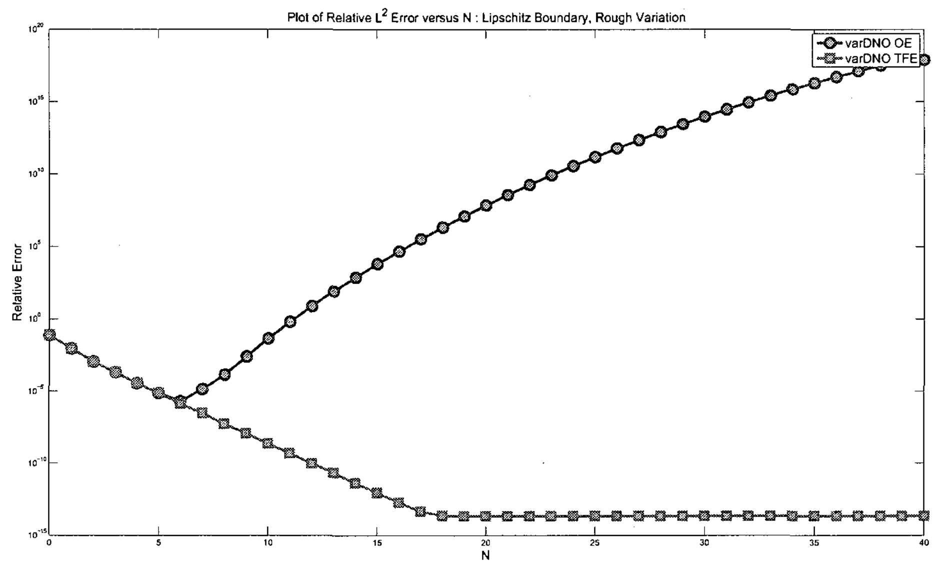

Figure 8. Lipschitz Surface Deformation, Rough Variation Direction

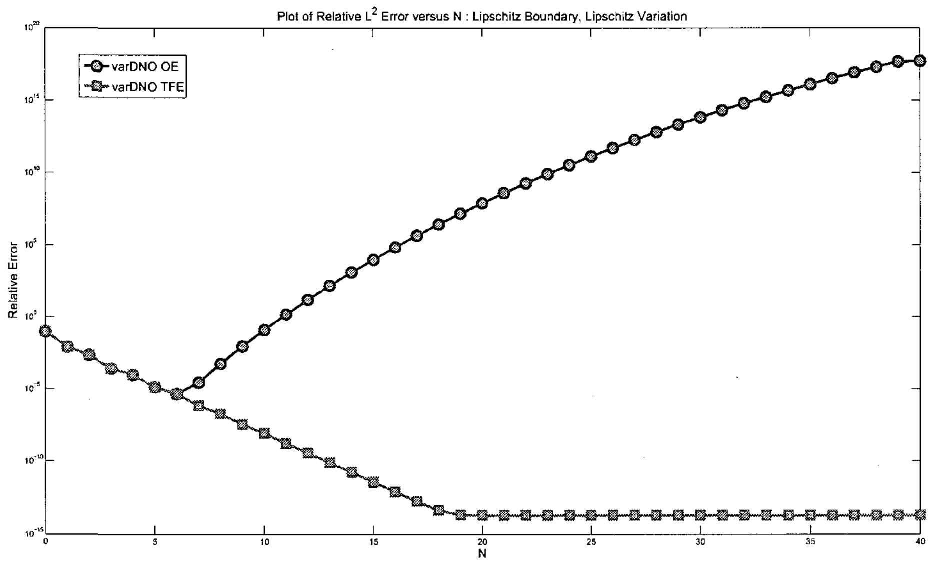

Figure 9. Lipschitz Surface Deformation, Lipschitz Variation Direction

CHAPTER 7

CONCLUSION

We conclude by providing additional context for this work, as well as mentioning advantages and disadvantages of the methods presented here. The use of the TFE methodology in this paper follows directly from similar usage by Nicholls & Reitich (NR01) regarding analyticity of the DNO itself.

7.1 Advantages

There are several noteworthy advantages to the TFE method. Prior to the change of variables, the FBP is faced with the problem of an evolving domain. As t progresses, the fluid moves, changing the shape of the free surface. This in turn changes the domain on which the problem is posed. Resolving this issue is a practical and theoretical difficulty that is avoided by the TFE method. Not only does the change of variables flatten the domain to one of separable geometry, but it makes this shape static.

Furthermore, when creating the series expansion in the OE method, successive terms are evaluated at the mean surface level $y = 0$. However, this is not the true boundary, as $g(x)$ is small but nonzero. While characteristic of many perturbation methods, this difficulty is overcome with TFE since the boundary of the domain is exactly $y = 0$.

Most striking is the nature of the recursions that result from the TFE method. They are of a form such that at the n-th order, the only terms present in the defining equations for u_n are of order $n-1$ and $n-2$. These terms typically contain derivatives of order 1 and 2, respectively, and therefore offer considerable advantage over the OE method. This method has derivatives of arbitrarily large order as n increases, which poses problems for both theoretical and numerical analyses. While the high order derivatives cancel in the OE method and allow some amount of convergence, instability in the numerical method will eventually cause exponential growth of error.

7.2 Disadvantages

The benefit of replacing a time-dependent domain with a flattened seperable geometry is not without cost. The surface deformation manifests itself, after the change of variables, through the nonhomogeneous terms on the right hand side of the PDE defining the velocity potential. Thus, the equations that must be solved at each order are of greatly increased complexity.

As a general note, while the interest in the DNO is the result of the surface formulation, the entire volumetric problem cannot itself be ignored. For example, when using the TFE method to compute the DNO and its variation, calls must be made to terms in the expansion of the field and its variation. These terms are indeed evaluated at the surface, but they must be calculated to some finite (nonzero) depth in order to accurately compute derivates. Methods exist (NR05; NR06) that attempt to reduce the computational complexity of this problem by imposing a so-called Transparent Boundary condition near the surface which, although it does not reduce the problem dimension by one, can greatly reduce the number of collocation points required to obtain a desired level of precision.

7.3 Future Directions

The framework which motivates this analysis is that of spectral stability of traveling waves. Since the variation of the DNO is present in the evolution equations governing perturbations of traveling waves, it is important to understand the spectrum of this operator. This spectrum can be approximated numerically by considering the eigenvalues of a finite (yet large) matrix representation of the variation of the DNO. Linear operator theory tells us that we can obtain a matrix representation of an operator by considering its action on the basis vectors of its domain. A future step for research along these directions involves obtaining such an approximation of the variation of the DNO, with a corresponding approximation of its spectrum.

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