A Theoretical and Numerical Analysis of the Faraday Wave Experiment

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THESIS

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To Mouhamed Ngom,

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SUMMARY

This thesis focuses on the well-posedness and stability of the Water Wave Equation in the context of the Faraday wave experiment. We prove that the solutions to the viscous water wave equation formulated by Dias, Dyachenko and Zakharov (DDZ) and supplemented with viscosity, surface tension and small vertical forcing is well-posed. We then derive some numerical stability results in the case of larger forcing and solve these equations numerically by using High Order Perturbation of Surfaces (HOPS) methods. We validate our code by first testing it in the case of traveling waves where exact solutions can be derived and by testing our approach against methods present in the literature.

CHAPTER 1

INTRODUCTION

(Previously published as Ngom, M. and Nicholls, D. P.: Well-posedness and analyticity of solutions to a water wave problem with viscosity. J. Differential Equations, 265:5031-5075, 2018.)

One of the central problems in fluid mechanics is the accurate modeling of the free–surface motion of a large body of water (e.g., a lake or an ocean) (55; 99; 1). It is not only a problem of classical interest (97; 3; 100; 17; 53), but also one of present importance due to its role in a number of applications from the formation and movement of sandbars, to the forces generated by waves on open–ocean oil rigs, to the propagation of tsunamis and the transport of pollutants. The "water wave equations" are the most faithful and successful model for this problem, but they have a surprisingly difficult and subtle well–posedness theory (103; 104; 13; 12; 56). We refer the interested reader to these papers, their extensive bibliographies, and the recent collection (18) for the state of the art in the field (in particular, see the chapters by Ambrose (11) and Wu (105)).

Due to the extremely important role of this model, we were inspired to find a new proof of well–posedness which did not rely on the sophisticated technology required in the papers mentioned above. While this has proven elusive, we demonstrate in this thesis that if a physically motivated viscosity is added, then a straightforward existence and uniqueness result can be established. For this we follow the lead of (8) where such a philosophy was pursued for a weakly nonlinear approximation of the full water wave problem.

Another motivation for this work is modeling the Faraday wave experiment (35) which considers the motion of the free air–fluid interface of a container of fluid which is being periodically shaken from below. As it is usually the case when studying the Faraday wave experiment, we produced a stability diagram of growth rate versus forcing and frequency similar to the ones provided by (102) where the governing equations were transformed into a Mathieu type equation. Here, we use a High Order Perturbation of Surfaces (HOPS) method (74) and a fourth order Runge-Kutta method to simulate the experiment.

The rest of this thesis is organized as follows: here in Chapter 1, we present our governing equations, then, in Chapter 2, we present our well-posedness theorem before showing in Chapter 3 our numerical results and simulations. These begin with the case of a traveling water solution and move to the case of the Faraday wave experiment.

1.1 Governing Equations

The well–known (55; 99; 1) equations governing the motion of two dimensional laterally periodic gravity–capillary water waves on a fluid of depth h are

$$
\begin{aligned}
&\Delta\varphi = 0, && -h < y < \eta,\\
&\partial_y\varphi = 0, && y = -h,\\
&\partial_t\eta = \partial_y\varphi - (\partial_x\eta)\partial_x\varphi, && y = \eta,\\
&\partial_t\varphi = -g\eta + \sigma\partial_x^2\eta + \sigma\partial_x\left[(\partial_x\eta)H(\partial_x\eta)\right] - \frac{1}{2}(\partial_x\varphi)^2 - \frac{1}{2}(\partial_y\varphi)^2, && y = \eta,
\end{aligned}
$$

where φ is the velocity potential ($\mathbf{u} = \nabla\varphi$), $y = \eta$ is the free air–fluid interface,

$$H(\partial_x \eta) := \frac{1}{\sqrt{1 + (\partial_x \eta)^2}} - 1,$$

and $g > 0$ and $\sigma > 0$ are the constants of gravity and surface tension, respectively. These are supplemented with the boundary conditions

$$\varphi(x + 2\pi, y, t) = \varphi(x, y, t), \quad \eta(x + 2\pi, t) = \eta(x, t),$$

and initial conditions

$$\eta(x, 0) = \eta^{(0)}(x), \quad \varphi(x, \eta^{(0)}, 0) = \xi^{(0)}(x),$$

where standard elliptic theory (32) reveals that specifying the initial velocity potential at the *surface* is sufficient. We supplement this with viscous terms first introduced by Dias, Dyachenko, and Zakharov (30) resulting in the "water wave equations with viscosity"

$$\Delta\varphi = 0, \qquad -h < y < \eta, \tag{1.1.1a}$$

$$\partial_y \varphi = 0, \qquad y = -h, \tag{1.1.1b}$$

$$\partial_t \eta = \partial_y \varphi + 2\mu \partial_x^2 \eta - (\partial_x \eta)\partial_x \varphi, \qquad y = \eta, \tag{1.1.1c}$$

$$\begin{aligned}\partial_t \varphi = &-g\eta + \sigma \partial_x^2 \eta - 2\mu \partial_y^2 \varphi + \sigma \partial_x \left[(\partial_x \eta) H(\partial_x \eta)\right] \\ &- \frac{1}{2}(\partial_x \varphi)^2 - \frac{1}{2}(\partial_y \varphi)^2, \qquad y = \eta,\end{aligned} \tag{1.1.1d}$$

$$\varphi(x, \eta, 0) = \xi^{(0)}(x), \tag{1.1.1e}$$

$$\eta(x, 0) = \eta^{(0)}(x), \tag{1.1.1f}$$

for a surface viscosity parameter $\mu > 0$. In these we have slightly modified Dias, Dyachenko, and Zakharov's equations by dropping the bottom viscosity terms included in (31). We will show that this problem is well–posed with analytic solutions.

Remark 1.1.1. We make two important observations: First, the mass

$$M = \int_0^{2\pi} \eta(x,t)\, dx,$$

is conserved by the flow. Indeed, it is well–known in the inviscid case (see, e.g., (26)) that

$$\partial_t[M]_{\mu=0} = \partial_t \int_0^{2\pi} \eta(x,t)\, dx = \int_0^{2\pi} \partial_t\eta(x,t)\, dx = \int_0^{2\pi} \left[\partial_y\varphi - (\partial_x\eta)\partial_x\varphi\right]_{y=\eta} = 0.$$

The addition of viscosity introduces only the term $\int_0^{2\pi} \mu\partial_x^2\eta\, dx$ to this computation, which is zero as it features an exact derivative and η is periodic.

Second, one can arrange for the mean of the *surface* velocity

$$\xi(x,t) := \varphi(x,\eta(x,t),t),$$

to remain zero if it is initially set that way. For this one must remember that the velocity potential is only meaningfully defined up to a time–dependent constant (55; 99; 1),

$$\varphi(x,y,t) = \tilde{\varphi}(x,y,t) + C(t).$$

With this we can consider the average surface velocity potentials

$$\Xi(t) = \int_0^{2\pi} \varphi(x,\eta,t)\, dx, \quad \tilde{\Xi}(t) = \int_0^{2\pi} \tilde{\varphi}(x,\eta,t)\, dx = \Xi(t) - 2\pi C(t).$$

So, if we choose $C(t) = \Xi(t)/(2\pi)$ and drop the tildes we are done. Thus we restrict our function spaces by requiring $M = \Xi = 0$.

1.2 Reformulation on a Fixed Domain

It was shown in (79; 81; 49) how a simple change of variables could be used to demonstrate the analyticity of Dirichlet–Neumann Operators (DNOs) with respect to sufficiently small and regular surface deformations $\eta = \varepsilon f$. The problem of computing DNOs for Laplace's equation is closely related to the water wave problem and, in fact, our equations could be equivalently restated at the fluid surface in terms of these operators (see, e.g., (29; 28; 64)), though we do not pursue it in this chapter, we do so in chapter 3.

To imitate this success for DNOs in the context of the water wave problem with viscosity, we follow the lead of (79) and perform the domain–flattening change of variables (known as σ–coordinates (95) and the C–Method (20))

$$x' = x, \quad y' = h\left(\frac{y-\eta}{h+\eta}\right), \quad t' = t,$$

from which we define

$$u(x', y', t') := \varphi\left(x', \left(\frac{h+\eta}{h}\right)y' + \eta, t'\right).$$

It is not difficult to show (93) that derivatives change as

$$M(x,t)\partial_x = M(x',t')\partial_{x'} + N(x',y',t')\partial_{y'},$$

$$M(x,t)\partial_t = M(x',t')\partial_{t'} + P(x',y',t')\partial_{y'},$$

$$M(x,t)\partial_y = h\partial_{y'},$$

where

$$M(x,t) = h + \eta(x,t),$$

$$N(x,y,t) = -(\partial_x \eta(x,t))(y+h),$$

$$P(x,y,t) = -(\partial_t \eta(x,t))(y+h).$$

The governing equations (Equation 1.1.1) transform, upon dropping the primes, to

$$\Delta u = F, \qquad -h < y < 0, \tag{1.2.1a}$$

$$\partial_y u = 0, \qquad y = -h, \tag{1.2.1b}$$

$$\partial_t \eta = \partial_y u + 2\mu \partial_x^2 \eta + Q, \qquad y = 0, \tag{1.2.1c}$$

$$\partial_t u = -g\eta + \sigma \partial_x^2 \eta - 2\mu \partial_y^2 u + R, \qquad y = 0, \tag{1.2.1d}$$

$$u(x,0,0) = \xi^{(0)}(x), \tag{1.2.1e}$$

$$\eta(x,0) = \eta^{(0)}(x). \tag{1.2.1f}$$

The form for F can be shown (93) to be

$$h^2 F = -\mathrm{div}\left[A^{(1)}(\eta)\nabla u\right] - \mathrm{div}\left[A^{(2)}(\eta)\nabla u\right] + (\partial_x \eta) B^{(0)} \cdot \nabla u + (\partial_x \eta) B^{(1)}(\eta) \cdot \nabla u,$$

where

$$A^{(1)}(\eta) = \begin{pmatrix} A^{(1),xx} & A^{(1),xy} \\ A^{(1),yx} & A^{(1),yy} \end{pmatrix} := \begin{pmatrix} 2h\eta & -h(y+h)\partial_x\eta \\ -h(y+h)\partial_x\eta & 0 \end{pmatrix},$$

$$A^{(2)}(\eta) = \begin{pmatrix} A^{(2),xx} & A^{(2),xy} \\ A^{(2),yx} & A^{(2),yy} \end{pmatrix} := \begin{pmatrix} \eta^2 & -(y+h)\eta(\partial_x\eta) \\ -(y+h)\eta(\partial_x\eta) & (y+h)^2(\partial_x\eta)^2 \end{pmatrix},$$

$$B^{(0)} = \begin{pmatrix} B^{(0),x} \\ B^{(0),y} \end{pmatrix} := \begin{pmatrix} h \\ 0 \end{pmatrix}, \quad B^{(1)}(\eta) = \begin{pmatrix} B^{(1),x} \\ B^{(1),y} \end{pmatrix} := \begin{pmatrix} \eta \\ -(y+h)\partial_x\eta \end{pmatrix}.$$

Therefore,

$$\begin{aligned} h^2F = &-\partial_x\left[2h\eta\partial_x u\right] + \partial_x\left[h(y+h)(\partial_x\eta)\partial_y u\right] + \partial_y\left[h(y+h)(\partial_x\eta)\partial_x u\right] \\ &-\partial_x\left[\eta^2\partial_x u\right] + \partial_x\left[(y+h)\eta(\partial_x\eta)\partial_y u\right] + \partial_y\left[(y+h)\eta(\partial_x\eta)\partial_x u\right] \\ &-\partial_y\left[(y+h)^2(\partial_x\eta)^2\partial_y u\right] + h(\partial_x\eta)\partial_x u + \eta(\partial_x\eta)\partial_x u - (y+h)(\partial_x\eta)^2\partial_y u. \end{aligned}$$

Furthermore, multiplying (Equation 1.1.1c) by M and evaluating at $y = 0$, we find

$$hQ = -\eta(\partial_t\eta) + 2\mu\eta(\partial_x^2\eta) - h(\partial_x\eta)\partial_x u - \eta(\partial_x\eta)\partial_x u + h(\partial_x\eta)^2\partial_y u,$$

where we have used the fact that, since η is independent of y, we have

$$\partial_t\eta = \partial_{t'}\eta, \quad \partial_x\eta = \partial_{x'}\eta.$$

Finally, multiplying (Equation 1.1.1d) by M^2 and evaluating at $y = 0$, we discover

$$\begin{aligned} h^2 R = &-2h\eta\partial_t u - \eta^2\partial_t u + h^2(\partial_t\eta)\partial_y u + h\eta(\partial_t\eta)\partial_y u - 2gh\eta^2 - g\eta^3 \\ &+ 2\sigma h\eta(\partial_x^2\eta) + \sigma\eta^2(\partial_x^2\eta) \\ &+ \sigma h^2\partial_x\left[(\partial_x\eta)H(\partial_x\eta)\right] + 2\sigma h\eta\partial_x\left[(\partial_x\eta)H(\partial_x\eta)\right] + \sigma\eta^2\partial_x\left[(\partial_x\eta)H(\partial_x\eta)\right] \\ &- \frac{1}{2}\left\{h^2(\partial_x u)^2 + 2h\eta(\partial_x u)^2 + \eta^2(\partial_x u)^2 - 2h^2(\partial_x\eta)(\partial_x u)\partial_y u \right. \\ &\left. -2h\eta(\partial_x\eta)(\partial_x u)\partial_y u + h^2(\partial_x\eta)^2(\partial_y u)^2\right\} - \frac{1}{2}h^2(\partial_y u)^2. \end{aligned}$$

1.2.1 A Boundary Perturbation Expansion

Our procedure for establishing well–posedness is to seek solutions of the form

$$\eta = \eta(x,t;\varepsilon) = \sum_{n=1}^{\infty} \eta_n(x,t)\varepsilon^n, \quad u = u(x,y,t;\varepsilon) = \sum_{n=1}^{\infty} u_n(x,y,t)\varepsilon^n, \tag{1.2.2}$$

given initial data $\{\eta^{(0)}(x), \xi^{(0)}(x)\}$. We will show that if this data lies in appropriate Sobolev spaces, then the $\{\eta_n, \varphi_n\}$ also exist in (different) Sobolev classes satisfying estimates which justify the strong convergence of the series (Equation 1.2.2). Upon insertion of these into (Equation 1.2.1) we find that, at each perturbation order,

$$\Delta u_n = F_n, \qquad -h < y < 0, \tag{1.2.3a}$$

$$\partial_y u_n = 0, \qquad y = -h, \tag{1.2.3b}$$

$$\partial_t \eta_n = \partial_y u_n + 2\mu \partial_x^2 \eta_n + Q_n, \qquad y = 0, \tag{1.2.3c}$$

$$\partial_t u_n = -g\eta_n + \sigma \partial_x^2 \eta_n - 2\mu \partial_y^2 u_n + R_n, \qquad y = 0. \tag{1.2.3d}$$

$$u_n(x, 0, 0) = \delta_{n,0} \xi^{(0)}(x), \tag{1.2.3e}$$

$$\eta_n(x, 0) = \delta_{n,0} \eta^{(0)}(x), \tag{1.2.3f}$$

where $\delta_{n,k}$ is the Kronecker delta function. In these F_n, Q_n, and R_n can be shown, using the notation $[\![\cdot]\!]_n$ from Appendix B, to be

$$\begin{aligned} h^2 F_n = & -\partial_x \left[2h [\![\eta \partial_x u]\!]_n\right] + \partial_x \left[h(y+h) [\![(\partial_x \eta) \partial_y u]\!]_n\right] + \partial_y \left[h(y+h) [\![(\partial_x \eta) \partial_x u]\!]_n\right] \\ & - \partial_x \left[[\![\eta^2 \partial_x u]\!]_n\right] + \partial_x \left[(y+h) [\![\eta (\partial_x \eta) \partial_y u]\!]_n\right] + \partial_y \left[(y+h) [\![\eta (\partial_x \eta) \partial_x u]\!]_n\right] \\ & - \partial_y \left[(y+h)^2 [\![(\partial_x \eta)^2 \partial_y u]\!]_n\right] + h [\![(\partial_x \eta) \partial_x u]\!]_n \\ & + [\![\eta (\partial_x \eta) \partial_x u]\!]_n - (y+h) [\![(\partial_x \eta)^2 \partial_y u]\!]_n , \end{aligned} \tag{1.2.4a}$$

$$hQ_n = - [\![\eta(\partial_t \eta)]\!]_n + 2\mu [\![\eta(\partial_x^2 \eta)]\!]_n - h [\![(\partial_x \eta) \partial_x u]\!]_n - [\![\eta (\partial_x \eta) \partial_x u]\!]_n + h [\![(\partial_x \eta)^2 \partial_y u]\!]_n , \tag{1.2.4b}$$

and

$$
\begin{aligned}
h^2 R_n = &-2h \left[\!\left[\eta \partial_t u \right]\!\right]_n - \left[\!\left[\eta^2 \partial_t u \right]\!\right]_n + h^2 \left[\!\left[(\partial_t \eta) \partial_y u \right]\!\right]_n + h \left[\!\left[\eta (\partial_t \eta) \partial_y u \right]\!\right]_n \\
&- 2gh \left[\!\left[\eta^2 \right]\!\right]_n - g \left[\!\left[\eta^3 \right]\!\right]_n + 2\sigma h \left[\!\left[\eta (\partial_x^2 \eta) \right]\!\right]_n + \sigma \left[\!\left[\eta^2 (\partial_x^2 \eta) \right]\!\right]_n \\
&+ \sigma h^2 \left[\!\left[\partial_x \left[(\partial_x \eta) H(\partial_x \eta) \right] \right]\!\right]_n + 2\sigma h \left[\!\left[\eta \partial_x \left[(\partial_x \eta) H(\partial_x \eta) \right] \right]\!\right]_n + \sigma \left[\!\left[\eta^2 \partial_x \left[(\partial_x \eta) H(\partial_x \eta) \right] \right]\!\right]_n \\
&- \frac{1}{2} \left\{ h^2 \left[\!\left[(\partial_x u)^2 \right]\!\right]_n + 2h \left[\!\left[\eta (\partial_x u)^2 \right]\!\right]_n + \left[\!\left[\eta^2 (\partial_x u)^2 \right]\!\right]_n - 2h^2 \left[\!\left[(\partial_x \eta)(\partial_x u) \partial_y u \right]\!\right]_n \right. \\
&\left. -2h \left[\!\left[\eta (\partial_x \eta)(\partial_x u) \partial_y u \right]\!\right]_n + h^2 \left[\!\left[(\partial_x \eta)^2 (\partial_y u)^2 \right]\!\right]_n \right\} - \frac{1}{2} h^2 \left[\!\left[(\partial_y u)^2 \right]\!\right]_n . \qquad (1.2.4c)
\end{aligned}
$$

Remark 1.2.1. For the expansion of H we recall that

$$
H(\psi) = \frac{1}{\sqrt{1+\psi^2}} - 1,
$$

which, upon squaring, can be written as

$$
(H^2 + 2H + 1)(1 + \psi^2) = 1,
$$

or

$$
H = -\frac{1}{2} \left\{ \psi^2 + H^2 + 2\psi^2 H + \psi^2 H^2 \right\} . \qquad (1.2.5)
$$

If we make the expansion, c.f. (Equation 1.2.2),

$$
\psi = \psi(x, t; \varepsilon) = \sum_{n=1}^{\infty} \psi_n(x, t) \varepsilon^n,
$$

then it is easy to see that

$$H = H(x,t;\varepsilon) = \sum_{n=2}^{\infty} H_n \varepsilon^n.$$

In fact, from (Equation 1.2.5) we have

$$H_n = -\frac{1}{2}\left\{ \left[\!\left[\psi^2\right]\!\right]_n + \left[\!\left[H^2\right]\!\right]_n + 2\left[\!\left[\psi^2 H\right]\!\right]_n + \left[\!\left[\psi^2 H^2\right]\!\right]_n \right\}, \tag{1.2.6}$$

and it is clear that H_n depends only on $\{\psi_1, \ldots, \psi_{n-1}\} = \{\partial_x \eta_1, \ldots, \partial_x \eta_{n-1}\}$.

CHAPTER 2

WELL-POSEDNESS

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2.1 Introduction

In this chapter, we present our well-posedness theorem for the above system of equations. Before proceeding, we point out that our method of proof is rather different from the standard techniques, e.g., described in (57; 8). Rather than seeking a fixed point of a contraction mapping, we follow the approach of Friedman and Reitich to free boundary problems, more specifically in the contexts of the classical Stefan problem (39) and the capillary drop problem (42). Friedman and Reitich's method is perturbative in nature, expanding the solution in a Taylor series in a parameter which characterizes the deformation of the free interface from a simple, separable geometry. Their proof uses, very strongly, the unique solvability of the governing equations on a *fixed*, *trivial* domain (using separation of variables) to show that higher order corrections satisfy appropriate bounds which demonstrate the strong convergence of the Taylor series for the solution. The difficulties here are certain algebra properties of the relevant function spaces and trace lemmas; these are different in the current context, but we show that their demonstrations can be extended.

Additionally, we point out that due to the nature of the function spaces we introduce, the conclusion of our theorem is not only the well–posedness of our model of viscous water waves, but also the very strong stability of our solutions. Our function spaces demand *exponential* decay in time with the rate determined by the value of the viscosity. Thus, not only do unique solutions exist, they persist globally in time and decay exponentially fast to zero.

The rest of this chapter is organized as follows. In § 2.2 we introduce the function spaces we require to establish our well–posedness result, together with crucial lemmas on algebra properties of functions in these spaces, and trace estimates on the same. In § 2.3 and 2.4 we state and prove fundamental estimates on the elliptic and parabolic problems which arise in the linearization of our governing equations about the trivial configuration which we analyze in § 2.5. In § 2.6 we state and prove an inductive lemma which enables the proof of our central well–posedness result which is established in § 2.7. We make concluding remarks in § 2.8. We collect the proofs of the trace lemma in § A and a lemma on products of analytic functions in § B.

2.2 Function Spaces

Following Friedman and Reitich (39; 42) we define, for the function $g = g(t)$, the norm

$$[g]_t^2 := \int_0^t e^{2\alpha u} |g(u)|^2 \, du,$$

for some $\alpha > 0$, and recall, for the function $f = f(x)$, the classical Sobolev norm (54; 2; 32), for real $s \geq 0$,

$$\|f\|^2_{H^s([0,2\pi])} := \sum_{p=-\infty}^{\infty} \langle p \rangle^{2s} \left| \hat{f}_p \right|^2, \quad \langle p \rangle^2 := 1 + |p|^2, \quad \hat{f}_p := \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-ipx} \, dx.$$

With these, for the function $U = U(x,t)$, we define, for real $s \geq 4$,

$$\begin{aligned} \|U\|^2_{X^s} &:= \left[\|U\|^2_{H^s([0,2\pi])} \right]^2_\infty + \left[\|\partial_t U\|^2_{H^{s-2}([0,2\pi])} \right]^2_\infty + \left[\|\partial_t^2 U\|^2_{H^{s-4}([0,2\pi])} \right]^2_\infty \\ &= \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2s} \left| \hat{U}_p(u) \right|^2 \, du + \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2(s-2)} \left| \partial_t \hat{U}_p(u) \right|^2 \, du \\ &\quad + \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2(s-4)} \left| \partial_t^2 \hat{U}_p(u) \right|^2 \, du, \end{aligned}$$

where the second time derivative is required for the algebra property (Lemma 2.2.1 below) to be valid, c.f. (39; 42).

In addition, in the next section we require volumetric norms. For the function $v = v(x,y)$, the classical Sobolev norm is (54; 2; 32), for integer $s \geq 0$,

$$\|v\|^2_{H^s([0,2\pi]\times[-h,0])} := \sum_{\ell=0}^{s} \sum_{p=-\infty}^{\infty} \langle p \rangle^{2(s-\ell)} \int_{-h}^0 \left| \partial_y^\ell \hat{v}_p(y) \right|^2 \, dy.$$

Finally, for the function $w = w(x, y, t)$, we define the following norm for integer $s \geq 4$,

$$\begin{aligned}
|||w|||_{V^s}^2 &:= \Big[\|w\|_{H^s([0,2\pi]\times[-h,0])}\Big]_\infty^2 + \Big[\|\partial_t w\|_{H^{s-2}([0,2\pi]\times[-h,0])}\Big]_\infty^2 \\
&\quad + \Big[\big\|\partial_t^2 w\big\|_{H^{s-4}([0,2\pi]\times[-h,0])}\Big]_\infty^2 \\
&= \int_0^\infty e^{2\alpha u} \sum_{\ell=0}^{s} \sum_{p=-\infty}^{\infty} \langle p\rangle^{2(s-\ell)} \int_{-h}^0 \Big|\partial_y^\ell \hat{w}_p(y,u)\Big|^2 \; dy\; du \\
&\quad + \int_0^\infty e^{2\alpha u} \sum_{\ell=0}^{s-2} \sum_{p=-\infty}^{\infty} \langle p\rangle^{2(s-2-\ell)} \int_{-h}^0 \Big|\partial_y^\ell \partial_t \hat{w}_p(y,u)\Big|^2 \; dy\; du \\
&\quad + \int_0^\infty e^{2\alpha u} \sum_{\ell=0}^{s-4} \sum_{p=-\infty}^{\infty} \langle p\rangle^{2(s-4-\ell)} \int_{-h}^0 \Big|\partial_y^\ell \partial_t^2 \hat{w}_p(y,u)\Big|^2 \; dy\; du.
\end{aligned}$$

With these norms we define the function spaces, for real $s \geq 0$,

$$H^s([0,2\pi]) := \Big\{ f(x) \in L^2([0,2\pi]) \;\Big|\; \int_0^{2\pi} f(x)\; dx = 0, \|f\|_{H^s([0,2\pi])} < \infty \Big\},$$

and, for real $s \geq 4$,

$$X^s([0,2\pi]\times[0,\infty)) := \Big\{ U(x,t) \in L^2([0,2\pi]\times[0,\infty)) \;\Big|\; \int_0^{2\pi} U(x,t)\; dx = 0, \|U\|_{X^s} < \infty \Big\},$$

and, for integer $s \geq 0$,

$$H^s([0,2\pi]\times[-h,0]) := \Big\{ v(x,y) \in L^2([0,2\pi]\times[-h,0]) \;\Big|\; \|v\|_{H^s([0,2\pi]\times[-h,0])} < \infty \Big\},$$

and, for integer $s \geq 4$,

$$V^s([0,2\pi] \times [-h,0] \times [0,\infty)) := \left\{ w(x,y,t) \in L^2([0,2\pi] \times [-h,0] \times [0,\infty)) \;\middle|\; |||w|||_{V^s} < \infty \right\}.$$

From this point we suppress the domain dependence unless there is danger of confusion.

Of fundamental importance to our proof are the following algebra properties which are straightforward generalizations of Friedman and Reitich's Theorem A.4 in (39).

Lemma 2.2.1. *If $s \geq 4$; $f, g \in X^s$; $v, w \in V^s$; then there is a constant $M > 0$ such that*

$$\|fg\|_{X^s} \leq M \|f\|_{X^s} \|g\|_{X^s}, \tag{2.2.1a}$$

$$|||fv|||_{V^s} \leq M \|f\|_{X^s} |||v|||_{V^s}, \tag{2.2.1b}$$

$$|||vw|||_{V^s} \leq M |||v|||_{V^s} |||w|||_{V^s}. \tag{2.2.1c}$$

Remark 2.2.2. While the lemma above is true for any real $s \geq 4$ by interpolation (39), we will only utilize (Equation 2.2.1b) and (Equation 2.2.1c) for integer s, and (Equation 2.2.1a) for integer s or $s = m + 1/2$ for m integer.

In addition, we require a temporal trace theorem due to Friedman and Reitich (39) (for the proof, see Appendix A) suitably modified to our space X^s.

Lemma 2.2.3. *If $s \geq 4$ and $\sigma(x,t) \in X^s$ then there exists a constant $C_t > 0$ such that*

$$\max\left\{\|\sigma(x,0)\|_{H^{s-1}}, \|\partial_t \sigma(x,0)\|_{H^{s-3}}\right\} \leq C_t \|\sigma\|_{X^s}. \tag{2.2.2}$$

Finally, we recall two auxiliary lemmas from (81).

Lemma 2.2.4. *If $s \geq 4$ and $w \in V^s$ then there exists a constant $Y = Y(s)$ such that*

$$\left\lVert\!\left| (y+h)w \right|\!\right\rVert_{V^s} < Y \left\lVert\!\left| w \right|\!\right\rVert_{V^s}.$$

Lemma 2.2.5. *There exists a universal constant $\Sigma > 0$ such that*

$$\max \left\{ \sum_{m=0}^{N} \frac{(N+1)^2}{(N-m+1)^2(m+1)^2}, \right.$$
$$\sum_{m=0}^{N}\sum_{\ell=0}^{m} \frac{(N+1)^2}{(N-m+1)^2(m-\ell+1)^2(\ell+1)^2},$$
$$\left. \sum_{m=0}^{N}\sum_{\ell=0}^{m}\sum_{q=0}^{\ell} \frac{(N+1)^2}{(N-m+1)^2(m-\ell+1)^2(\ell-q+1)^2(q+1)^2} \right\} < \Sigma.$$

2.3 A Fundamental Lemma for the Elliptic Problem

To prove our well–posedness result, Theorem 2.7.1, we must establish the following elliptic estimate which generalizes the results found in (79) to the spaces V^s and X^s.

Theorem 2.3.1. *Given an integer $s \geq 4$, if $F \in V^{s+1}$ and $\psi \in X^{s+5/2}$, then there exists a unique solution of*

$$\Delta w = F, \qquad -h < y < 0, \tag{2.3.1a}$$

$$w(x,0,t) = \psi(x,t), \tag{2.3.1b}$$

$$\partial_y w(x,-h,t) = 0, \tag{2.3.1c}$$

in V^{s+3} *satisfying*

$$\max\left\{\|w(x,0,t)\|_{X^{s+5/2}}, |||w|||_{V^{s+3}}\right\} \leq K_e \left\{|||F|||_{V^{s+1}} + \|\psi\|_{X^{s+5/2}}\right\}, \tag{2.3.2}$$

where $K_e > 0$ *is a universal constant.*

Proof. In a recent publication a similar result for the Helmholtz equation was shown and we follow those developments here. To begin, we use the lateral periodicity of the solution to express

$$\{w(x,y,t), F(x,y,t), \psi(x,t)\} = \sum_{p=-\infty}^{\infty} \left\{\hat{w}_p(y,t), \hat{F}_p(y,t), \hat{\psi}_p(t)\right\} e^{ipx},$$

and note that (Equation 2.3.1) then demands that

$$\partial_y^2 \hat{w}_p - |p|^2 \hat{w}_p = \hat{F}_p, \qquad -h < y < 0, \tag{2.3.3a}$$

$$\hat{w}_p(0,t) = \hat{\psi}_p(t), \tag{2.3.3b}$$

$$\partial_y \hat{w}_p(-h,t) = 0. \tag{2.3.3c}$$

Existence and Uniqueness: To show the existence and uniqueness of a solution we appeal to the classical results of Keller (52), later extended in the “Integrated Solution Method” of

Zhang (107; 108) (see also (21)). Using the notation of (21) we consider, after a trivial change of variables $y \to y + h$, the problem

$$
\begin{aligned}
&\mathbf{u}'(y) + \mathbf{M}(y)\mathbf{u}(y) = \mathbf{f}(y), && 0 < y < h, \\
&\mathbf{A}_0\mathbf{u} = \mathbf{r}_0, && y = 0, \\
&\mathbf{B}_1\mathbf{u} = \mathbf{s}_1, && y = h,
\end{aligned}
$$

where

$$\mathbf{f}(y) \in \mathbf{C}^m, \quad \mathbf{r}_0 \in \mathbf{C}^{m_1}, \quad \mathbf{s}_1 \in \mathbf{C}^{m_2},$$

are vector fields ($m = m_1 + m_2$). Further,

$$\mathbf{M}(y) \in \mathbf{C}^{m\times m}, \quad \mathbf{A}_0 \in \mathbf{C}^{m_1\times m}, \quad \mathbf{B}_1 \in \mathbf{C}^{m_2\times m},$$

are full rank matrices. Let $\mathbf{\Phi}(y)$ be the fundamental matrix solution of the system

$$\mathbf{\Phi}'(y) + \mathbf{M}(y)\mathbf{\Phi}(y) = 0, \quad \mathbf{\Phi}(0) = I_m,$$

where I_m is the $m \times m$ identity matrix. Keller shows (52) that the two–point value problem above has a unique solution if and only if

$$\det \begin{pmatrix} \mathbf{A}_0 \\ \mathbf{B}_1\mathbf{\Phi}(h) \end{pmatrix} \neq 0.$$

In this instance we have $m = 2$, $m_1 = m_2 = 1$, and

$$\mathbf{u} = \begin{pmatrix} \hat{w}_p \\ \partial_y \hat{w}_p \end{pmatrix}, \quad \mathbf{M}(y) = \begin{pmatrix} 0 & -1 \\ -|p|^2 & 0 \end{pmatrix}, \quad \mathbf{f}(y) = \begin{pmatrix} 0 \\ \hat{F}_p \end{pmatrix},$$

$$\mathbf{A}_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \mathbf{r}_0 = 0, \quad \mathbf{s}_1 = \hat{\psi}_p.$$

There are two cases of p to consider.

1. The case $p = 0$: Here we may show that

$$\mathbf{\Phi}(y) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix},$$

and

$$\det \begin{pmatrix} \mathbf{A}_0 \\ \mathbf{B}_1 \mathbf{\Phi}(h) \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ 1 & h \end{pmatrix} = -1 \neq 0.$$

Thus, a unique solution exists in this case.

2. The case $p \neq 0$: Here one can see that

$$\mathbf{\Phi}(y) = \begin{pmatrix} \cosh(|p|\, y) & \sinh(|p|\, y)/\,|p| \\ |p| \sinh(|p|\, y) & \cosh(|p|\, y) \end{pmatrix},$$

and

$$\det\begin{pmatrix} \mathbf{A}_0 \\ \\ \mathbf{B}_1\mathbf{\Phi}(h) \end{pmatrix} = \det\begin{pmatrix} 0 & 1 \\ \\ \cosh(|p|\,h) & \sinh(|p|\,h)/\,|p| \end{pmatrix} = -\cosh(|p|\,h) \neq 0.$$

Again, a unique solution exists in this case.

We note that existence and uniqueness of solutions can also be verified by simply (but less elegantly) writing down the exact solution as in (79).

Estimates: In order to accommodate Sobolev spaces with very low smoothness, we consider the slightly generalized form of (Equation 2.3.1)

$$\begin{aligned} &\Delta w = \partial_x F^{(1)} + \partial_y F^{(2)} + F^{(3)}, && -h < y < 0, \\ &w(x,0,t) = \psi(x,t), \\ &\partial_y w(x,-h,t) = 0, \end{aligned}$$

which, upon Fourier transform, becomes

$$\begin{aligned} &\partial_y^2 \hat{w}_p - |p|^2\,\hat{w}_p = (ip)\hat{F}_p^{(1)} + \partial_y \hat{F}_p^{(2)} + \hat{F}_p^{(3)}, && -h < y < 0, \\ &\hat{w}_p(0,t) = \hat{\psi}_p(t), \\ &\partial_y \hat{w}_p(-h,t) = 0. \end{aligned}$$

Following the developments of (79), we set

$$\hat{w}_p(y,t) = \hat{w}_p^{(0)}(y,t) + \hat{w}_p^{(1)}(y,t) + \hat{w}_p^{(2)}(y,t) + \hat{w}_p^{(3)}(y,t),$$

where, for $j = 0, 1, 2, 3,$

$$\partial_y^2 \hat{w}_p^{(j)} - |p|^2\, \hat{w}_p^{(j)} = \delta_{j,1}(ip)\hat{F}_p^{(1)} + \delta_{j,2}\partial_y \hat{F}_p^{(2)} + \delta_{j,3}\hat{F}_p^{(3)}, \qquad -h < y < 0, \tag{2.3.4a}$$

$$\hat{w}_p^{(j)}(0,t) = \delta_{j,0}\hat{\psi}_p(t), \tag{2.3.4b}$$

$$\partial_y \hat{w}_p^{(j)}(-h,t) = 0. \tag{2.3.4c}$$

It can be shown (see Lemma A.2 of (75)) from the solution formula for $\hat{w}_p(y,t)$, that the following volumetric estimates hold for the unique solution when $\ell = 0, 1,$

$$\left\| \partial_y^\ell \hat{w}_p^{(0)}(y,t) \right\|_{L^2(dy)}^2 \le K_e \langle p \rangle^{-1+2\ell} \left| \hat{\psi}_p(t) \right|^2, \tag{2.3.5a}$$

$$\left\| \partial_y^\ell \hat{w}_p^{(1)}(y,t) \right\|_{L^2(dy)}^2 \le K_e \langle p \rangle^{-2+2\ell} \left\| \hat{F}_p^{(1)}(y,t) \right\|_{L^2(dy)}^2, \tag{2.3.5b}$$

$$\left\| \partial_y^\ell \hat{w}_p^{(2)}(y,t) \right\|_{L^2(dy)}^2 \le K_e \langle p \rangle^{-2+2\ell} \left\| \hat{F}_p^{(2)}(y,t) \right\|_{L^2(dy)}^2, \tag{2.3.5c}$$

$$\left\| \partial_y^\ell \hat{w}_p^{(3)}(y,t) \right\|_{L^2(dy)}^2 \le K_e \langle p \rangle^{-4+2\ell} \left\| \hat{F}_p^{(3)}(y,t) \right\|_{L^2(dy)}^2, \tag{2.3.5d}$$

for some $K_e > 0$. In addition, the subsequent boundary estimates hold for $\ell = 0, 1$

$$\left|\partial_y^\ell \hat{w}_p^{(0)}(0,t)\right|^2 \leq K_e \langle p\rangle^{2\ell} \left|\hat{\psi}_p(t)\right|^2, \tag{2.3.6a}$$

$$\left|\partial_y^\ell \hat{w}_p^{(1)}(0,t)\right|^2 \leq K_e \langle p\rangle^{-1+2\ell} \left\|\hat{F}_p^{(1)}(y,t)\right\|_{L^2(dy)}^2, \tag{2.3.6b}$$

$$\left|\partial_y^\ell \hat{w}_p^{(2)}(0,t)\right|^2 \leq K_e \langle p\rangle^{-1+2\ell} \left\|\hat{F}_p^{(2)}(y,t)\right\|_{L^2(dy)}^2, \tag{2.3.6c}$$

$$\left|\partial_y^\ell \hat{w}_p^{(3)}(0,t)\right|^2 \leq K_e \langle p\rangle^{-3+2\ell} \left\|\hat{F}_p^{(3)}(y,t)\right\|_{L^2(dy)}^2, \tag{2.3.6d}$$

with $K_e > 0$ sufficiently large. Furthermore, as the governing equations (Equation 2.3.4) treat time as a parameter, by simply applying time derivatives one can achieve the same estimates with $\{\hat{w}_p^{(j)}, \hat{F}_p^{(j)}, \hat{\psi}_p\}$ replaced by $\{\partial_t \hat{w}_p^{(j)}, \partial_t \hat{F}_p^{(j)}, \partial_t \hat{\psi}_p\}$, and $\{\partial_t^2 \hat{w}_p^{(j)}, \partial_t^2 \hat{F}_p^{(j)}, \partial_t^2 \hat{\psi}_p\}$.

Consider the H^1–type norm of w

$$\left[\|w\|_{H^1}^2\right]_\infty^2 = \int_0^\infty e^{2\alpha u} \sum_{\ell=0}^{1} \sum_{p=-\infty}^{\infty} \langle p\rangle^{2(1-\ell)} \left\|\partial_y^\ell \hat{w}_p(y,u)\right\|_{L^2(dy)}^2 \, du,$$

and the H^{-1} analogue for F

$$\left[\|F\|_{H^{-1}}^2\right]_\infty^2 = \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \sum_{j=1}^{3} \left\|\hat{F}_p^{(j)}(y,u)\right\|_{L^2(dy)}^2 \, du, \quad F = \partial_x F^{(1)} + \partial_y F^{(2)} + F^{(3)},$$

for some $F^{(j)} \in L^2([-h,0])$ (see Chapter 5 of Evans (32)). Using the estimates above (and being a little wasteful in our estimate of $F^{(3)}$) we have

$$\begin{aligned}
\left[\|w\|_{H^1}^2\right]_\infty^2 &\leq \sum_{j=0}^{3} \left[\left\|w^{(j)}\right\|_{H^1}^2\right]_\infty^2 \\
&\leq \int_0^\infty e^{2\alpha u} \sum_{\ell=0}^{1} \sum_{p=-\infty}^{\infty} \langle p\rangle^{2(1-\ell)} K_e \left\{ \langle p\rangle^{-1+2\ell} \left|\hat{\psi}_p(u)\right|^2 \right. \\
&\qquad \left. + \sum_{j=1}^{3} \langle p\rangle^{-2+2\ell} \left\|\hat{F}_p^{(j)}(y,u)\right\|_{L^2(dy)}^2 \right\} du.
\end{aligned}$$

Rearranging this we find

$$\begin{aligned}
\left[\|w\|_{H^1}^2\right]_\infty^2 &\leq K_e \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \langle p\rangle^{1} \left|\hat{\psi}_p(u)\right|^2 du \\
&\qquad + K_e \int_0^\infty e^{2\alpha u} \sum_{p=-\infty}^{\infty} \sum_{j=1}^{3} \left\|\hat{F}_p^{(j)}(y,u)\right\|_{L^2(dy)}^2 du \\
&\leq K_e \left\{ \|\psi\|_{H^{1/2}}^2 + \|F\|_{H^{-1}}^2 \right\}.
\end{aligned}$$

Either by conducting the tedious manipulations to produce the higher–order analogues of (Equation 2.3.5) and (Equation 2.3.6), or by proceeding as in Chapter 6 of Evans (32), we can deduce

$$\left[\|w\|_{H^{s+3}}^2\right]_\infty^2 \leq CK_e \left\{ \left[\|F\|_{H^{s+1}}^2\right]_\infty^2 + \left[\|\psi\|_{H^{s+5/2}}^2\right]_\infty^2 \right\}.$$

Applying this estimate to $\partial_t w$ and $\partial_t^2 w$, and recalling that $s \geq 4$, we discover

$$\left[\|\partial_t w\|_{H^{s+1}}^2\right]_\infty^2 \leq CK_e \left\{ \left[\|\partial_t F\|_{H^{s-1}}^2\right]_\infty^2 + \left[\|\partial_t \psi\|_{H^{s+1/2}}^2\right]_\infty^2 \right\},$$

and

$$\left[\left\|\partial_t^2 w\right\|_{H^{s-1}}^2\right]_\infty^2 \leq C K_e \left\{ \left[\left\|\partial_t^2 F\right\|_{H^{s-3}}^2\right]_\infty^2 + \left[\left\|\partial_t^2 \psi\right\|_{H^{s-1/2}}^2\right]_\infty^2 \right\},$$

which, upon summation, delivers the conclusion of the theorem. □

2.4 A Fundamental Lemma for the Parabolic Problem

To state our next result we recall the definition of an order–k Fourier multiplier.

Definition 2.4.1. Suppose that $\psi \in L^2([0, 2\pi])$ then the equation

$$m(D)\psi(x) := \sum_{p=-\infty}^{\infty} m(p)\hat{\psi}_p e^{ipx},$$

defines the Fourier multiplier $m(D)$. If, for some $k \in \mathbf{R}$, we have for any s real

$$\|m(D)\psi\|_{H^s} \leq C \|\psi\|_{H^{s+k}},$$

then we say that $m(D)$ is order–k.

Remark 2.4.2. The classical derivative, ∂_x, is clearly an order–one Fourier multiplier with symbol (iD). Of relevance to the current contribution are the order–one multiplier

$$G_0 := |D| \tanh(h |D|),$$

which is the flat–interface DNO for Laplace's equation on a strip (29; 28; 64), the order–three–halves $(i\omega_D)$ operator

$$(i\omega_D)\psi(x) = \sum_{p=-\infty}^{\infty} (i\omega_p)\hat{\psi}_p e^{i\alpha_p x} := \sum_{p=-\infty}^{\infty} i\sqrt{(g + \sigma\,|p|^2)\left|\hat{G}_{0,p}\right|}\,\hat{\psi}_p e^{i\alpha_p x},$$

which comes from the dispersion relation for water waves (55; 99; 1), and the order–two operator $(-\,|D|^2) = \partial_x^2$.

We require the following parabolic estimate for our inductive proof to proceed.

Theorem 2.4.3. *Given a real number $s \geq 4$, if $Q \in X^{s+1}$, $R \in X^{s+1/2}$, $\eta^{(0)} \in H^{s+2}$, and $\xi^{(0)} \in H^{s+3/2}$ then there exists a unique solution of*

$$\partial_t \eta = G_0[\xi] + 2\mu\partial_x^2\eta + Q, \qquad \eta(x,0) = \eta^{(0)}(x), \tag{2.4.1a}$$

$$\partial_t \xi = -g\eta + \sigma\partial_x^2\eta - 2\mu\,|D|^2\,\xi + R, \qquad \xi(x,0) = \xi^{(0)}(x), \tag{2.4.1b}$$

satisfying

$$\max\left\{\|\eta\|_{X^{s+3}}, \|\partial_t\eta\|_{X^{s+1}}, \|\xi\|_{X^{s+5/2}}, \|\partial_t\xi\|_{X^{s+1/2}}\right\} \\ \leq K_p\left\{\|Q\|_{X^{s+1}} + \|R\|_{X^{s+1/2}} + \left\|\eta^{(0)}\right\|_{H^{s+2}} + \left\|\xi^{(0)}\right\|_{H^{s+3/2}}\right\}, \tag{2.4.2}$$

where $K_p > 0$ is a universal constant.

To establish this we require the following result from Friedman and Reitich (42) (Lemma 7.1).

Lemma 2.4.4. *Consider the initial value problem*

$$\dot{B}(t) + (K + iM)B(t) = F(t), \qquad t > 0,$$

$$B(0) = B_0,$$

where $K, M \in \boldsymbol{R}$; $K > 0$; *and* $F \in L^2(0,T)$ *for any* $T > 0$. *If* $0 < \alpha < K$ *then the following inequalities hold*

$$[B]_t^2 \leq \frac{2}{(K-\alpha)^2}\,[F]_t^2 + \frac{|B_0|^2}{K-\alpha}, \tag{2.4.3a}$$

$$\left[\dot{B}\right]_t^2 \leq 2\left(\frac{2K^2}{(K-\alpha)^2} + 1\right)[F]_t^2 + \frac{2K^2}{K-\alpha}\,|B_0|^2\,. \tag{2.4.3b}$$

To prove Theorem 2.4.3 we establish a similar result for a decoupled version of (Equation 2.4.1).

Theorem 2.4.5. *Given a real number* $s \geq 4$, *if* $W \in X^s$, $f \in H^{s+1}$ *then there exists a unique solution of*

$$\partial_t U = \left[-2\mu\,|D|^2 \pm (i\omega_D)\right] U + W, \quad U(x,0) = f(x), \tag{2.4.4}$$

satisfying

$$\max\left\{\|U\|_{X^{s+2}}, \|\partial_t U\|_{X^s}\right\} \leq \tilde{K}_p\left\{\|W\|_{X^s} + \|f\|_{H^{s+1}}\right\}, \tag{2.4.5}$$

where $\tilde{K}_p > 0$ *is a universal constant.*

Proof. We focus on the case of (Equation 2.4.4) with the minus sign in front of $(i\omega_D)$; the other case is handled similarly. We expand W and f in Fourier series

$$W(x,t) = \sum_{p=-\infty}^{\infty} \hat{W}_p(t)e^{ipx}, \quad f(x) = \sum_{p=-\infty}^{\infty} \hat{f}_p e^{ipx},$$

where $\hat{W}_0(t) \equiv 0$ and $\hat{f}_0 = 0$ by the definitions of X^s and H^s, respectively, and seek a solution

$$U(x,t) = \sum_{p=-\infty}^{\infty} \hat{U}_p(t)e^{ipx}.$$

Inserting these into (Equation 2.4.4) we find

$$\partial_t \hat{U}_p(t) = -\Omega(p)\hat{U}_p(t) + \hat{W}_p(t), \tag{2.4.6a}$$

$$\hat{U}_p(0) = \hat{f}_p, \tag{2.4.6b}$$

where

$$\Omega(p) := \left(2\mu\,|p|^2 + i\omega_p\right) = \mathcal{O}(p^2), \quad p \to \infty.$$

It is clear that $\hat{U}_0(t) \equiv 0$ is the unique solution in the case $p = 0$ so we now concentrate on $p \neq 0$. Using Lemma 2.4.4 we find

$$\left[\hat{U}_p\right]_\infty^2 \leq C_{0,W}(p)\left[\hat{W}_p\right]_\infty^2 + C_{0,f}(p)\left|\hat{f}_p\right|^2, \tag{2.4.7a}$$

$$\left[\partial_t\hat{U}_p\right]_\infty^2 \leq C_{1,W}(p)\left[\hat{W}_p\right]_\infty^2 + C_{1,f}(p)\left|\hat{f}_p\right|^2, \tag{2.4.7b}$$

with, since $p \neq 0$,

$$K = K_p := 2\mu |p|^2 > 0, \quad 0 < \alpha < 2\mu |1|^2 \leq \min_{p\neq 0} K_p,$$

for some α. In these

$$\begin{aligned}
C_{0,W}(p) &:= \frac{2}{(2\mu |p|^2 - \alpha)^2} = \mathcal{O}(p^{-4}), && p \to \infty \\
C_{0,f}(p) &:= \frac{1}{(2\mu |p|^2 - \alpha)} = \mathcal{O}(p^{-2}), && p \to \infty \\
C_{1,W}(p) &:= 2\left[\frac{2(2\mu |p|^2)^2}{(2\mu |p|^2 - \alpha)^2} + 1\right] = \mathcal{O}(1), && p \to \infty \\
C_{1,f}(p) &:= \frac{2(2\mu |p|^2)^2}{(2\mu |p|^2 - \alpha)} = \mathcal{O}(p^2), && p \to \infty.
\end{aligned}$$

Differentiating (Equation 2.4.6a) once with respect to t yields

$$\partial_t(\partial_t \hat{U}_p)(t) = -\Omega(p)(\partial_t \hat{U}_p)(t) + \partial_t \hat{W}_p(t), \tag{2.4.8a}$$

$$\partial_t \hat{U}_p(0) = -\Omega(p)\hat{f}_p + \hat{W}_p(0), \tag{2.4.8b}$$

where we have used (Equation 2.4.6a) and (Equation 2.4.6b) for the initial condition. Again, appealing to Lemma 2.4.4 we find

$$\left[\partial_t \hat{U}_p\right]_\infty^2 \leq C_{0,W}(p)\left[\partial_t \hat{W}_p\right]_\infty^2 + C_{0,f}(p)\,|\Omega(p)|^2 \left|\hat{f}_p\right|^2 + C_{0,f}(p)\left|\hat{W}_p(0)\right|^2, \tag{2.4.9a}$$

$$\left[\partial_t^2 \hat{U}_p\right]_\infty^2 \leq C_{1,W}(p)\left[\partial_t \hat{W}_p\right]_\infty^2 + C_{1,f}(p)\,|\Omega(p)|^2 \left|\hat{f}_p\right|^2 + C_{1,f}(p)\left|\hat{W}_p(0)\right|^2. \tag{2.4.9b}$$

Differentiating (Equation 2.4.8a) once with respect to t yields

$$\partial_t(\partial_t^2 \hat{U}_p)(t) = -\Omega(p)(\partial_t^2 \hat{U}_p)(t) + \partial_t^2 \hat{W}_p, \tag{2.4.10a}$$

$$\partial_t^2 \hat{U}_p(0) = \Omega(p)^2 \hat{f}_p - \Omega(p)\hat{W}_p(0) + \partial_t \hat{W}_p(0), \tag{2.4.10b}$$

where we have used (Equation 2.4.8a) and (Equation 2.4.8b) for the initial condition. Appealing to Lemma 2.4.4 as before we find

$$\begin{aligned}
\left[\partial_t^2 \hat{U}_p\right]_\infty^2 &\leq C_{0,W}(p) \left[\partial_t^2 \hat{W}_p\right]_\infty^2 + C_{0,f}(p) \left|\Omega(p)\right|^4 \left|\hat{f}_p\right|^2 \\
&\quad + C_{0,f}(p) \left|\Omega(p)\right|^2 \left|\hat{W}_p(0)\right|^2 + C_{0,f}(p) \left|\partial_t \hat{W}_p(0)\right|^2, \qquad (2.4.11a) \\
\left[\partial_t^3 \hat{U}_p\right]_\infty^2 &\leq C_{1,W}(p) \left[\partial_t^2 \hat{W}_p\right]_\infty^2 + C_{1,f}(p) \left|\Omega(p)\right|^4 \left|\hat{f}_p\right|^2 \\
&\quad + C_{1,f}(p) \left|\Omega(p)\right|^2 \left|\hat{W}_p(0)\right|^2 + C_{1,f}(p) \left|\partial_t \hat{W}_p(0)\right|^2. \qquad (2.4.11b)
\end{aligned}$$

If we multiply (Equation 2.4.7a) by $\langle p\rangle^{2s+4}$, (Equation 2.4.9a) by $\langle p\rangle^{2s}$, and (Equation 2.4.11a) by $\langle p\rangle^{2s-4}$ and sum over p we find

$$\begin{aligned}
\|U\|_{X^{s+2}} &= \sum_{p=-\infty}^{\infty} \left[\langle p\rangle^{2s+4} \left[\hat{U}_p\right]_\infty^2 + \langle p\rangle^{2s} \left[\partial_t \hat{U}_p\right]_\infty^2 + \langle p\rangle^{2s-4} \left[\partial_t^2 \hat{U}_p\right]_\infty^2\right] \\
&\leq \sum_{p=-\infty}^{\infty} \Bigg[C_{0,W}(p) \left\{\langle p\rangle^{2s+4} \left[\hat{W}_p\right]_\infty^2 + \langle p\rangle^{2s} \left[\partial_t \hat{W}_p\right]_\infty^2 + \langle p\rangle^{2s-4} \left[\partial_t^2 \hat{W}_p\right]_\infty^2\right\} \\
&\quad + C_{0,f}(p) \Big\{\langle p\rangle^{2s+4} \left|\hat{f}_p\right|^2 + \langle p\rangle^{2s} \left|\Omega(p)\right|^2 \left|\hat{f}_p\right|^2 + \langle p\rangle^{2s} \left|\hat{W}_p(0)\right|^2 \\
&\quad + \langle p\rangle^{2s-4} \left|\Omega(p)\right|^4 \left|\hat{f}_p\right|^2 + \langle p\rangle^{2s-4} \left|\Omega(p)\right|^2 \left|\hat{W}_p(0)\right|^2 + \langle p\rangle^{2s-4} \left|\partial_t \hat{W}_p(0)\right|^2\Big\}\Bigg].
\end{aligned}$$

From this we easily find that

$$\|U\|_{X^{s+2}} \leq K_0 \left[\|W\|_{X^s} + \|f\|_{H^{s+1}} + \|W(\cdot,0)\|_{H^{s-1}} + \|\partial_t W(\cdot,0)\|_{H^{s-3}}\right]. \tag{2.4.12}$$

In a similar way, if we multiply (Equation 2.4.7b) by $\langle p\rangle^{2s}$, (Equation 2.4.9b) by $\langle p\rangle^{2s-4}$, and (Equation 2.4.11b) by $\langle p\rangle^{2s-8}$ and sum over p we find

$$\|\partial_t U\|_{X^s} \leq K_0 \left[\|W\|_{X^s} + \|f\|_{H^{s+1}} + \|W(\cdot,0)\|_{H^{s-1}} + \|\partial_t W(\cdot,0)\|_{H^{s-3}}\right]. \tag{2.4.13}$$

We now appeal to Lemma 2.2.3 (which requires $s \geq 4$) and use this and estimate (Equation 2.4.12) to deliver

$$\|U\|_{X^{s+2}} \leq \tilde{K}_p \left[\|W\|_{X^s} + \|f\|_{H^{s+1}}\right],$$

and (Equation 2.4.13) to give

$$\|\partial_t U\|_{X^s} \leq \tilde{K}_p \left[\|W\|_{X^s} + \|f\|_{H^{s+1}}\right],$$

for some $\tilde{K}_p > 0$ and we are done. $\square$

Proof. (Theorem 2.4.3) To establish this result we express our initial value problem, (Equation 2.4.1), on the Fourier side by using

$$\eta(x,t) = \sum_{p=-\infty}^{\infty} \hat{\eta}_p(t) e^{ipx}, \quad \xi(x,t) = \sum_{p=-\infty}^{\infty} \hat{\xi}_p(t) e^{ipx},$$

which, upon insertion into (Equation 2.4.1), delivers

$$\partial_t \hat{\eta}_p = \hat{G}_{0,p}\hat{\xi}_p - 2\mu\,|p|^2\,\hat{\eta}_p + \hat{Q}_p, \qquad \hat{\eta}_p(0) = \widehat{\eta^{(0)}}_p, \tag{2.4.14a}$$

$$\partial_t \hat{\xi}_p = -(g + \sigma\,|p|^2)\hat{\eta}_p - 2\mu\,|p|^2\,\hat{\xi}_p + \hat{R}_p, \qquad \hat{\xi}_p(0) = \widehat{\xi^{(0)}}_p, \tag{2.4.14b}$$

or

$$\partial_t \begin{pmatrix} \hat{\eta}_p \\ \hat{\xi}_p \end{pmatrix} = \begin{pmatrix} -2\mu\,|p|^2 & \hat{G}_{0,p} \\ -(g+\sigma\,|p|^2) & -2\mu\,|p|^2 \end{pmatrix} \begin{pmatrix} \hat{\eta}_p \\ \hat{\xi}_p \end{pmatrix} + \begin{pmatrix} \hat{Q}_p \\ \hat{R}_p \end{pmatrix}.$$

If we use $\omega_p = \sqrt{(g + \sigma\,|p|^2)\hat{G}_{0,p}}$, define

$$\hat{P}_p := \begin{pmatrix} -i\omega_p & i\omega_p \\ g+\sigma\,|p|^2 & g+\sigma\,|p|^2 \end{pmatrix}, \quad \hat{P}_p^{-1} = \frac{1}{2}\begin{pmatrix} -1/(i\omega_p) & 1/(g+\sigma\,|p|^2) \\ 1/(i\omega_p) & 1/(g+\sigma\,|p|^2) \end{pmatrix},$$

and make the change of variables

$$\begin{pmatrix} \hat{\zeta}_p \\ \hat{\chi}_p \end{pmatrix} := \hat{P}_p^{-1} \begin{pmatrix} \hat{\eta}_p \\ \hat{\xi}_p \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\hat{\eta}_p/(i\omega_p) + \hat{\xi}_p/(g+\sigma\,|p|^2) \\ \hat{\eta}_p/(i\omega_p) + \hat{\xi}_p/(g+\sigma\,|p|^2) \end{pmatrix},$$

we find

$$\partial_t \hat{\zeta}_p = \left(-2\mu\,|p|^2 + i\omega_p\right)\hat{\zeta}_p + \hat{V}_p$$

$$\partial_t \hat{\chi}_p = \left(-2\mu\,|p|^2 - i\omega_p\right)\hat{\chi}_p + \hat{W}_p,$$

where

$$\begin{pmatrix} \hat{V}_p \\ \\ \hat{W}_p \end{pmatrix} := \hat{P}_p^{-1} \begin{pmatrix} \hat{Q}_p \\ \\ \hat{R}_p \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\hat{Q}_p/(i\omega_p) + \hat{R}_p/(g + \sigma\,|p|^2) \\ \\ \hat{Q}_p/(i\omega_p) + \hat{R}_p/(g + \sigma\,|p|^2) \end{pmatrix}.$$

We note that

$$\eta = -(i\omega_D)\zeta + (i\omega_D)\chi,$$

$$\xi = (g - \sigma\partial_x^2)\zeta + (g - \sigma\partial_x^2)\chi,$$

so

$$\max\left\{\|\eta\|_{X^{s+3}}, \|\partial_t\eta\|_{X^{s+1}}, \|\xi\|_{X^{s+5/2}}, \|\partial_t\xi\|_{X^{s+1/2}}, \right\}$$
$$\leq C\left\{\|\zeta\|_{X^{s+9/2}} + \|\partial_t\zeta\|_{X^{s+5/2}} + \|\chi\|_{X^{s+9/2}} + \|\partial_t\chi\|_{X^{s+5/2}}\right\}.$$

From Theorem 2.4.5 we have

$$\max\left\{\|\eta\|_{X^{s+3}}, \|\partial_t\eta\|_{X^{s+1}}, \|\xi\|_{X^{s+5/2}}, \|\partial_t\xi\|_{X^{s+1/2}}\right\}$$
$$\leq C\tilde{K}_P\left\{\|V\|_{X^{s+5/2}} + \left\|\zeta^{(0)}\right\|_{H^{s+7/2}} + \|W\|_{X^{s+5/2}} + \left\|\chi^{(0)}\right\|_{H^{s+7/2}}\right\}.$$

Now, since

$$\begin{aligned}
V &= \frac{1}{2}\left\{(i\omega_D)^{-1}Q + (g-\sigma\partial_x^2)^{-1}R\right\}, \\
W &= \frac{1}{2}\left\{(i\omega_D)^{-1}Q + (g-\sigma\partial_x^2)^{-1}R\right\}, \\
\zeta^{(0)} &= \frac{1}{2}\left\{(i\omega_D)^{-1}\eta^{(0)} + (g-\sigma\partial_x^2)^{-1}\xi^{(0)}\right\}, \\
\chi^{(0)} &= \frac{1}{2}\left\{(i\omega_D)^{-1}\eta^{(0)} + (g-\sigma\partial_x^2)^{-1}\xi^{(0)}\right\},
\end{aligned}$$

we have

$$\begin{aligned}
&\max\left\{\|\eta\|_{X^{s+3}}, \|\partial_t\eta\|_{X^{s+1}}, \|\xi\|_{X^{s+5/2}}, \|\xi\|_{X^{s+1/2}}\right\} \\
&\qquad\qquad \le C\tilde{K}_P\tilde{C}\left\{\|Q\|_{X^{s+1}} + \|R\|_{X^{s+1/2}} + \left\|\eta^{(0)}\right\|_{H^{s+2}} + \left\|\xi^{(0)}\right\|_{H^{s+3/2}}\right\},
\end{aligned}$$

and we are done if we choose $K_P = C\tilde{K}_P\tilde{C}$. □

2.5 A Fundamental Lemma for the Linearized Water Wave Problem with Viscosity

We require the following estimate of the linearization of water wave problem (Equation 1.2.1) in order to proceed.

Lemma 2.5.1. *Given an integer $s \geq 4$, if $F \in V^{s+1}$, $Q \in X^{s+1}$, $R \in X^{s+1/2}$, $\eta^{(0)} \in H^{s+2}$, and $\xi^{(0)} \in H^{s+3/2}$ then there exists a unique solution of*

$$\Delta u = F, \qquad -h < y < 0, \tag{2.5.1a}$$

$$\partial_y u = 0, \qquad y = -h, \tag{2.5.1b}$$

$$\partial_t \eta = \partial_y u + 2\mu\partial_x^2\eta + Q, \qquad y = 0, \tag{2.5.1c}$$

$$\partial_t u = -g\eta + \sigma\partial_x^2\eta - 2\mu\partial_y^2 u + R, \qquad y = 0, \tag{2.5.1d}$$

$$u(x,0,0) = \xi^{(0)}(x), \tag{2.5.1e}$$

$$\eta(x,0) = \eta^{(0)}(x), \tag{2.5.1f}$$

satisfying

$$\begin{aligned} &\max\left\{\|\eta\|_{X^{s+3}}, \|\partial_t\eta\|_{X^{s+1}}, \|u(x,0,t)\|_{X^{s+5/2}}, \|\partial_t u(x,0,t)\|_{X^{s+1/2}}, \|\!|u|\!\|_{V^{s+3}}\right\} \\ &\qquad \leq K\left\{\|\!|F|\!\|_{V^{s+1}} + \|Q\|_{X^{s+1}} + \|R\|_{X^{s+1/2}} + \left\|\eta^{(0)}\right\|_{H^{s+2}} + \left\|\xi^{(0)}\right\|_{H^{s+3/2}}\right\}, \end{aligned} \tag{2.5.2}$$

for a universal constant $K > 0$.

Proof. Using the periodicity of solutions we write

$$\{u, F\} = \{u, F\}(x, y, t) = \sum_{p=-\infty}^{\infty} \{\hat{u}_p, \hat{F}_p\}(y, t)e^{ipx},$$
$$\{\eta, Q, R\} = \{\eta, Q, R\}(x, t) = \sum_{p=-\infty}^{\infty} \{\hat{\eta}_p, \hat{Q}_p, \hat{R}_p\}(t)e^{ipx},$$
$$\{\eta^{(0)}, \xi^{(0)}\} = \{\eta^{(0)}, \xi^{(0)}\}(x) = \sum_{p=-\infty}^{\infty} \{\widehat{\eta^{(0)}}_p, \widehat{\xi^{(0)}}_p\}e^{ipx},$$

which transforms (Equation 2.5.1) into

$$\partial_y^2 \hat{u}_p - |p|^2 \hat{u}_p = \hat{F}_p, \qquad -h < y < 0, \tag{2.5.3a}$$

$$\partial_y \hat{u}_p = 0, \qquad y = -h, \tag{2.5.3b}$$

$$\partial_t \hat{\eta}_p = \partial_y \hat{u}_p - 2\mu |p|^2 \hat{\eta}_p + \hat{Q}_p, \qquad y = 0, \tag{2.5.3c}$$

$$\partial_t \hat{u}_p = -g\hat{\eta}_p - \sigma |p|^2 \hat{\eta}_p - 2\mu \partial_y^2 \hat{u}_p + \hat{R}_p, \qquad y = 0, \tag{2.5.3d}$$

$$\hat{u}_p(0, 0) = \widehat{\xi^{(0)}}_p, \tag{2.5.3e}$$

$$\hat{\eta}_p(0) = \widehat{\eta^{(0)}}_p. \tag{2.5.3f}$$

We now decompose the solution into two parts

$$\{\hat{u}_p, \hat{\eta}_p\} = \{U^P, H^P\} + \{U^E, H^E\},$$

which essentially solve the parabolic (§ 2.4) and elliptic (§ 2.3) problems respectively, and we have suppressed the p subscript for clarity. More specifically, $\{U^P, H^P\}$ solves (Equation 2.5.3) in the case $\hat{F}_p \equiv 0$,

$$\partial_y^2 U^P - |p|^2 U^P = 0, \qquad -h < y < 0, \tag{2.5.4a}$$

$$\partial_y U^P = 0, \qquad y = -h, \tag{2.5.4b}$$

$$\partial_t H^P = \partial_y U^P - 2\mu |p|^2 H^P + \hat{Q}_p, \qquad y = 0, \tag{2.5.4c}$$

$$\partial_t U^P = -gH^P - \sigma |p|^2 H^P - 2\mu \partial_y^2 U^P + \hat{R}_p, \qquad y = 0, \tag{2.5.4d}$$

$$U^P(0,0) = \widehat{\xi^{(0)}}_p, \tag{2.5.4e}$$

$$H^P(0) = \widehat{\eta^{(0)}}_p, \tag{2.5.4f}$$

while $\{U^E, H^E\}$ solves (Equation 2.5.3) where $\hat{Q}_p \equiv \hat{R}_p \equiv \widehat{\xi^{(0)}}_p \equiv \widehat{\eta^{(0)}}_p \equiv 0$,

$$\partial_y^2 U^E - |p|^2 U^E = \hat{F}_p, \qquad -h < y < 0, \tag{2.5.5a}$$

$$\partial_y U^E = 0, \qquad y = -h, \tag{2.5.5b}$$

$$\partial_t H^E = \partial_y U^E - 2\mu |p|^2 H^E, \qquad y = 0, \tag{2.5.5c}$$

$$\partial_t U^E = -gH^E - \sigma |p|^2 H^E - 2\mu \partial_y^2 U^E, \qquad y = 0, \tag{2.5.5d}$$

$$U^E(0,0) = 0, \tag{2.5.5e}$$

$$H^E(0) = 0. \tag{2.5.5f}$$

It is not difficult to show that the solution of (Equation 2.5.4a) and (Equation 2.5.4b) is

$$U^P(y,t) = U^P(0,t)\frac{\cosh(|p|\,(y+h))}{\cosh(|p|\,h)}.$$

Upon insertion of this form into (Equation 2.5.4c)–(Equation 2.5.4f) we find

$$\begin{aligned}
&\partial_t H^P = |p|\tanh(h\,|p|)U^P - 2\mu\,|p|^2\,H^P + \hat{Q}_p, && y=0,\\
&\partial_t U^P = -(g+\sigma\,|p|^2)H^P - 2\mu\,|p|^2\,U^P + \hat{R}_p, && y=0,\\
&U^P(0,0) = \widehat{\xi^{(0)}}_p,\\
&H^P(0) = \widehat{\eta^{(0)}}_p.
\end{aligned}$$

Upon inverse Fourier transform we find that this equation is identical to that appearing in Theorem 2.4.3 with $\hat{\xi}_p(t) = U^P(0,t)$. From this we learn that

$$\begin{aligned}
\max\left\{\left\|H^P\right\|_{X^{s+3}}, \left\|\partial_t H^P\right\|_{X^{s+1}}, \left\|U^P(x,0,t)\right\|_{X^{s+5/2}}, \left\|\partial_t U^P(x,0,t)\right\|_{X^{s+1/2}}\right\}&\\
\leq K_p\left\{\|Q\|_{X^{s+1}} + \|R\|_{X^{s+1/2}} + \left\|\eta^{(0)}\right\|_{H^{s+2}} + \left\|\xi^{(0)}\right\|_{H^{s+3/2}}\right\}.&
\end{aligned} \tag{2.5.6}$$

Turning to (Equation 2.5.5) it is easy to see that (Equation 2.5.5c)–(Equation 2.5.5f) demand that

$$U^E(0,t) \equiv H^E(t) \equiv 0,$$

so that we are left to solve

$$
\begin{aligned}
&\partial_y^2 U^E - |p|^2 U^E = \hat{F}_p, && -h < y < 0,\\
&\partial_y U^E = 0, && y = -h,\\
&U^E = 0, && y = 0.
\end{aligned}
$$

However, upon inverse Fourier transform, we realize that this is simply the system of equations in Theorem 2.3.1 with $\psi \equiv 0$. Thus, we conclude that

$$
\max\left\{\left\|U^E(x,0,t)\right\|_{X^{s+5/2}}, \left\|\left|U^E\right|\right\|_{V^{s+3}}\right\} \leq K_e \,\|\|F\|\|_{V^{s+1}}. \tag{2.5.7}
$$

Combining (Equation 2.5.6) and (Equation 2.5.7) to estimate $\hat{\eta}_p = H^P + H^E$ and $\hat{u}_p = U^P + U^E$ we realize (Equation 2.5.2) for some $K > 0$. □

2.6 An Inductive Lemma

To complete the proof of our theorem we require the following recursive estimates.

Lemma 2.6.1. *For an integer $s \geq 4$, suppose for some $C, B > 0$ we have*

$$
\begin{aligned}
\max\{&\|\eta_n\|_{X^{s+3}}, \|\partial_t \eta_n\|_{X^{s+1}}, \|u_n(x,0,t)\|_{X^{s+5/2}}, \|\partial_t u_n(x,0,t)\|_{X^{s+1/2}}, \|\|u_n\|\|_{V^{s+3}}\}\\
&\leq C\frac{B^{n-1}}{(n+1)^2}, \quad \forall n < N,
\end{aligned}
$$

then the functions F_N, Q_N and R_N satisfy

$$\max\left\{|||F_N|||_{V^{s+1}}, \|Q_N\|_{X^{s+1}}, \|R_N\|_{X^{s+1/2}}\right\} \leq C_i C \left\{ \frac{B^{N-2}}{(N+1)^2} + \frac{B^{N-3}}{(N+1)^2} + \frac{B^{N-4}}{(N+1)^2} \right\},$$

for a universal constant $C_i > 0$.

Proof. To begin, we consider (Equation 1.2.4a) and estimate

$$\begin{aligned} h^2 |||F_N|||_{V^{s+1}} \leq{}& 2h\, |||[\![\eta \partial_x u]\!]_N|||_{V^{s+2}} + hY \left|\left|\left| [\![(\partial_x \eta)\partial_y u]\!]_N \right|\right|\right|_{V^{s+2}} \\ &+ hY\, |||[\![(\partial_x \eta)\partial_x u]\!]_N|||_{V^{s+2}} + \left|\left|\left| [\![\eta^2 \partial_x u]\!]_N \right|\right|\right|_{V^{s+2}} \\ &+ Y \left|\left|\left| [\![\eta(\partial_x \eta)\partial_y u]\!]_N \right|\right|\right|_{V^{s+2}} + Y\, |||[\![\eta(\partial_x \eta)\partial_x u]\!]_N|||_{V^{s+2}} \\ &+ Y^2 \left|\left|\left| [\![(\partial_x \eta)^2 \partial_y u]\!]_N \right|\right|\right|_{V^{s+2}} + h\, |||[\![(\partial_x \eta)\partial_x u]\!]_N|||_{V^{s+1}} \\ &+ |||[\![\eta(\partial_x \eta)\partial_x u]\!]_N|||_{V^{s+1}} + Y \left|\left|\left| [\![(\partial_x \eta)^2 \partial_y u]\!]_N \right|\right|\right|_{V^{s+1}}. \end{aligned}$$

From Theorem B.0.1 we find, since $s+2, s+1 \geq 4$,

$$\begin{aligned} h^2 |||F_N|||_{V^{s+1}} \leq{}& \{2hC[\eta, \partial_x u] + hYC[\partial_x \eta, \partial_y u] \\ &+ hYC[\partial_x \eta, \partial_x u] + hC[\partial_x \eta, \partial_x u]\} \frac{B^{N-2}}{(N+1)^2} \\ &+ \{C[\eta, \eta, \partial_x u] + YC[\eta, \partial_x \eta, \partial_y u] + YC[\eta, \partial_x \eta, \partial_x u] \\ &+ Y^2 C[\partial_x \eta, \partial_x \eta, \partial_y u] + C[\eta, \partial_x \eta, \partial_x u] + YC[\partial_x \eta, \partial_x \eta, \partial_y u]\} \frac{B^{N-3}}{(N+1)^2}, \end{aligned}$$

where we have used $\eta \in X^{s+3}$ and $u \in V^{s+3}$. Since we have chosen the same constant C for the estimates above we find

$$h^2 \left|\left|\left| F_N \right|\right|\right|_{V^{s+1}} \leq (3h + 2hY)C^2 M\Sigma \frac{B^{N-2}}{(N+1)^2} + (2 + 3Y + Y^2)C^3 M^2 \Sigma \frac{B^{N-3}}{(N+1)^2},$$

and we are done provided

$$C_i > \frac{1}{h^2} \max\{(3h + 2hY)CM\Sigma, (2 + 3Y + Y^2)C^2 M^2 \Sigma\}.$$

We continue by considering (Equation 1.2.4b) and estimate

$$h \left\| Q_N \right\|_{X^{s+1}} \leq \left\| [\![\eta(\partial_t \eta)]\!]_N \right\|_{X^{s+1}} + 2\mu \left\| [\![\eta(\partial_x^2 \eta)]\!]_N \right\|_{X^{s+1}} + h \left\| [\![(\partial_x \eta)\partial_x u]\!]_N \right\|_{X^{s+1}} + \left\| [\![\eta(\partial_x \eta)\partial_x u]\!]_N \right\|_{X^{s+1}} + h \left\| [\![(\partial_x \eta)^2 \partial_y u]\!]_N \right\|_{X^{s+1}},$$

From Theorem B.0.1 we find, since $s + 1 \geq 4$,

$$h \left\| Q_N \right\|_{X^{s+1}} \leq \left\{ C[\eta, \partial_t \eta] + 2\mu C[\eta, \partial_x^2 \eta] + hC[\partial_x \eta, \partial_x u] \right\} \frac{B^{N-2}}{(N+1)^2} + \left\{ C[\eta, \partial_x \eta, \partial_x u] + hC[\partial_x \eta, \partial_x \eta, \partial_y u] \right\} \frac{B^{N-3}}{(N+1)^2},$$

where we have used $\eta_n \in X^{s+3}$, $\partial_t \eta_n \in X^{s+1}$, and $u_n(x,0,t) \in X^{s+5/2}$. Again, as we have chosen the same constant C for the estimates above, we find

$$h\,\|Q_N\|_{X^{s+1}} \leq (1+2\mu+h)C^2 M\Sigma \frac{B^{N-2}}{(N+1)^2} + (1+h)C^3 M^2 \Sigma \frac{B^{N-3}}{(N+1)^2},$$

and we are done provided

$$C_i > \frac{1}{h}\max\{(1+2\mu+h)CM\Sigma, (1+h)C^2M^2\Sigma\}.$$

Finally, we consider R_N and for this we require the following estimate on H_N from (Equation 1.2.6). If

$$\|\eta_n\|_{X^{s+3}} \leq C\frac{B^{n-1}}{(n+1)^2}, \quad \forall n < N,$$

then

$$\|H_N\|_{X^{s+2}} \leq C_i C \frac{B^{N-2}}{(N+1)^2}.$$

This can be established either by an argument analogous to the one given here for $\{F_N, Q_N, R_N\}$, or by simply appealing to the fact that the composition of two analytic functions is analytic. With this fact we return to (Equation 1.2.4c) and estimate

$$
\begin{aligned}
h^2 \|R_N\|_{X^{s+1/2}} \leq {} & 2h \|[\![\eta\partial_t u]\!]_N\|_{X^{s+1/2}} + \left\| [\![\eta^2\partial_t u]\!]_N \right\|_{X^{s+1/2}} + h^2 \left\|[\![(\partial_t\eta)\partial_y u]\!]_N\right\|_{X^{s+1/2}} \\
& + h \left\|[\![\eta(\partial_t\eta)\partial_y u]\!]_N\right\|_{X^{s+1/2}} + 2gh \left\| [\![\eta^2]\!]_N \right\|_{X^{s+1/2}} + g \left\| [\![\eta^3]\!]_N \right\|_{X^{s+1/2}} \\
& + 2\sigma h \left\| [\![\eta(\partial_x^2\eta)]\!]_N \right\|_{X^{s+1/2}} + \sigma \left\| [\![\eta^2(\partial_x^2\eta)]\!]_N \right\|_{X^{s+1/2}} \\
& + \sigma h^2 \|[\![\partial_x \left[(\partial_x\eta)H(\partial_x\eta)\right]]\!]_N\|_{X^{s+1/2}} + 2\sigma h \|[\![\eta\partial_x \left[(\partial_x\eta)H(\partial_x\eta)\right]]\!]_N\|_{X^{s+1/2}} \\
& + \sigma \left\| [\![\eta^2\partial_x \left[(\partial_x\eta)H(\partial_x\eta)\right]]\!]_N \right\|_{X^{s+1/2}} \\
& + \frac{1}{2} \Big\{ h^2 \left\| [\![(\partial_x u)^2]\!]_N \right\|_{X^{s+1/2}} + 2h \left\| [\![\eta(\partial_x u)^2]\!]_N \right\|_{X^{s+1/2}} \\
& + \left\| [\![\eta^2(\partial_x u)^2]\!]_N \right\|_{X^{s+1/2}} + 2h \left\|[\![(\partial_x\eta)(\partial_x u)\partial_y u]\!]_N\right\|_{X^{s+1/2}} \\
& + 2h \left\|[\![\eta(\partial_x\eta)(\partial_x u)\partial_y u]\!]_N\right\|_{X^{s+1/2}} + h^2 \left\| [\![(\partial_x\eta)^2(\partial_y u)^2]\!]_N \right\|_{X^{s+1/2}} \Big\} \\
& + \frac{1}{2} h^2 \left\| [\![(\partial_y u)^2]\!]_N \right\|_{X^{s+1/2}} .
\end{aligned}
$$

Once again using Theorem B.0.1 we find, since $s+1/2 \geq 4$,

$$\begin{aligned}h^2 \left\|R_N\right\|_{X^{s+1/2}} \leq & \left\{2hC[\eta,\partial_t u] + h^2 C[\partial_t\eta,\partial_y u] + 2ghC[\eta,\eta] + 2\sigma hC[\eta,\partial_x^2\eta]\right. \\ & \left. +\sigma h^2 C[\partial_x\eta, H] + \frac{h^2}{2}C[\partial_x u,\partial_x u] + \frac{h^2}{2}C[\partial_y u,\partial_y u]\right\} \frac{B^{N-2}}{(N+1)^2} \\ & + \left\{C[\eta,\eta,\partial_t u] + hC[\eta,\partial_t\eta,\partial_y u] + gC[\eta,\eta,\eta] + \sigma C[\eta,\eta,\partial_x^2\eta]\right. \\ & \left. +2\sigma hC[\eta,\partial_x\eta,H] + hC[\eta,\partial_x u,\partial_x u] + hC[\partial_x\eta,\partial_x u,\partial_y u]\right\} \frac{B^{N-3}}{(N+1)^2} \\ & + \left\{\sigma C[\eta,\eta,\partial_x\eta,H] + \frac{1}{2}C[\eta,\eta,\partial_x u,\partial_x u] + hC[\eta,\partial_x\eta,\partial_x u,\partial_y u]\right. \\ & \left. +\frac{h^2}{2}C[\partial_x\eta,\partial_x\eta,\partial_y u,\partial_y u]\right\} \frac{B^{N-4}}{(N+1)^2},\end{aligned}$$

where we have used $\eta_n \in X^{s+3}$, $\partial_t\eta_n \in X^{s+1}$, $u_n(x,0,t) \in X^{s+5/2}$, and $\partial_t u_n(x,0,t) \in X^{s+1/2}$. Finally, as above, since we have chosen the same constant C for the estimates above we find

$$\begin{aligned}h^2 \left\|R_N\right\|_{X^{s+1/2}} \leq & \left(2(1+g+\sigma)h + (2+\sigma)h^2\right) \frac{B^{N-2}}{(N+1)^2} + (1+3h+\sigma+g+2\sigma h) \frac{B^{N-3}}{(N+1)^2} \\ & + \left(\sigma + \frac{1}{2} + h + \frac{h^2}{2}\right) \frac{B^{N-4}}{(N+1)^2},\end{aligned}$$

and we are done provided

$$\begin{aligned}C_i > \frac{1}{h^2}\max & \left\{\left(2(1+g+\sigma)h + (2+\sigma)h^2\right) CM\Sigma, (1+3h+\sigma+g+2\sigma h)\, C^2M^2\Sigma,\right. \\ & \left.\left(\sigma + \frac{1}{2} + h + \frac{h^2}{2}\right) C^3M^3\Sigma\right\}.\end{aligned}$$

$\square$

2.7 Well–Posedness Proof

At last we are in a position to establish our main result.

Theorem 2.7.1. *Given an integer $s \geq 4$, if $\eta^{(0)} \in H^{s+2}$ and $\xi^{(0)} \in H^{s+3/2}$ then there exists a unique solution, of* (Equation 1.2.1) *of the form* (Equation 1.2.2) *satisfying*

$$\max\{\|\eta_n\|_{X^{s+3}}, \|\partial_t\eta_n\|_{X^{s+1}}, \|u_n(x,0,t)\|_{X^{s+5/2}}, \|\partial_t u_n(x,0,t)\|_{X^{s+1/2}}, |||u_n|||_{V^{s+3}}\} \leq C\frac{B^{n-1}}{(n+1)^2}, \quad \forall n > 0, \quad (2.7.1)$$

for universal constants $C, B > 0$.

Proof. We work by induction on n and begin at order $n = 1$ where (Equation 1.2.3) gives us

$$\begin{aligned}
&\Delta u_1 = 0, && -h < y < 0,\\
&\partial_y u_1 = 0, && y = -h,\\
&\partial_t\eta_1 = \partial_y u_1 + 2\mu\partial_x^2\eta_1, && y = 0,\\
&\partial_t u_1 = -g\eta_1 + \sigma\partial_x^2\eta_1 - 2\mu\partial_y^2 u_1, && y = 0,\\
&u_1(x,0,0) = \xi^{(0)}(x), &&\\
&\eta_1(x,0) = \eta^{(0)}(x). &&
\end{aligned}$$

This can be solved explicitly and we set

$$C \quad := \quad \max\left\{ \|\eta_1\|_{X^{s+3}} , \|\partial_t \eta_1\|_{X^{s+1}} , \|u_1(x,0,t)\|_{X^{s+5/2}} , \|\partial_t u_1(x,0,t)\|_{X^{s+1/2}} , |||u_1|||_{V^{s+3}} \right\},$$

which, of course, depends upon $\left\|\eta^{(0)}\right\|_{H^{s+2}}$ and $\left\|\xi^{(0)}\right\|_{H^{s+3/2}}$. We now assume estimate (Equation 2.7.1) for all $n < N$, and apply Lemma 2.5.1 to (Equation 1.2.3) at order N to realize

$$\begin{aligned} &\max\left\{ \|\eta_N\|_{X^{s+3}} , \|\partial_t \eta_N\|_{X^{s+1}} , \|u_N(x,0,t)\|_{X^{s+5/2}} , \|\partial_t u_N(x,0,t)\|_{X^{s+1/2}} , |||u_N|||_{V^{s+3}} \right\} \\ &\qquad \leq K \left\{ \|Q_N\|_{X^{s+1}} + \|R_N\|_{X^{s+1/2}} + |||F_N|||_{V^{s+1}} \right\}, \end{aligned}$$

where we have used that $\eta^{(0)} \equiv \xi^{(0)} \equiv 0$ for $n > 1$. From Lemma 2.6.1 we have

$$\begin{aligned} &\max\left\{ \|\eta_N\|_{X^{s+3}} , \|\partial_t \eta_N\|_{X^{s+1}} , \|u_N(x,0,t)\|_{X^{s+5/2}} , \|\partial_t u_N(x,0,t)\|_{X^{s+1/2}} , |||u_N|||_{V^{s+3}} \right\} \\ &\qquad \leq K C_i C \left\{ \frac{B^{N-2}}{(N+1)^2} + \frac{B^{N-3}}{(N+1)^2} + \frac{B^{N-4}}{(N+1)^2} \right\}, \end{aligned}$$

and we are done if we choose $B > \max\{KC_i, 1\}/3$. $\square$

Remark 2.7.2. Before closing, we remark on a limitation of our method of proof. There is clearly a very specific choice of function spaces for the unknowns: $\eta_n \in X^{s+3}$, $\xi_n \in X^{s+5/2}$, $u_n \in V^{s+3}$. One can wonder if these can be changed. However, we believe that these choices are fixed for the following reasons:

1. Since u_n represents the field in the solution of an elliptic equation and ξ_n is its trace, it must be the case that

$$u_n \in V^t \qquad \Longleftrightarrow \qquad \xi_n \in X^{t-1/2}.$$

2. Our change of variables induces a relationship between the field and the surface deformation, namely, since $\Delta u_n = F_n$ and F_n involves the second derivative of η_n, c.f. (Equation 1.2.4a), it appears that

$$u_n \in V^t \qquad \Longleftrightarrow \qquad \eta_n \in X^t.$$

3. Finally, the parabolic estimate *with* capillarity features the balance

$$\eta_n \in X^t \qquad \Longleftrightarrow \qquad \xi_n \in X^{t-1/2}.$$

So, if we select $t = s + 3$ we can satisfy all three demands. However, if we drop the capillarity term the final balance in Point 3 becomes

$$\eta_n \in X^t \qquad \Longleftrightarrow \qquad \xi_n \in X^{t+1/2},$$

and our argument falls apart. However, it is quite possible that a different change of variables or a more subtle analysis could "improve" the relationship in Point 2 and allow us to consider pure gravity waves.

Remark 2.7.3. We mentioned that another motivation for this work is the goal of modeling the Faraday wave experiment. We believe that the viscous water waves problem presented here will be a reasonable model of this physical problem provided that a periodically modulated gravity is introduced, e.g., g replaced by $g+g\phi(t)$, $\phi(t+\Omega)=\phi(t)$. If the forcing is small, e.g. $\phi=\mathcal{O}(\varepsilon)$, then our new theorem can be used to establish existence and uniqueness of solutions; a couple of additional terms appear in the definition of R_n which are readily estimated. However, this is not completely satisfying as our results conclude that solutions decay exponentially as time evolves which is *not* the interesting regime of the Faraday wave experiment. To address the situation where $\phi=\mathcal{O}(1)$ requires an analysis of a new linearized parabolic problem which has the character of a Mathieu equation. We save such considerations for a future publication.

2.8 Conclusions

In this chapter we have established the existence and uniqueness of solutions to the capillary–gravity water wave problem supplemented with physically motivated viscosity. Our method of proof follows that of Friedman and Reitich in the contexts of the classical Stefan problem (39) and the capillary drop problem (42) which produces somewhat different results than those which can be attained by more standard techniques. It should be noted that due to the nature of the function spaces, the conclusion of the theorem is not only the well–posedness of our model, but also the stability of our solutions. More specifically, we discover *exponential* decay in time with the rate determined by the value of the viscosity. Thus, not only do unique solutions exist, they persist globally in time and decay exponentially fast to zero.

CHAPTER 3

NUMERICAL SIMULATIONS

3.1 Introduction

In this chapter, we present our numerical results. First, we are going to rewrite our equations using the Zakharov, Craig and Sulem ((29), (106)) surface formulation and give a formal definition of the Dirichlet-Neumann Operator (DNO). We then use Field Expansions ((74)) to compute the DNO and a fourth order Runge-Kutta time stepping method to solve the equations. We validate our code by comparing the solutions we obtain to the ones that can be derived in the case of traveling waves solutions. Finally, we use our solver to perform a study on the Faraday Wave experiment.

3.2 The Surface formulation of Zakharov, Craig and Sulem

We restate our problem in terms of Zakharov's canonical conjugate variables

$$\eta(x,t), \qquad \xi(x,t) := \varphi(x,\eta(x,t),t),$$

with the use of the Dirichlet-Neumann operator

$$G : \xi \to [\partial_y \varphi - (\partial_x \eta)\partial_x \varphi](x,\eta(x,t),t),$$

which maps Dirichlet data to its Neumann counterpart. In terms of these, (Equation 1.1.1c-Equation 1.1.1f) can be written as

$$\partial_t \eta = G_0[\xi] + 2\mu\partial_x^2\eta + Q(x,t) \tag{3.2.1a}$$

$$\partial_t \xi = -g\eta + \sigma\partial_x^2\eta + 2\mu\partial_x^2\xi + R(x,t) \tag{3.2.1b}$$

$$\xi(x,\eta,0) = \xi^{(0)}(x), \tag{3.2.1c}$$

$$\eta(x,0) = \eta^{(0)}(x), \tag{3.2.1d}$$

where we have used the fact that, by continuity $\Delta\varphi = 0$ is true at the fluid air interface which means that $\partial_y^2\varphi = -\partial_x^2\varphi = -\partial_x^2\xi$. Q and R are redefined from Chapter 1 to be

$$Q = G(\eta)[\xi] - G_0[\xi],$$

$$R = A(\eta)(\partial_x\xi)^2 - (G(\eta)[\xi])^2 - 2(\partial_x\xi)(\partial_x\eta)G(\eta)[\xi] + \sigma\partial_x\left[(\partial_x\eta)H(\partial_x\eta)\right]$$

and

$$A = A(\eta) := \frac{1}{2(1+(\partial_x\eta)^2)}. \tag{3.2.3a}$$

Here, G_0 expresses the lowest order behavior of the DNO

$$G_0[\xi] = \sum_{p=-\infty}^{\infty} \hat{G}_0\hat{\xi}_p e^{i(\frac{2\pi}{d})px} = \sum_{p=-\infty}^{\infty} |p|\tanh(h|p|)\hat{\xi}_p e^{i(\frac{2\pi}{d})px},$$

where $\hat{\xi}_p$ is the p-th Fourier coefficient of $\xi(x)$. To simplify the computations, we suppose $\frac{2pi}{d} = 1$ in the rest of this chapter.

3.3 2D Traveling Water Waves

We seek to test our codes which simulate the evolution of water waves in the Faraday wave experiment. For this, we use the traveling wave solutions in the absence of viscosity, surface tension and forcing to verify the accuracy of our code. We note that these can be computed to very high accuracy by the method outlined below. Traveling waves play an important role in applications like the propagation of tsunamis or the transport of pollutants. M. Kakleas and D. Nicholls (51) have simulated traveling waves solutions in the case of infinite depth for a weakly nonlinear version of the model, where only linear and quadratic terms were considered. Here, we present solutions to the full model for water with depth h using our Field Expansions method to compute the Dirichlet-Neumann operator and a Runge-Kutta 4 scheme to derive the solution.

3.3.1 Derivation of the exact solutions

With the viscosity $\mu = 0$ and the surface tension $\sigma = 0$, (Equation 3.2.1) become:

$$\partial_t \eta = G_0[\xi] + Q, \qquad \eta(x,0) = \eta^{(0)}(x), \tag{3.3.1a}$$

$$\partial_t \xi = -g\eta + R, \qquad \xi(x,0) = \xi^{(0)}(x). \tag{3.3.1b}$$

To get traveling waves, we seek solutions of the form

$$u = \begin{pmatrix} \eta \\ \\ \xi \end{pmatrix} = f(x+ct),$$

where c is the speed of the wave. The above equation becomes:

$$c\partial_x \eta - G_0 \xi = Q, \qquad y = 0, \tag{3.3.2a}$$

$$c\partial_x \xi + g\eta = R, \qquad y = 0. \tag{3.3.2b}$$

We then use Taylor expansions and seek solutions of the form:

$$u = \sum_{n=1}^{\infty} u_n(x)\varepsilon^n, \quad c = c_0 + \sum_{n=1}^{\infty} c_n \varepsilon^n,$$

where

$$u_n = \begin{pmatrix} \eta_n \\ \\ \xi_n \end{pmatrix}.$$

1 **At order 1**, (Equation 3.3.2) becomes

$$\begin{cases} c_0 \partial_x \eta_1 - G_0 \xi_1 = 0, \\ \\ c_0 \partial_x \xi_1 + g\eta_1 = 0, \end{cases}$$

or written in matrix form,

$$\begin{pmatrix} c_0\partial_x & -G_0 \\ g & c_0\partial_x \end{pmatrix} \begin{pmatrix} \eta_1 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Using bifurcation theory, we seek c_0 such that

$$B_0 = \begin{pmatrix} c_0\partial_x & -G_0 \\ g & c_0\partial_x \end{pmatrix},$$

has a non-trivial null space (otherwise, the equations only have trivial solutions).

As the solutions are spatially periodic, we apply Fourier transformations to η_1 and ξ_1 and obtain

$$\begin{pmatrix} \eta_1 \\ \xi_1 \end{pmatrix} = \sum_{p=-\infty}^{\infty} \begin{pmatrix} \hat{\eta}_{1,p} \\ \hat{\xi}_{1,p} \end{pmatrix} e^{ipx},$$

which gives

$$\hat{B}_{0,p} = \begin{pmatrix} ipc_0 & -|p|\tanh(h|p|) \\ g & ipc_0 \end{pmatrix}.$$

Since we are looking for nontrivial solutions, we consider parameter values where the determinant function

$$\Delta_p = -p^2 c_0^2 + g|p|\tanh(h|p|), \tag{3.3.4}$$

is equal to zero. For any $\tilde{p} \in \mathbf{Z}\backslash\{0\}$, we can choose $c_0 = \sqrt{\frac{g\ \tanh(h|\tilde{p}|)}{|\tilde{p}|}}$ so that $\Delta_{\tilde{p}} = 0$. Then, we pick for $\rho \in \mathbf{R}$

$$\begin{pmatrix} \hat{\eta}_{1,\tilde{p}} \\ \hat{\xi}_{1,\tilde{p}} \end{pmatrix} = \rho \begin{pmatrix} |\tilde{p}|\tanh(h|\tilde{p}|) \\ i\tilde{p}c_0 \end{pmatrix},$$

$$\begin{pmatrix} \hat{\eta}_{1,-\tilde{p}} \\ \hat{\xi}_{1,-\tilde{p}} \end{pmatrix} = \begin{pmatrix} \bar{\hat{\eta}}_{1,\tilde{p}} \\ \bar{\hat{\xi}}_{1,\tilde{p}} \end{pmatrix},$$

$$\begin{pmatrix} \hat{\eta}_{1,p} \\ \\ \hat{\xi}_{1,p} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \quad \text{for } p \neq \pm\tilde{p}.$$

2 **For orders $\mathbf{n > 1}$**, (Equation 3.3.2) gives us the equations

$$c_0 \partial_x \eta_n + \sum_{k=1}^{n-1} c_k \partial_x \eta_{n-k} - |D| \tanh(h|D|) \xi_n = Q_n, \tag{3.3.5a}$$

$$c_0 \partial_x \xi_n + \sum_{k=1}^{n-1} c_k \partial_x \xi_{n-k} + g \eta_n = R_n, \tag{3.3.5b}$$

where, $Q_1 \equiv R_1 \equiv 0$ and, for $n \geq 2$ we have

$$\begin{aligned} Q_n &= G_n, \\ R_n &= \sum_{m=1}^{n} \sum_{l=1}^{m-1} A_{n-m} (\partial_x \xi_{m-l})(\partial_x \xi_l) - \sum_{m=1}^{n} \sum_{l=0}^{m} A_{n-m} (\partial_x \xi_{m-l})(\partial_x \xi_l) \\ &\quad - 2 \sum_{m=1}^{n} \sum_{l=1}^{m-1} \sum_{q=0}^{l-1} A_{n-m} (\partial_x \xi_{m-l})(\partial_x \eta_{l-q})(G_q) \end{aligned}$$

and

$$A(\eta) = \sum_{n=0}^{\infty} A_n \varepsilon^n, \quad G(\eta)[\xi] = \sum_{n=0}^{\infty} G_n \varepsilon^n.$$

Remark 3.3.1. From the definition (Equation 3.2.3a) of A, we get

$$2A + 2(\partial_x \eta)^2 A = 1,$$

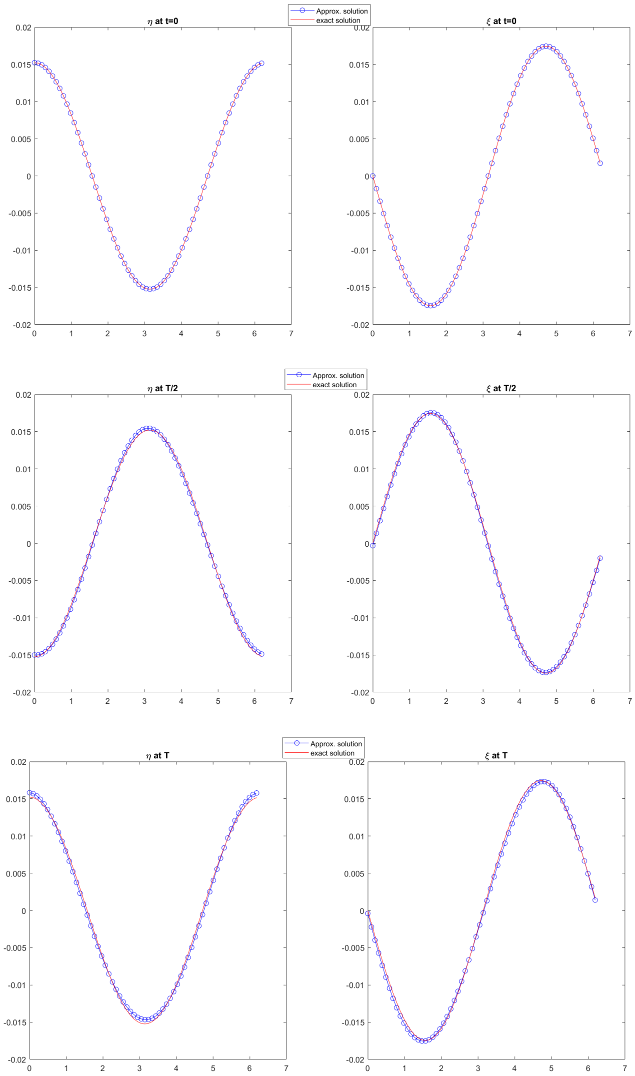

FIGURE 1: NUMERICAL APPROXIMATION OF A TWO-DIMENSIONAL TRAVELING WAVE SOLUTION FOR $\varepsilon = \frac{1}{100}$ AT $t = 0$, $t = \frac{T}{2}$, $t = T$ USING $N = 8$ TAYLOR ORDERS IN THE APPROXIMATION OF THE DNO.

which gives $A_0 = \frac{1}{2}$, $A_1 = 0$, and, for $n \geq 2$,

$$A_n = -\sum_{m=1}^{n-1} \sum_{l=0}^{m-1} (\partial_x \eta_{n-m})(\partial_x \eta_{m-l}) A_l.$$

Upon going to the Fourier side,

$$\hat{B}_{0,p} \begin{pmatrix} \hat{\eta}_{n,p} \\ \\ \hat{\xi}_{n,p} \end{pmatrix} = \begin{pmatrix} \hat{Q}_{n,p} - \sum_{k=1}^{k=n-1} ipc_k \hat{\eta}_{n-k,p} \\ \\ \hat{R}_{n,p} - \sum_{k=1}^{k=n-1} ipc_k \hat{\xi}_{n-k,p} \end{pmatrix} = \begin{pmatrix} \tilde{\hat{Q}}_{n,p} - ipc_{n-1} \hat{\eta}_{1,p} \\ \\ \tilde{\hat{R}}_{n,p} - ipc_{n-1} \hat{\xi}_{1,p} \end{pmatrix}.$$

We now consider three cases of $p \in \mathbf{Z}$:

a. **If $\mathbf{p} \neq \mathbf{0}, \pm\tilde{\mathbf{p}}$**, then $\hat{B}_{0,p}$ is invertible and the solution is easily derived.

b. **If $\mathbf{p} = \mathbf{0}$**, then we obtain $g\hat{\eta}_{n,p} = \hat{R}_{n,p}$ which is easily solved. We then set $\hat{\xi}_{n,p} = 0$ since the velocity potential is only meaningful up to a constant.

c. **If $\mathbf{p} = \pm\tilde{\mathbf{p}}$**, for example, if $p = \tilde{p}$, the matrix $\hat{B}_{0,\tilde{p}}$ is singular but, using Gaussian elimination, the equation is solvable if

$$c_{n-1} = \frac{g\tilde{\hat{Q}}_{n,\tilde{p}} - (ic_0\tilde{p})\tilde{\hat{R}}_{n,\tilde{p}}}{g(i\tilde{p})\hat{\eta}_{1,\tilde{p}} - (ic_0\tilde{p})i\tilde{p}\hat{\xi}_{1,\tilde{p}}}.$$

We then follow the approach of Stokes (100) to get the uniqueness of the solution by taking η_n orthogonal to η_1 in L^2. Since η_1 is only supported by wavenumbers $p = \pm\tilde{p}$, this is obtained by setting $\hat{\eta}_{n,\pm\tilde{p}} = 0$. We then obtain

$$\hat{\xi}_{n,\tilde{p}} = \frac{\tilde{\hat{R}}_{n,\tilde{p}} - c_{n-1}(i\tilde{p})\hat{\xi}_{1,\tilde{p}}}{ic_0\tilde{p}}, \quad \hat{\xi}_{n,-\tilde{p}} = \bar{\hat{\xi}}_{n,\tilde{p}}.$$

With these, we find approximations of the traveling wave solutions with the truncated series

$$u^N(x) = \sum_{n=0}^{N} u_n(x)\varepsilon^n.$$

3.3.2 Solving the Water Wave Equations

We now describe our procedure for solving (Equation 3.3.2). Following Trefethen (101), we utilize a Fourier collocation approach where we assume

$$\{\eta, \xi\}(x,t) = \sum_{p=-\frac{N_x}{2}}^{\frac{N_x}{2}-1} \{\hat{\eta}_p(t), \hat{\xi}_p(t)\}e^{ipx}$$

and demand that (Equation 3.3.5)be true at the equally-spaced gridpoints $\{x_j\}_{j=0}^{N_x-1}$, $x_j = \frac{2\pi j}{N_x}$. The only novelty of our approach is the treatment of the DNO that we approximate using Field Expansions (74).

Then, we use a fourth order Runge-Kutta method to approximate the solution to (Equation 3.3.2). Runge-Kutta methods are a family of iterative methods used to approximate solutions of ordinary differential equations of the form

$$\partial_t y = f(y(t), t), \qquad y(t_0) = y_0.$$

For a step size h, the fourth order Runge-Kutta method is based on the following iterations

$$y_{n+1} = y_n + \frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4,$$

where

$$\begin{aligned}
k_1 &= hf(y_n, t_n),\\
k_2 &= hf(y_n + \frac{k_1}{2}, t_n + \frac{h}{2}),\\
k_3 &= hf(y_n + \frac{k_2}{2}, t_n + \frac{h}{2}),\\
k_4 &= hf(y_n + k_3, t_n + h).
\end{aligned}$$

3.3.3 Simulation of two-dimensional traveling waves

In figure (1), we show the solutions at different times t with initial conditions given by the traveling wave solution at $t = 0$. As expected, the solution comes back to its initial position at the final time T which corresponds to one full period.

3.4 3D Traveling Water Waves

3.4.1 Derivation of the exact solutions

We now simulate 3D traveling waves solutions. Here $x = (x_1, x_2)$ and we derivate the exact solution by following Nicholls and Reitich (85). As before, we are looking for solutions of the form

$$u = \sum_{n=1}^{\infty} u_n(x)\varepsilon^n,$$

where

$$u_n = \begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix}.$$

We seek traveling waves with speed $c = (c_{x_1}, c_{x_2}) = \sum_{n=0}^{\infty}(c_n^{x_1}, c_n^{x_2})\varepsilon^n$ such that

$$u(x_1, x_2, t) = f(x_1 + c_1 t, x_2 + c_2 t).$$

Upon going on the Fourier side to get

$$\begin{pmatrix} \eta_n \\ \xi_n \end{pmatrix} = \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \begin{pmatrix} \hat{\eta}_{n,p,q} \\ \hat{\xi}_{n,p,q} \end{pmatrix} e^{ipx_1 + iqx_2}.$$

We now have

$$\hat{B}_{0,p,q} = \begin{pmatrix} i(c_0^{x_1}p + c_0^{x_2}q) & -\sqrt{p^2+q^2}\tanh(h\sqrt{p^2+q^2}) \\ g & i(c_0^{x_1}p + c_0^{x_2}q) \end{pmatrix},$$

and $\Delta_{p,q} = -(c_0^{x_1}p + c_0^{x_2}q)^2 + g * \sqrt{p^2+q^2}\tanh(h\sqrt{p^2+q^2})$. We show below the procedure to compute a 3D traveling wave solution:

1 **At order 1** For $(p,q) \neq (0,0)$, if we reason as in the 2D case, we will pick $(\tilde{p},\tilde{q})$ such that $\Delta_{\tilde{p},\tilde{q}} = 0$. From that, we would get for $\rho \in \mathbf{R}$,

$$\begin{pmatrix} \hat{\eta}_{1,\tilde{p},\tilde{q}} \\ \hat{\xi}_{1,\tilde{p},\tilde{q}} \end{pmatrix} = \rho \begin{pmatrix} \sqrt{\tilde{p}^2+\tilde{q}^2}\tanh(h\sqrt{\tilde{p}^2+\tilde{q}^2}) \\ i(c_0^{x_1}\tilde{p} + c_0^{x_2}\tilde{q}) \end{pmatrix}, \quad \begin{pmatrix} \hat{\eta}_{1,-\tilde{p},-\tilde{q}} \\ \hat{\xi}_{1,-\tilde{p},-\tilde{q}} \end{pmatrix} = \begin{pmatrix} \bar{\hat{\eta}}_{1,\tilde{p},\tilde{q}} \\ \bar{\hat{\xi}}_{1,\tilde{p},\tilde{q}} \end{pmatrix},$$

$$\begin{pmatrix} \hat{\eta}_{1,p,q} \\ \hat{\xi}_{1,p,q} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{for } (p,q) \neq (\tilde{p},\tilde{q}).$$

However, as noted in (85), these solutions are two dimensional waveforms in a rotated set of coordinates (see figure(2)) .

To get true three-dimensional solutions, we follow (85) and begin by selecting two lineary independant wavenumbers $k_1 = (\tilde{p},\tilde{q})$ and $k_2 = (\tilde{q},-\tilde{p})$. We then solve $\Delta_{k_j} = 0$, $j = 1,2$ to obtain

$$c_0^{x_1} = \frac{-(\tilde{p}+\tilde{q})\sqrt{g\hat{G}_{0,p,\tilde{\tilde{q}}}}}{-(\tilde{p}^2+\tilde{q}^2)}, \quad c_0^{x_1} = \frac{(\tilde{p}-\tilde{q})\sqrt{g\hat{G}_{0,p,\tilde{\tilde{q}}}}}{-(\tilde{p}^2+\tilde{q}^2)}.$$

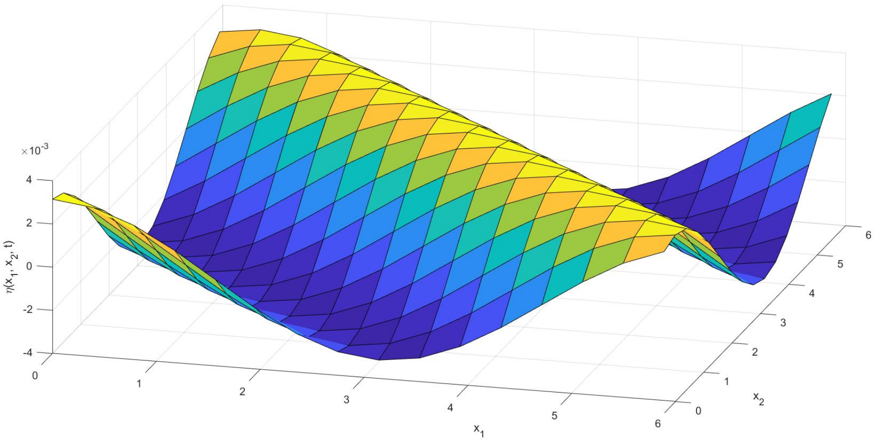

FIGURE 2: ILLUSTRATION OF A NONTRULY 3D TRAVELING WAVE SOLUTION FOR $(\tilde{p}, \tilde{q}) = (1, 1)$

Then, we pick, for $\rho_1, \rho_2 \in \mathbf{R}$

$$\begin{pmatrix} \hat{\eta}_{1,\tilde{p},\tilde{q}} \\ \\ \hat{\xi}_{1,\tilde{p},\tilde{q}} \end{pmatrix} = \rho_1 \begin{pmatrix} \sqrt{\tilde{p}^2 + \tilde{q}^2}\tanh(h\sqrt{\tilde{p}^2 + \tilde{q}^2}) \\ \\ i(c_0^{x_1}\tilde{p} + c_0^{x_2}\tilde{q}) \end{pmatrix}, \quad \begin{pmatrix} \hat{\eta}_{1,-\tilde{p},-\tilde{q}} \\ \\ \hat{\xi}_{1,-\tilde{p},-\tilde{q}} \end{pmatrix} = \begin{pmatrix} \bar{\hat{\eta}}_{1,\tilde{p},\tilde{q}} \\ \\ \bar{\hat{\xi}}_{1,\tilde{p},\tilde{q}} \end{pmatrix},$$

$$\begin{pmatrix} \hat{\eta}_{1,\tilde{q},-\tilde{p}} \\ \\ \hat{\xi}_{1,\tilde{q},-\tilde{p}} \end{pmatrix} = \rho_2 \begin{pmatrix} \sqrt{\tilde{p}^2 + \tilde{q}^2}\tanh(h\sqrt{\tilde{p}^2 + \tilde{q}^2}) \\ \\ i(c_0^{x_1}\tilde{q} - c_0^{x_2}\tilde{p}) \end{pmatrix}, \quad \begin{pmatrix} \hat{\eta}_{1,-\tilde{q},\tilde{p}} \\ \\ \hat{\xi}_{1,-\tilde{q},\tilde{p}} \end{pmatrix} = \begin{pmatrix} \bar{\hat{\eta}}_{1,\tilde{q},-\tilde{p}} \\ \\ \bar{\hat{\xi}}_{1,\tilde{q},-\tilde{p}} \end{pmatrix},$$

$$\begin{pmatrix} \hat{\eta}_{1,p,q} \\ \\ \hat{\xi}_{1,p,q} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \quad \text{for } (p, q) \neq (\tilde{p}, \tilde{q}),\ (p, q) \neq (\tilde{q}, -\tilde{p}).$$

2 **For orders n > 1**, we have the equations

$$\hat{B}_{0,p,q}\begin{pmatrix}\hat{\eta}_{n,p,q}\\ \\ \hat{\xi}_{n,p,q}\end{pmatrix} = \begin{pmatrix}\hat{\tilde{Q}}_{p,q} - i(c^{x_1}_{n-1}p + c^{x_2}_{n-1}q)\hat{\eta}_{1,p,q}\\ \\ \hat{\tilde{R}}_{p,q} - i(c^{x_1}_{n-1}p + c^{x_2}_{n-1}q)\hat{\xi}_{1,p,q}\end{pmatrix}.$$

a. **If** $(p,q) \neq (0,0)$ then $\hat{B}_{0,p,q}$ is invertible and the solution is easily derived

b. **If** $(p,q) = (0,0)$ then we obtain $g\hat{\eta}_{n,p,q} = \hat{\tilde{Q}}_{p,q}$ which is easily solved. We then set $\hat{\xi}_{n,p,q} = 0$.

c. **If** $(p,q) = \pm(\tilde{p},\tilde{q})$ **or** $\pm(\tilde{q},-\tilde{p})$, then, as for the 2D case, we use Gaussian elimination to get

$$\begin{aligned}
\tilde{p}c^{x_1}_{n-1} + \tilde{q}c^{x_2}_{n-1} =& \frac{g\hat{\tilde{Q}}_{n,\tilde{p},\tilde{q}} - i(c^{x_1}_0\tilde{p} + c^{x_2}_0\tilde{q})\hat{\tilde{R}}_{n,\tilde{p},\tilde{q}}}{g(i\hat{\eta}_{1,\tilde{p},\tilde{q}} - (i(c^{x_1}_0\tilde{p} + c^{x_2}_0\tilde{q}))(i(\tilde{p}+\tilde{q}))\hat{\xi}_{1,\tilde{p},\tilde{q}}} = rhs_1,\\
\tilde{q}c^{x_1}_{n-1} - \tilde{p}c^{x_2}_{n-1} =& \frac{g\hat{\tilde{Q}}_{n,\tilde{q},-\tilde{p}} - i(c^{x_1}_0\tilde{q} - c^{x_2}_0\tilde{p})\hat{\tilde{R}}_{n,\tilde{q},-\tilde{p}}}{g(i\hat{\eta}_{1,\tilde{q},-\tilde{p}} - (i(c^{x_1}_0\tilde{q} - c^{x_2}_0\tilde{p}))(i(\tilde{q}-\tilde{p}))\hat{\xi}_{1,\tilde{q},-\tilde{p}}} = rhs_2,
\end{aligned}$$

Which, upon solving gives us

$$\begin{aligned}
c^{x_1}_{n-1} =& \frac{-rhs_1 \times \tilde{p} - rhs_2 \times \tilde{q}}{-(\tilde{p}^2 + \tilde{q}^2)},\\
c^{x_2}_{n-1} =& \frac{rhs_2 \times \tilde{p} - rhs_1 \times \tilde{q}}{-(\tilde{p}^2 + \tilde{q}^2)}.
\end{aligned}$$

Then, following Stokes again, set $\hat{\eta}_{n,\tilde{p},\tilde{q}} = \hat{\eta}_{n,-\tilde{p},-\tilde{q}} = \hat{\eta}_{n,\tilde{q},-\tilde{p}} = \hat{\eta}_{n,-\tilde{q},\tilde{p}} = 0$ and obtain

$$\begin{aligned}
\hat{\xi}_{n,\tilde{p},\tilde{q}} =& \frac{\tilde{\hat{R}}_{n,\tilde{p},\tilde{q}} - i(c_{n-1}^{x_1}\tilde{p} + c_{n-1}^{x_2}\tilde{q})\hat{\xi}_{1,\tilde{p},\tilde{q}}}{i(c_0^{x_1}\tilde{p} + c_0^{x_2}\tilde{q})}, \quad \hat{\xi}_{n,-\tilde{p},-\tilde{q}} = \bar{\hat{\xi}}_{n,\tilde{p},\tilde{q}} \\
\hat{\xi}_{n,\tilde{q},-\tilde{p}} =& \frac{\tilde{\hat{R}}_{n,\tilde{q},-\tilde{p}} - i(c_{n-1}^{x_1}\tilde{q} - c_{n-1}^{x_2}\tilde{p})\hat{\xi}_{1,\tilde{q},-\tilde{p}}}{i(c_0^{x_1}\tilde{q} - c_0^{x_2}\tilde{p})}, \quad \hat{\xi}_{n,-\tilde{q},\tilde{p}} = \bar{\hat{\xi}}_{n,\tilde{q},-\tilde{p}}.
\end{aligned}$$

In the same spirit as before, we now approximate the exact traveling wave solutions with the truncated series

$$u^N(x_1, x_2) = \sum_{n=0}^{N} u_n(x_1, x_2)\varepsilon^n.$$

3.4.2 Simulation of three-dimensional traveling waves

Here we adapt the scheme given above to approximate a two-dimensional traveling wave to the three- dimensional case. We assume that $u_n(x_1, x_2)$ is approximated with

$$u_n(x_1, x_2, t) = \sum_{p=-\frac{N_x}{2}}^{\frac{N_x}{2}-1} \sum_{q=-\frac{N_y}{2}}^{\frac{N_y}{2}-1} \hat{u}_{n,p,q} e^{i(px_1+qx_2}$$

and demand that (Equation 3.3.2) be true at the equally-spaced gridpoints

$$\left(\{x_{1j}\}_{j=0}^{N_x-1}, \{x_{2j}\}_{j=0}^{N_y-1}\right), \quad (x_{1j}, x_{2j}) = \left(\frac{2\pi j}{N_x}, \frac{2\pi j}{N_y}\right).$$

We make sure to pick $(\tilde{p}, \tilde{q})$ such that $(\tilde{q}, -\tilde{p})$ is in the truncated wavenumbers domain. As in the two-dimensional case, we approximate the DNO using Field Expansions and use a Runge-

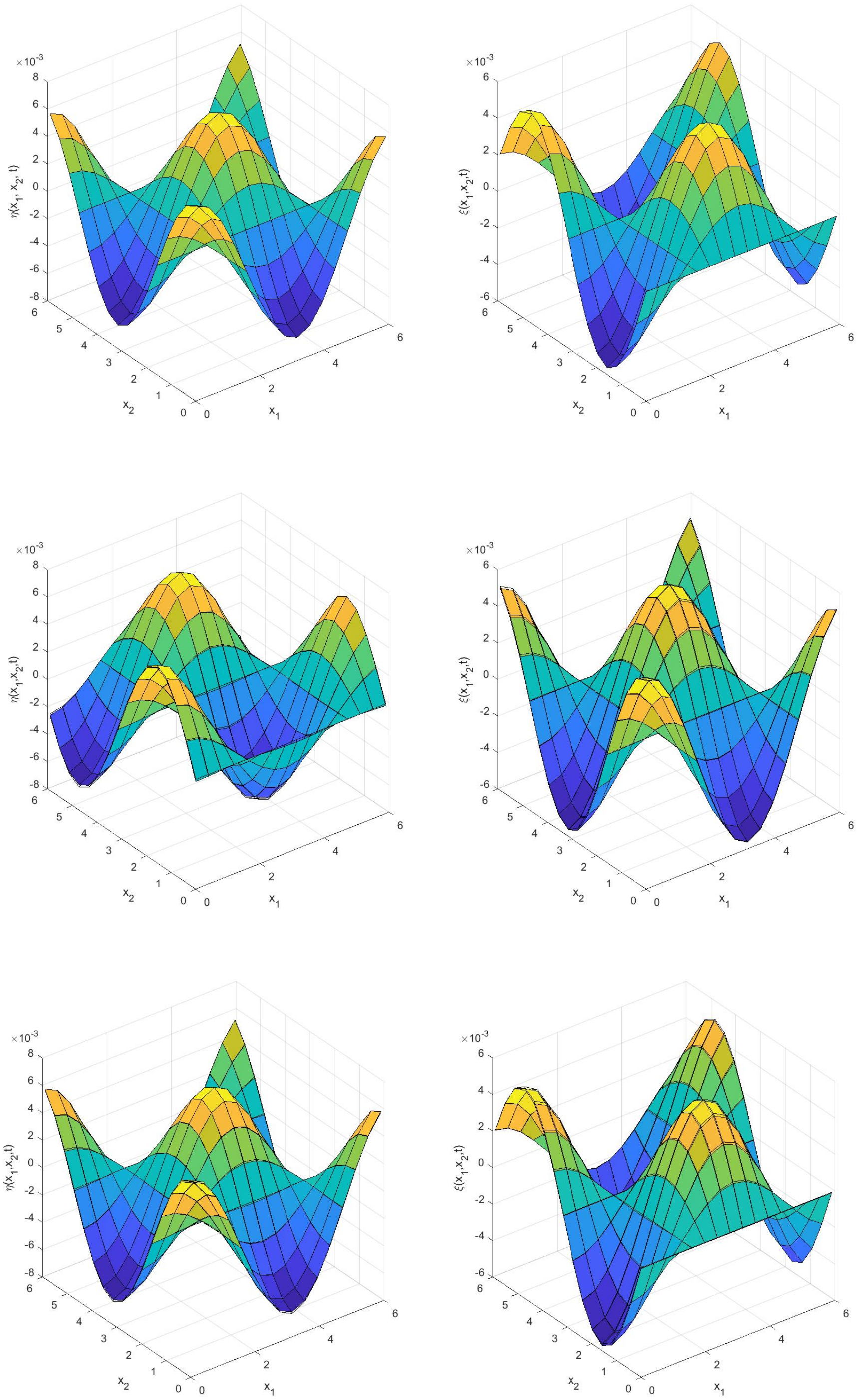

FIGURE 3: NUMERICAL APPROXIMATION OF A 3D TRAVELING WAVE SOLUTION FOR $\varepsilon = \frac{1}{100}$ AT $t = 0$, $t = \frac{3T}{4}$, $t = T$ USING $N = 4$ TAYLOR ORDERS IN THE APPROXIMATION OF THE DNO AND $(\tilde{p}, \tilde{q}) = (1, 1)$

Kutta method to approximate the solutions of (Equation 3.3.2).

Figure (3) shows the solutions obtained by our method and those obtained by truncating the exact solutions for $(\tilde{p},\tilde{q}) = (1,1)$. The figure shows that our method fits the truncated exact solution satisfactorily. Also, as expected, the solution come back to its initial position at the final time T which corresponds to one full period. Figure (4) shows an approximated traveling wave solution at different times for $(\tilde{p},\tilde{q}) = (2,2)$.

Given the satisfactory results we obtain in the case of traveling wave solutions, we feel justified in beginning our study of the Faraday Wave Experiment.

3.5 The Faraday Wave Experiment

Faraday waves are obtained when a vessel containing two immiscible fluids is vertically forced into motion. The shaking induces a pattern of standing waves at the interface of the two fluids. They were first noticed by Faraday in 1831, and Miles and Henderson provided an extensive review on Faraday waves in 1990. The patterns can have differents shapes from simple squares to eight-fold patterns (Christiansen, Alstrom and Levinsen (1992)), twelve-fold quasipatterns (Edwards and Fauve, 1994), triangles (Muller, 1993), and even spatiotemporal chaos (Kudrolli and Gollub, 1996). Benjamin and Ursell (14) studied the stability of the free surface between inviscid fluids theoretically by showing that the equations of the system are equivalent to a Mathieu equation. However, they observed large discrepancies between their results in the case of ideal fluids and the experimental results for viscous fluids. For this reason,

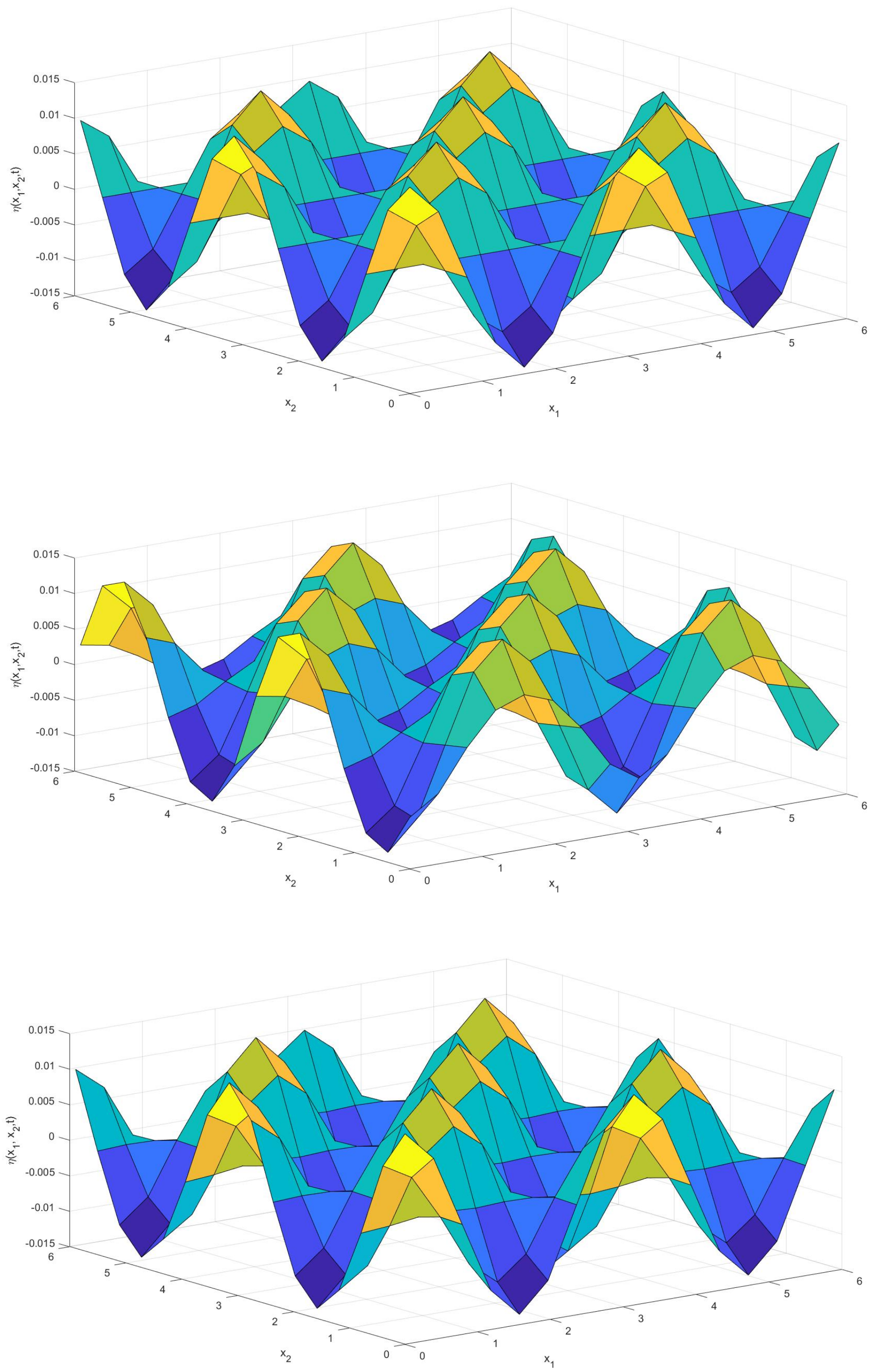

FIGURE 4: NUMERICAL APPROXIMATION OF A 3D TRAVELING WAVE SOLUTION FOR $\varepsilon = \frac{1}{100}$ AT $t = 0,\ t = \frac{2T}{3},\ t = T$ USING $N = 4$ TAYLOR ORDERS IN THE APPROXIMATION OF THE DNO AND $(\tilde{p}, \tilde{q}) = (2, 2)$

Tuckerman and Kumar (102) extended the work of Benjamin and Ursell to viscous fluids in 1994.

Here we consider a vessel containing a layer of (viscous) water of height h and air above, and are interested in the stability of the free surface of the water. We simulate the Faraday wave experiment (35) by supplementing our equations with a periodically varying gravity in the form of a single frequency sinusoid so that Equation 3.2.1 become:

$$\partial_t \eta = G_0 \xi + 2\mu \partial_x^2 \eta + Q, \tag{3.5.1a}$$

$$\partial_t \xi = -g(1 - 2F\cos(\Omega t))\eta + \sigma \partial_x^2 \eta + 2\mu \partial_x^2 \xi + R, \tag{3.5.1b}$$

where $G_0 = |D| \tanh(h|D|)$.

First we derive a Mathieu equation and compare its solutions to the one obtained by using our algorithm to validate it. Then, we use our solver to derive stability results similar to the ones obtained by Tuckermann and Kumar.

3.5.1 Derivation of a Mathieu type Equation

Following Benjamin and Ursell, we linearize (Equation 3.5.1) and show how a Mathieu equation can be obtained. First, we make the change of variables

$$\Omega t = 2\tau \implies \tau = \frac{\Omega}{2} t,$$

which transforms (Equation 3.5.1) into

$$\partial_\tau \eta = \frac{2}{\Omega} G_0 \xi + \frac{4\mu}{\Omega} \partial_x^2 \eta + \frac{2}{\Omega} Q, \tag{3.5.2a}$$

$$\partial_t \xi = -g \frac{2}{\Omega} (1 - 2F \cos(\Omega t)) \eta + \frac{2\sigma}{\Omega} \partial_x^2 \eta + \frac{4\mu}{\Omega} \partial_x^2 \xi + \frac{2}{\Omega} R. \tag{3.5.2b}$$

3.5.1.1 Linear Solutions and Spatial Periodicity

If we consider $\{\eta, \xi\} = O(\varepsilon)$, then, at leading order, the governing equations (Equation 3.5.2) become (recall that Q and R contain the nonlinear terms)

$$\partial_\tau \eta = \frac{2}{\Omega} G_0 \xi + \frac{4\mu}{\Omega} \partial_x^2 \eta, \tag{3.5.3a}$$

$$\partial_t \xi = -g \frac{2}{\Omega} (1 - 2F \cos(\Omega t)) \eta + \frac{2\sigma}{\Omega} \partial_x^2 \eta + \frac{4\mu}{\Omega} \partial_x^2 \xi. \tag{3.5.3b}$$

We seek spatially d-periodic solutions. Defining the spatial frequency $\kappa := \frac{2\pi}{d}$ we have

$$\{\eta(x,\tau), \xi(x,\tau)\} = \sum_{p=-\infty}^{\infty} \{\hat{\eta}(x,\tau), \hat{\xi}(x,\tau)\} e^{i\kappa p x},$$

and find

$$\partial_\tau \begin{pmatrix} \hat{\eta}_p \\ \hat{\xi}_p \end{pmatrix} = \begin{pmatrix} -\frac{4\mu}{\Omega}|p|^2 & \frac{2}{\Omega}\hat{G}_{0,p} \\ -\frac{2g}{\Omega}[1 - 2F\cos(2\tau)] - \sigma|p|^2 & -\frac{4\mu}{\Omega}|p|^2 \end{pmatrix} \begin{pmatrix} \hat{\eta}_p \\ \hat{\xi}_p \end{pmatrix}.$$

To simplify things, we factor out the time-decay with the change of variables

$$u(x,\tau) = e^{2\alpha\tau}\eta(x,\tau), \quad v(x,\tau) = e^{2\alpha\tau}\xi(x,\tau), \quad \alpha := \frac{2\mu|p|^2}{\Omega},$$

resulting in

$$\partial_\tau \begin{pmatrix} \hat{u}_p \\ \hat{v}_p \end{pmatrix} = \begin{pmatrix} 0 & \frac{2}{\Omega}\hat{G}_{0,p} \\ -\frac{2g}{\Omega}[1-2F\cos(2\tau)] - \sigma|p|^2 & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_p \\ \hat{v}_p \end{pmatrix}.$$

Taking the time derivative of the first equation and using the second equation, we find

$$\partial_\tau^2 \hat{u}_p + \left(\frac{2}{\Omega}\right)^2 g\hat{G}_{0,p}\left(1 - \frac{\sigma\Omega|p|^2}{2} - 2F\cos(2\tau)\right)\hat{u}_p = 0,$$

or defining

$$\omega_0^2 = \omega_0^2(p; g, h, \kappa) := g\hat{G}_{0,p},$$

this becomes

$$\partial_\tau^2 \hat{u}_p + \left(\frac{2\omega_0}{\Omega}\right)^2 \left(1 - \frac{\sigma\Omega|p|^2}{2} - 2F\cos(2\tau)\right)\hat{u}_p = 0. \tag{3.5.4}$$

Upon defining

$$a = \left(\frac{2\omega_0}{\Omega}\right)^2 \left(1 - \frac{\sigma\Omega|p|^2}{2}\right), \quad q = aF,$$

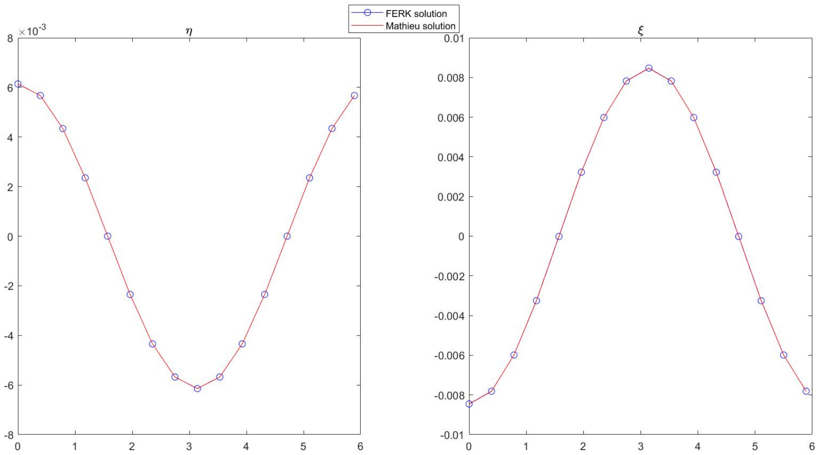

Figure 5: A comparision between the solutions obtained by solving a Mathieu equation and the solutions obtained using our algorithm for $N = 16$ Taylor orders, $N_x = 16$ Fourier terms, $\varepsilon = \frac{1}{100}$ and $F = 0$.

we discover that (Equation 3.5.4) is the canonical Mathieu equation

$$\partial_\tau^2 \hat{u}_p + (a - 2q\cos(2\tau))\hat{u}_p = 0. \tag{3.5.5}$$

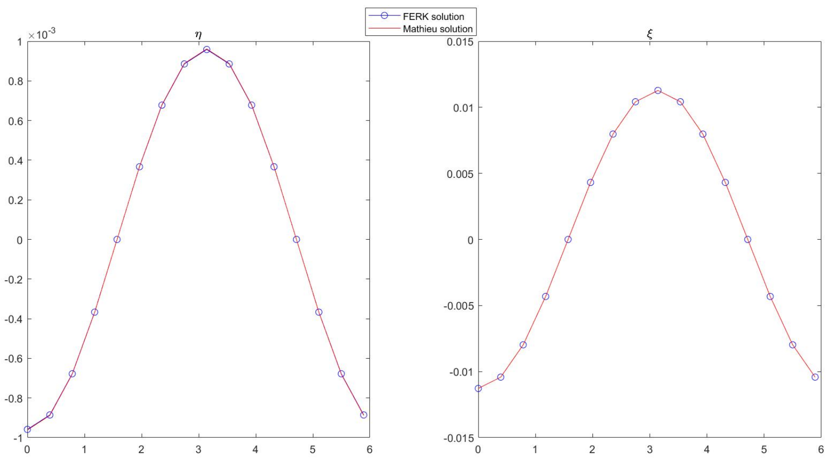

FIGURE 6: A COMPARISION BETWEEN THE SOLUTIONS OBTAINED BY SOLVING A MATHIEU EQUATION AND THE SOLUTIONS OBTAINED USING OUR ALGORITHM FOR $N = 16$ TAYLOR ORDERS, $N_x = 16$ FOURIER TERMS, $\varepsilon = \frac{1}{100}$ AND $F = 0.23$.

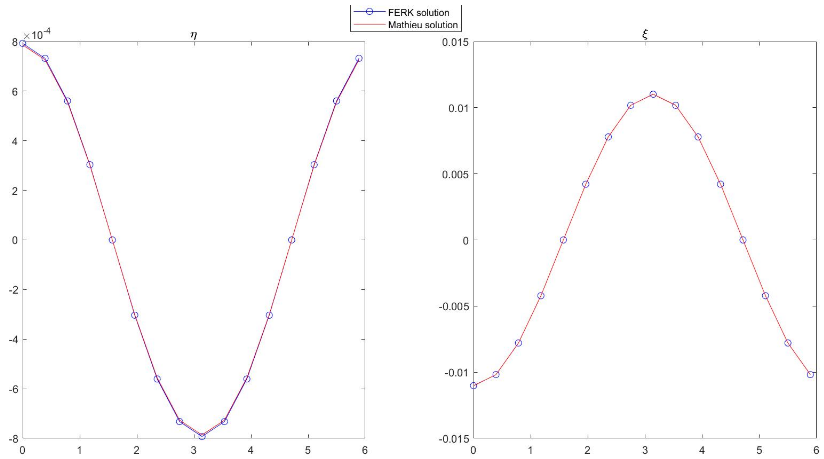

FIGURE 7: A COMPARISION BETWEEN THE SOLUTIONS OBTAINED BY SOLVING A MATHIEU EQUATION AND THE SOLUTIONS OBTAINED USING OUR ALGORITHM FOR $N = 16$ TAYLOR ORDERS, $N_x = 16$ FOURIER TERMS, $\varepsilon = \frac{1}{100}$ AND $F = 0.39$.

3.5.1.2 Comparison with HOPS methods

We compare the solutions obtained by solving the above Mathieu equation to the one obtain by using Field Expansions to simulate the DNO and a fourth order Runge-Kutta method. The following figures (5, 6, 7, 8) show the two solutions at the final time T of our experiment which corresponds to one full period.

Our method replicates the behavior of the solution obtained by solving the Mathieu equation. One advantage of our method is that it is faster and more flexible than the one proposed by Tuckermann et al. In fact, in the literature, people have used Floquet analysis to obtain a

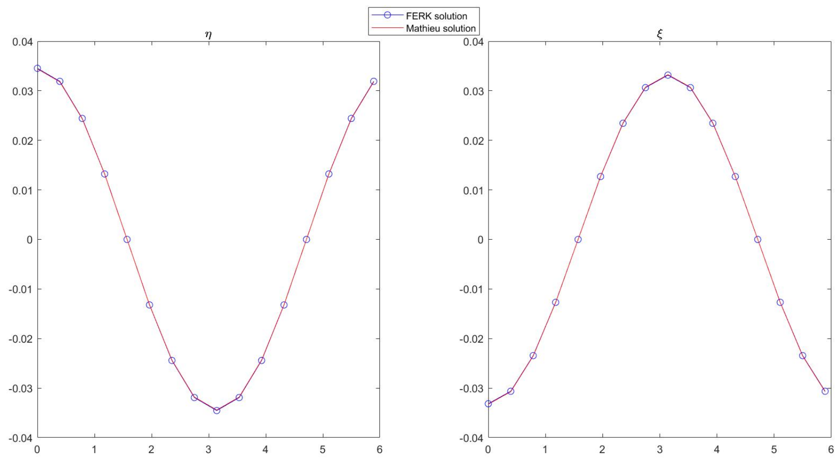

Figure 8: A comparision between the solutions obtained by solving a Mathieu equation and the solutions obtained using our algorithm for $N = 16$ Taylor orders, $N_x = 16$ Fourier terms, $\varepsilon = \frac{1}{100}$ and $F = 1.30$

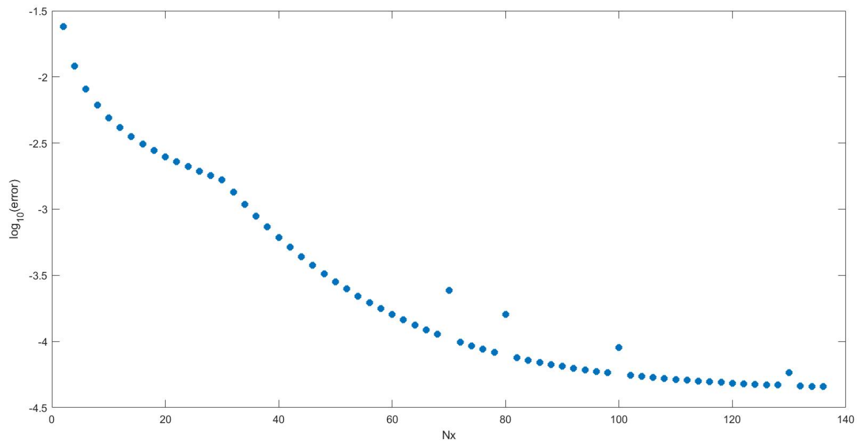

Figure 9: Relative error between the Mathieu equation solution and our algorithm solution for different N_x Fourier terms, $N = 16$ Taylor orders, $\varepsilon = \frac{1}{100}$ and $F = 1.30$

very computationally expensive eigenvalue problem that they then solved to derive the solutions to their equations (102).

In figure (9), we show the $\log_{10}$ of the relative errors (using the supremum norm) with respect to the number of Fourier terms (N_x). As expected, the errors decrease when N_x increases except at a few isolated values of N_x where the error slightly goes up.

F	Error in η	Error in ξ
0	$2.94e^{-5}$	$9.53e^{-5}$
0.23	$2.82e^{-3}$	$1.54e^{-3}$
0.39	$7.96e^{-3}$	$4.90e^{-3}$
1.30	$3.11e^{-3}$	$2.80e^{-3}$

TABLE I: Relative Errors for different values of the Amplitude F of the forcing for $N = 16$ Taylor Orders, $N_x = 16$ Fourier terms and $\varepsilon = \frac{1}{100}$

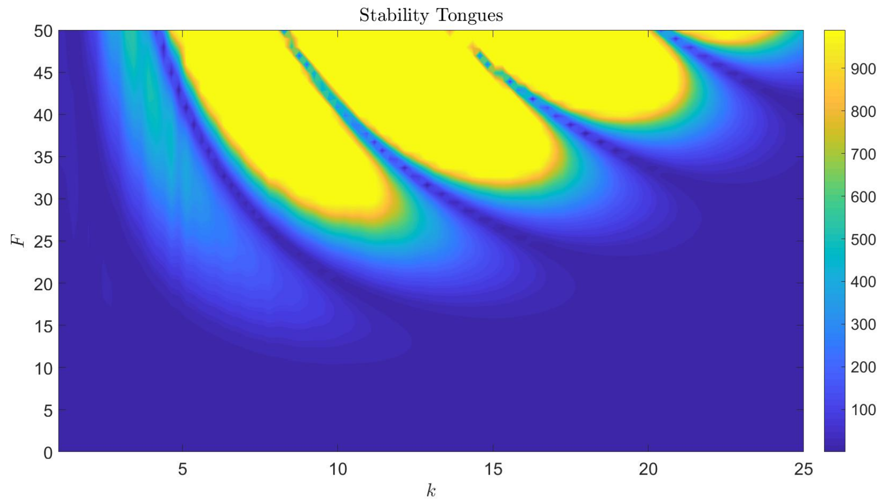

Figure 10: Stability tongues for water and air, F is the amplitude of the vertical forcing and k is the wavenumber.

3.5.2 Numerical Stability

In this section, we reproduce in figure (10) the tongue-like stability zones obtained by Tuckermann and Kumar (1994) by using our algorithm as opposed to solving a Mathieu equation. We consider a vessel containing water at 20 degres Celsius up to a height $h = 29mm$ and the rest of the vessel is filled with air. The frequency of the forcing is $100Hz$.

Inside the tongues (yellow) the solutions are unstable (growing or blowing up) and they stable outside of them (blue). The amplitude of the forcing at the interface of these regions is called the critical forcing. This is consistent with the stability tongues obtained by Tuckermann et al. We can see that the equations are stable for small forcing amplitude regardless of the wavenumber k. As in Tuckermann et al., we also see that the critical forcing increases with the wavenumber k which is expected since the viscous dissipation increases with k.

As it is usual when studying the Faraday wave experiment, we have produced an instability diagram with respect to the parameters of the model. Our method is not a standard one as we use Field Expansions to compute the DNO and a fourth order Runge-Kutta method to study the stability of our model instead of solving a Mathieu like equation. In the future, we would like to numerically simulate patterns at the water/air interface.

CHAPTER 4

CONCLUSION

In this thesis, we have provided a novel approach to prove the well-posedness of a water wave equation with viscosity and surface tension. Our goal was to extend this proof to the case of the Faraday wave experiment where the liquid is subject to a vertical forcing. We were able to do so in the case of small forcing but not in the case of large forcing which we expect to achieve in the future.

We then solved the equations numerically by using Field Expansions and a fourth order Runge-Kutta method. First we studied traveling waves solutions when the viscosity and the surface tension are set to zero. In this case, we can derive the exact solutions by hand. We built a model for the full problem and provided satisfactory approximations to the exact solutions in the two dimensional and the three dimensional cases. Finally, we looked at the Faraday wave experiment by adding vertical forcing to our equations. First, we transformed our equations into the traditional Mathieu equation which allowed us to use classical methods to solve this equation. We used the solutions obtained from the Mathieu equation and compared them to the ones obtained using our new method. We then computed an instability diagram for the equations with respect to the parameters F (forcing) and k (wavenumber). This is important as it tells us how to pick our parameters when we perform the Faraday wave experiment which is a goal for us.

APPENDICES

Appendix A

PROOF OF THE TRACE INEQUALITY

(Previously published as Ngom, M. and Nicholls, D. P.: Well-posedness and analyticity of solutions to a water wave problem with viscosity. J. Differential Equations, 265:5031-5075, 2018.)

The goal in this appendix is the proof of Lemma 2.2.3.

Proof. [Lemma 2.2.3] We begin by showing that

$$\|\sigma(x,0)\|_{H^{s-1}} \leq C_t \|\sigma\|_{X^s}. \tag{A.0.1}$$

Following (39) we specify a function $\rho(x,t)$, defined for $0 \leq x \leq 2\pi$ and $-\infty < t < \infty$, which agrees with σ for $0 \leq t \leq 1$ and vanishes for $t \leq -1$ and $t \geq 2$, such that

$$\begin{aligned}
M^2 &:= \int_{-1}^{2} \sum_{p=-\infty}^{\infty} \left[\langle p \rangle^{2s} \left|\hat{\rho}_p(u)\right|^2 + \langle p \rangle^{2s-4} \left|\partial_t \hat{\rho}_p(u)\right|^2 + \langle p \rangle^{2s-8} \left|\partial_t^2 \hat{\rho}_p(u)\right|^2 \right] du \\
&= \int_{-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left[\langle p \rangle^{2s} \left|\hat{\rho}_p(u)\right|^2 + \langle p \rangle^{2s-4} \left|\partial_t \hat{\rho}_p(u)\right|^2 + \langle p \rangle^{2s-8} \left|\partial_t^2 \hat{\rho}_p(u)\right|^2 \right] du \\
&= \int_{-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left\{ \langle p \rangle^{2s} + \langle \tau \rangle^2 \langle p \rangle^{2s-4} + \langle \tau \rangle^4 \langle p \rangle^{2s-8} \right\} \left|\tilde{\rho}_p(\tau)\right|^2 \, d\tau \\
&\leq C \|\sigma\|_{X^s}^2,
\end{aligned}$$

Appendix A (Continued)

where $\tilde{\rho}_p(\tau)$ is the space–time Fourier transform of ρ, and the penultimate equality comes from Parseval's relation. Since $\|\sigma(x,0)\|_{H^{s-1}} = \|\rho(x,0)\|_{H^{s-1}}$, to prove (Equation A.0.1) it suffices to show that

$$\|\rho(x,0)\|_{H^{s-1}} \le C_t M.$$

Now, by interpolation (2), we have

$$\|\rho(x,0)\|_{H^{s-1}}^2 \le C \left\{ \|\rho(x,0)\|_{L^2}^2 + \left\|\partial_x^{s-1}\rho(x,0)\right\|_{L^2}^2 \right\},$$

and, from the classical trace theorem (2), we can bound the right hand side to deliver

$$\begin{aligned} \|\rho(x,0)\|_{H^{s-1}}^2 &\le C \left\{ \|\rho\|_{H^{1/2}(dx,dt)}^2 + \left\|\partial_x^{s-1}\rho\right\|_{H^{1/2}(dx,dt)}^2 \right\} \\ &\le C \int_{-\infty}^{\infty} \sum_{p=-\infty}^{\infty} \left(1 + |p|^2 + |\tau|^2\right)^{1/2} \left(1 + |p|^{2(s-1)}\right) |\tilde{\rho}_p(\tau)|^2 \, d\tau, \end{aligned} \qquad \text{(A.0.2)}$$

Next, since $\sqrt{1+x^2} \le 1 + x$ for any $x > 0$, we have

$$\begin{aligned} \left[1 + |p|^2 + |\tau|^2\right]^{1/2} &\le 1 + \left[|p|^2 + |\tau|^2\right]^{1/2} = 1 + |p| \left[1 + \left(\frac{|\tau|}{|p|}\right)^2\right]^{1/2} \\ &\le 1 + |p| \left[1 + \frac{|\tau|}{|p|}\right] = 1 + |p| + |\tau|. \end{aligned}$$

Appendix A (Continued)

Thus we can conclude that

$$\begin{aligned}
\|\rho(x,0)\|^2_{H^{s-1}} &\le C\int_{-\infty}^{\infty}\sum_{p=-\infty}^{\infty}(1+|p|+|\tau|)\left(1+|p|^{2(s-1)}\right)|\tilde{\rho}_p(\tau)|^2\ d\tau \\
&= C\int_{-\infty}^{\infty}\sum_{p=-\infty}^{\infty}\left(1+|p|^{2s-2}+|p|+|p|^{2s-1}+|\tau|+|\tau|\,|p|^{2s-2}\right)|\tilde{\rho}_p(\tau)|^2\ d\tau.
\end{aligned}$$

Now, all of the terms on the right hand side will be bounded by M^2 provided, for $|p|\,,|\tau|>1$,

$1 \le C\,|p|^{2s}$	requires $s \ge 0$
$|p|^{2s-2} \le C\,|p|^{2s}$	requires $s \ge 0$
$|p| \le C\,|p|^{2s}$	requires $s \ge 1/2$
$|p|^{2s-1} \le C\,|p|^{2s}$	requires $s \ge 0$
$|\tau| \le C\,|\tau|^2\,|p|^{2s-4}$	requires $s \ge 2$,

and

$$|\tau|\,|p|^{2s-2} \le C\,|\tau|^2\,|p|^{2s-4}\,,$$

which requires more analysis. We note that

$$|\tau|\,|p|^{2s-2} = |\tau|\,|p|^a\,|p|^b \le \frac{1}{2}\left(|\tau|^2\,|p|^{2a}+|p|^{2b}\right),$$

Appendix A (Continued)

where $a+b=2s-2$. For our estimate we set $2a=2s-4$ which demands that $b=2s-2-a=2s-2-2s+4=2$. In light of this we have the estimate

$$|\tau|\,|p|^{2s-2} \leq \frac{1}{2}\left(|\tau|^2\,|p|^{2s-4}+|p|^4\right),$$

and we are done provided that $2s \geq 4$, or $s \geq 2$.

We now move to establishing

$$\|\partial_t\sigma(x,0)\|_{H^{s-3}} \leq C_t\,\|\sigma\|_{X^s}\,. \tag{A.0.3}$$

The proof is identical to that presented above save that we must bound

$$\left(1+|p|^2+|\tau|^2\right)^{1/2}\left(1+|p|^{2(s-3)}\right)|\tau|^2,$$

c.f., (Equation A.0.2). All of the terms on the right hand side will be bounded by M provided

$\|\tau\|^2 \leq C\,\|\tau\|^2\,\|p\|^{2s-4}$	requires $s \geq 2$
$\|\tau\|^2\,\|p\|^{2s-6} \leq C\,\|\tau\|^2\,\|p\|^{2s-4}$	requires $s \geq 0$
$\|\tau\|^2\,\|p\| \leq C\,\|\tau\|^2\,\|p\|^{2s-4}$	requires $s \geq 5/2$
$\|\tau\|^2\,\|p\|^{2s-5} \leq C\tau^2\,\|p\|^{2s-4}$	requires $s \geq 0$
$\|\tau\|^3 \leq C\,\|\tau\|^4\,\|p\|^{2s-8}$	requires $s \geq 4$,

Appendix A (Continued)

and

$$|\tau|^3 |p|^{2s-6} \leq C |\tau|^4 |p|^{2s-8},$$

which requires more analysis. We note that, from Hölder's Inequality,

$$|\tau|^3 |p|^{2s-6} = |\tau|^3 |p|^a |p|^b \leq \frac{3}{4}\left(|\tau|^3 |p|^a\right)^{4/3} + \frac{1}{4}\left(|p|^b\right)^4 = \frac{3}{4}|\tau|^4 |p|^{4a/3} + \frac{1}{4}|p|^{4b}$$

where $a + b = 2s - 6$. For our estimate we set $4a/3 = 2s - 8$, or $a = (3/2)s - 6$, which demands that $b = 2s - 6 - a = 2s - 6 - (3/2)s + 6 = (1/2)s$, or $4b = 2s$. In light of this we have the estimate

$$|\tau|^3 |p|^{2s-6} \leq \frac{3}{4}|\tau|^4 |p|^{2s-8} + \frac{1}{4}|p|^{2s}$$

and we are done provided that $s \geq 4$. $\square$

Appendix B

PRODUCTS OF ANALYTIC FUNCTIONS

(Previously published as Ngom, M. and Nicholls, D. P.: Well-posedness and analyticity of solutions to a water wave problem with viscosity. J. Differential Equations, 265:5031-5075, 2018.)

In this section we collect some identities involving the products of analytic functions in terms of their Taylor series. To begin let suppose that A, B, C, D are analytic functions of ε so that the following Taylor series are convergent

$$D = D(\varepsilon) = \sum_{n=1}^{\infty} D_n \varepsilon^n, \quad E = E(\varepsilon) = \sum_{n=1}^{\infty} E_n \varepsilon^n, \tag{B.0.1a}$$

$$F = F(\varepsilon) = \sum_{n=1}^{\infty} F_n \varepsilon^n, \quad G = G(\varepsilon) = \sum_{n=1}^{\infty} G_n \varepsilon^n. \tag{B.0.1b}$$

It is not difficult to see that

$$D(\varepsilon)E(\varepsilon) = \sum_{n=2}^{\infty} [\![DE]\!]_n \, \varepsilon^n, \quad [\![DE]\!]_n := \sum_{m=1}^{n-1} D_{n-m} E_m, \tag{B.0.2a}$$

and

$$D(\varepsilon)E(\varepsilon)F(\varepsilon) = \sum_{n=3}^{\infty} [\![DEF]\!]_n \, \varepsilon^n, \quad [\![DEF]\!]_n := \sum_{m=2}^{n-1} \sum_{\ell=1}^{m-1} D_{n-m} E_{m-\ell} F_\ell, \tag{B.0.2b}$$

Appendix B (Continued)

and

$$D(\varepsilon)E(\varepsilon)F(\varepsilon)G(\varepsilon) = \sum_{n=4}^{\infty} [\![DEFG]\!]_n \varepsilon^n, \quad [\![DEFG]\!]_n := \sum_{m=3}^{n-1} \sum_{\ell=1}^{m-1} \sum_{q=1}^{\ell-1} D_{n-m} E_{m-\ell} F_{\ell-q} G_q. \tag{B.0.2c}$$

For the results above to be true, the quantities $\{D, E, F, G\}$ need not be scalars and may be members of *any* normed linear space, Z. From these expansions we can prove the following fundamental result provided that the norm, $\|\cdot\|_Z$, satisfies the algebra property

$$\|DE\|_Z \leq M \|D\|_Z \|E\|_Z, \tag{B.0.3}$$

for some $M > 0$, e.g., the spaces H^s, X^s, and V^s provided that s is large large enough ($s \geq 4$ is certainly sufficient), c.f. Lemma 2.2.1.

Theorem B.0.1. *Suppose that $D, E, F, G \in Z$, a normed linear space with norm satisfying the algebra property* (Equation B.0.3)*. If D, E, F, G are all analytic in ε with Taylor series expansions* (Equation B.0.1) *such that*

$$\|D_n\|_Z < C_D \frac{B^{n-1}}{(n+1)^2}, \quad \|E_n\|_Z < C_E \frac{B^{n-1}}{(n+1)^2},$$
$$\|F_n\|_Z < C_F \frac{B^{n-1}}{(n+1)^2}, \quad \|G_n\|_Z < C_G \frac{B^{n-1}}{(n+1)^2},$$

Appendix B (Continued)

for constants $C_D, C_E, C_F, C_G, B > 0$*. Then* $DE, DEF, DEFG \in Z$ *are all analytic in* ε *as well, satisfying*

$$\|[\![DE]\!]_n\|_Z < C[D,E]\frac{B^{n-2}}{(n+1)^2}, \quad \|[\![DEF]\!]_n\|_Z < C[D,E,F]\frac{B^{n-3}}{(n+1)^2}, \tag{B.0.4a}$$

$$\|[\![DEFG]\!]_n\|_Z < C[D,E,F,G]\frac{B^{n-4}}{(n+1)^2}, \tag{B.0.4b}$$

where

$$C[D,E] = C_D C_E M\Sigma, \quad C[D,E,F] = C_D C_E C_F M^2\Sigma,$$

$$C[D,E,F,G] = C_D C_E C_F C_G M^3\Sigma,$$

c.f. Lemma 2.2.5.

Appendix B (Continued)

Proof. The proof is straightforward and we only present it for the final estimate (Equation B.0.4b). From (Equation B.0.2c) we have

$$\begin{aligned}
\|[\![DEFG]\!]_n\|_Z &\leq \sum_{m=3}^{n-1}\sum_{\ell=1}^{m-1}\sum_{q=1}^{\ell-1} \|D_{n-m}E_{m-\ell}F_{\ell-q}G_q\|_Z \\
&\leq \sum_{m=3}^{n-1}\sum_{\ell=1}^{m-1}\sum_{q=1}^{\ell-1} M^3 \|D_{n-m}\|_Z \|E_{m-\ell}\|_Z \|F_{\ell-q}\|_Z \|G_q\|_Z \\
&\leq \sum_{m=3}^{n-1}\sum_{\ell=1}^{m-1}\sum_{q=1}^{\ell-1} M^3 C_D \frac{B^{n-m-1}}{(n-m+1)^2} C_E \frac{B^{m-\ell-1}}{(m-\ell+1)^2} \\
&\quad \times C_F \frac{B^{\ell-q-1}}{(\ell-q+1)^2} C_G \frac{B^{q-1}}{(q+1)^2} \\
&\leq C_D C_E C_F C_G M^3 \frac{B^{n-4}}{(n+1)^2} \\
&\quad \times \sum_{m=3}^{n-1}\sum_{\ell=1}^{m-1}\sum_{q=1}^{\ell-1} \frac{(n+1)^2}{(n-m+1)^2(m-\ell+1)^2(\ell-q+1)^2(q+1)^2} \\
&\leq C_D C_E C_F C_G M^3 \Sigma \frac{B^{n-4}}{(n+1)^2},
\end{aligned}$$

from Lemma 2.2.5. The estimates (Equation B.0.4) can readily used in an inductive proof of the analyticity of all of the products DE, DEF, and $DEFG$. $\square$

Appendix C

PERMISSIONS FOR THE INCLUSION OF PUBLISHED WORKS

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VITA

Marième Ngom

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Education

2014-2019 **PhD in Applied Mathematics**, *University of Illinois at Chicago*, Chicago, Illinois, Studying Partial Differential Equations.
Minor: Statistics

2013-2014 **Master of Science in Mathematics**, *University of Paris-Sud*, Orsay, France, Specialized in Partial Differential Equations and Scientific Computing.

2010-2013 **Master of Engineering in Computer Science and Applied Mathematics**, *INP-ENSEEIHT & French Meteorology University*, Toulouse, France, Specialized in High Performance Computing and Forecast in Stochastic Dynamical Systems.

2007–2010 **Classes Préparatoires aux Grandes Écoles**, *Fénelon Sainte-Marie*, Paris, France, 3-years of intensive undergraduate courses to prepare competitive entrance to National Engineering Schools.
Specialized in Mathematics and Physics.

Computer Skills

Programming Languages: Matlab, Java, C, Python, Fortran, R, FreeFem++

Libraries: MPI, BLAS, LAPACK

Experience

2014-Now **Teaching Assistant**, University of Illinois at Chicago, Chicago, Illinois.
- Teaching and grading undergraduate level Mathematics classes (Precalculus, Calculus 1, Calculus for Business Majors, Introduction to Differential Equations, Statistical Theory, Numerical Analysis)

2014, April-August **Intern**, Electricité de France, Paris, France.
- Successfully implemented a code that aimed at reconstructing copper deposit layers on steam generator tubes by using signals from Eddy current coil.

2013, April-September **Research Intern**, Lawrence Berkeley National Laboratory/European Center for Research and Advanced Training in Scientific Computing, Berkeley, California/Toulouse, France.
- Established the stability of a block low-rank factorization and of the factorization of matrices having Hierarchically Semi-Separable representations,
- Tested the stability with Fortran codes in a parallel setting.

2012, June-August **Intern**, Numerical Optimization Group of the University of Florence, Florence, Italia.
- Studied preconditioners for sequences of diagonally modified linear systems as well as limited-memory preconditioners for large scale linear systems with multiple right-hand sides.

Publications

Marième Ngom and David P. Nicholls, *"Well-Posedness and Analyticity of Solutions to a Water Wave Problem with Viscosity"*, Journal of Differential Equations Volume 365, Issue 10, 5031-5065 (2018).

Alfredo Buttari, Serge Gratton, Xiaoye S. Li, Marième Ngom, François-Henry Rouet, David Titley-Peloquin and Clément Weisbecker, *"Error Analysis of the Block Low-Rank LU factorization of dense matrices"*, Technical Report, IRIT-CERFACS, RT-APO-13-7 (2013).

Talks

"Error Analysis of Block Low-Rank factorizations", *Argonne National Laboratory Seminar, Lemont, Il (December 2018).*

"Well-Posedness and Analyticity of Solutions to a Water Wave Problem with Viscosity", *Chicago area SIAM student chapter, Illinois Institute of Technology, Chicago, Il (April 2018)* .

"Numerical Weather Prediction", *Association for Women in Mathematics, University of Illinois at Chicago, Chicago, Il (March 2016).*

"Error Analysis of the Block Low-Rank LU factorization of dense matrices", *Low-rank Workshop, LSTC, Livermore, Ca (August 2013).*

Languages

Fluent in: French, English, Wolof