A Discontinuous Galerkin Method for the CEV Process

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M.S. (University of Illinois at Chicago) 2009

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics (Applied Mathematics)
in the Graduate College of the
University of Illinois at Chicago, 2013

Chicago, Illinois

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To my parents,

Philip & Susan Sward.

ACKNOWLEDGMENTS

"I don't understand why you buy eggs at seven cents a piece in Malta and sell them for five cents."
"I do it to make a profit."
"But how can you make a profit? You lose two cents an egg."
- Joseph Heller, *Catch-22*

I would like to thank my advisor, David Nicholls, and all of my committee members: Oleg Bondarenko, Charles Knessl, Irina Nenciu, and Jan Verschelde. This work would not have been possible without the support of my family and friends. A very special thank you to Jared Drewniak, Ingrid Larsson, Amanda Moore, Joshua Packard, Daniel Shea, Randall Stading, and Alan Ward. I would also like to thank my fellow young mathematicians whom I have worked with collaboratively over the past 6 years, including: Deniz Bilman, John Goes, Marc Kjerland, Hao Liang, Rachel Reyes, Jonathan Schnieder, and Miao Xu. I would also like to thank my high school calculus teacher, Robert Ali, who began my journey down the rabbit hole 11 years ago.

This thesis should not be seen as an endorsement of the current monetary system, but an exercise in mathematics. Stock markets are not wealth generators, but economic force-multipliers. True economic advancement is measured through scientific discoveries that lead to an improved standard of living for the entire human population. The future of mankind is not in the stars, the future of the stars is in mankind.

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LIST OF ABBREVIATIONS

CEV	Constant Elasticity of Variance
DG	Discontinuous-Galerkin
FMV	Fair Market Value
FDM	Finite Difference Method
FEM	Finite Element Method
ITM	In-the-Money
OTM	Out-of-the-Money
PDE	Partial Differential Equation
UIC	University of Illinois at Chicago

SUMMARY

This thesis considers the Black-Scholes and CEV financial models for underlying asset movement, and their associated PDEs for valuing options. A Discontinuous-Galerkin Method for approximating solutions to these PDEs is formulated in one and two dimensions. The focus in the one-dimensional case is on valuing the European and American Put option, with comparisons to the Binomial Method, Finite Difference Methods, and exact formulas in the case of the European option. In two dimensions, examination is placed on the weighted put basket option. A list of tables and figures are provided to display comparison and accuracy. Codes were written in Matlab.

CHAPTER 1

INTRODUCTION

1.1 The Study of Options

An option, sometimes referred to as a "derivative", is a financial contract that derives its value from an underlying asset. Often, the underlying asset is a stock (as will be studied in this thesis), but it may be other things, such as a future or an index. The commonly traded options for stocks are calls and puts, with American or European exercise rights.

A call option gives the buyer the right, but not the obligation, to purchase the underlying asset at a set price (called the strike price) on a set date (called the expiry date). If the option is American, the buyer may exercise this purchase at any time before the expiry, whereas a European option may only be exercised on the expiry date. If the buyer exercises the option, the asset is "called away" from the writer (seller) of the option. On the day of expiry, the value of the call option is equal to max(S - E, 0), where E is the strike price and S is the value of the asset on the day of expiry. If the underlying asset is trading above the strike price, the call option is said to be "in-the-money".

A put option gives the buyer the right, but not the obligation, to sell the underlying asset at the strike price on the expiry date. Once again, if the option is American, the holder of the option may exercise at any time before the expiry date. If the holder of the option exercises, the asset is "put" to the writer of the option. On the day of expiry, the value of the put option is equal to max(E - S, 0), with E being the strike and S being the value of the underlying asset on the day of expiry.

Typically, in real market situations, options at expiration are settled in cash for their value, and the underlying assets may never actually change ownership. This cash settlement behavior has lead to the development of "exotic" options such as: Asian options (where the option value is based on the average value of the underlying asset over the length of the contract), barrier options (where the underlying asset must reach one, or many, "barrier" prices in order for the option to activate or deactivate), digital options (which may pay a fixed amount of cash if the underlying is trading above or below a strike price). A barrier option is an example of a path-dependent option, where the payoff is not simply determined by the price of the underlying asset at expiry, but the path that was taken to reach that price.

One may also consider options with more than one underlying asset, so-called basket options. A basket option is typically a cheaper alternative for hedging multiple assets over a portfolio of individual calls and puts. If the basket components correlate negatively, movements in the value of one component neutralize opposite movements of another component. Basket options are popular for hedging foreign exchange risk, where each underlying may be priced in a different currency.

This thesis studies the valuation of vanilla put options with American and European exercise styles, as well as basket put options. It is well known that the value of an American call is the same as the value of a European call (a proof is provided in Appendix A), but this is not the case for put options. The early-exercise feature of an American Put leads to a free-boundary problem, tracking the location where it first becomes optimal to exercise early. The goal in studying options is to determine the "fair market value" of the option before the expiry date.

1.2 Underlying Assumptions and Models

An arbitrage is a trade (or series of trades) that requires zero cost and yields a risk-free profit. For example, if an asset can be purchased below market value and be immediately sold at (or above) market value with no risk, the trader has made a risk-free profit. For an option at the expiry date, the fair market value of the option must be equal to its payoff function in order to eliminate an arbitrage opportunity. The study of options is dedicated to determining the value of the option prior to the expiry date so that no arbitrage trades can be made.

It is standard practice to assume that a trader may borrow or save cash at a risk-free interest rate, r. This can be thought of as a bank account, but is generally taken to be the interest rate for a treasury bond. Whether or not a bond can really be considered "risk-free" is an open question. It is also assumed that trades can be conducted instantaneously, in any denomination with no transaction costs, and that there are always buyers and sellers for an asset at fair market value.

What is then needed is a mathematical model for the movement of the underlying asset. The most famous framework is to assume the asset price follows a lognormal distribution, or equivalently, to assume the asset undergoes Geometric Brownian Motion. This leads to the famous Black-Scholes PDE for valuing options. The lognormal approach assumes a constant volatility parameter, σ, which attempts to encapsulate the variation of the asset movement over time. A generalization of this approach is to assume that the volatility is proportional to the price of the asset to some power, this is the Constant Elasticity of Variance (CEV) model [7]. Quite a number of alternative models for asset movement have been considered, including Merton's Jump Diffusion Model (which attempts to account for randomly occurring "jumps" in the asset price), stochastic volatility models (the Heston model, among others), as well as models that incorporate a stochastic interest rate (see [6]). The Black-Scholes and CEV models are considered in this thesis.

1.3 Numerical Methods

The fun does not end here, however! Once an option type and model have been chosen, there is then the matter of pricing the option itself. Many numerical methods have been developed for this purpose, and entire books have been dedicated to this topic alone ([2], [3], [6]) . For European calls and puts, exact solutions are known for the Black-Scholes and CEV models ([6], [7]). Pricing American puts are more difficult, as there are no known exact formulas. If one is to take a PDE approach (as is done in this thesis) one can make use of approximation methods for PDEs, including Finite Difference Methods (FDM) ([3], [4]), Finite Element Methods (FEM) [3], Discontinous-Galerkin Methods (DG as is discussed in this thesis) [1] and asymptotic methods [10]. Numerical methods that do not rely on a PDE formulation include binomial and Monte Carlo Methods.

Binomial Methods involve the generation of an asset price binary tree, where each node in the tree has two children: One corresponding to a downward price movement with a certain probability, and the other corresponding to an upward price movement with the complement probability, and moving from one level to the next represents a small change in time. The final nodes of the tree (the bottom level) then correspond to "all" possible asset prices that can be obtained in the given time to expiry. The payoffs (and hence the values) of an option are thus known at the bottom of the tree, and one can work backwards through the tree using the risk-free probabilities and appropriate discounting of the interest rate in order to obtain the value of the option at any time prior to the expiry date. Binomial Methods are capable of pricing American options, since working backwards through the tree easily allows for the checking of early exercise opportunities. A Binomial Method for the CEV process was developed by Lu and Hsu [7] and is used in this thesis for comparing convergence to the DG Method.

Taking the PDE approach, there are a number of approximation methods that may be considered. The Finite Difference approach involves approximating the partial derivatives with first or second (or higher) order Taylor methods on a uniform mesh of the domain. FD Methods can be computationally expensive, and are not ideally suited for problems on geometrically complicated domains. The Finite Element Method is an improved technique that computes interpolation functions on entire subsets of the domain (known as elements), and utilizes information from adjacent elements to enforce conditions such as continuity between boundaries. FEMs are flexible enough to conform to complicated domains and accuracy can be increased both by refining the element size or increasing the order of the interpolating polynomials. An abstraction to the FEM approach is the class of Galerkin Methods, of which, the Discontinuous-Galerkin Method is considered here. As the name implies, DG does not enforce a continuity condition across elements, but transmits information from adjacent elements through the use of a flux term. This flux term can be chosen at implementation, often with respect to the inherent physical dynamics of the underlying process. In wave-like equations for example, an upwind flux term numerically simulates the direction of propagation of the wave.

DG Methods offer a variety of computational benefits. First, they allow much in the way of flexibility for element sizing and spacing. The elements need not be uniformly sized, and smaller elements can be grouped in locations of particular interest in the domain. For options valuation, a grouping of smaller elements localized near the strike price provides for increased accuracy in computing valuations in this region. Second, DGMs can be parallelized to greatly speed up computation times over traditional methods. DGMs also allow for tracking of the free-boundary for American options, and compute the option value on the entire stock-space, as opposed to the Binomial Method which computes a valuation for a single stock value only.

CHAPTER 2

PREVIOUS WORK

This thesis follows the framework developed by Hesthaven and Warburton [1] in formulation of the DG Method for parabolic equations. A detailed study of valuing options in the Black-Scholes setting is provided by Higham [2] as well as Wilmott, Howison and Dewyne [3], Hull [5] and Wilmott [6]. A study of the CEV process is provided by Wong and Zhao [8], Hsu, Lin and Lee in [13] and Wilmott [6] briefly gives formulas for valuing European options under CEV. Lu and Hsu [7] provide a binomial method for pricing options in the CEV environment. Knessl and Xu [10] provide asymptotic analysis for the American put under the CEV process. Glasserman [15] provides a Monte Carlo method for a specific choice of the exponent parameter in CEV. Barrier option types under CEV have been considered by Hu and Knessl as well as by Lo, Tang, Ku and Hui [12].

A useful reference on Finite Difference Methods is provided by Strikwerda [4], and Finite Element Methods (applied to options) by Achdon and Pironneau [11]. Cont, Lantos and Pironneau provide a Galerkin Method using Black-Scholes solutions as basis functions for CEV in [9], but do not provide an extension to basket options. Basket options under CEV are considered by Alexander and Venkatramanan [14] as well as by Xu and Zheng [16] for local volatility jump-diffusion models.

CHAPTER 3

THE CEV PROCESS

CEV stands for Constant Elasticity of Variance. It is a stochastic model used in finance for pricing options with an underlying stock price. The CEV model describes a process, $S(t)$, which evolves according to the following stochastic differential equation:

$$dS = \mu S dt + \sigma S^{\alpha} dW. \tag{3.1}$$

Here, μ is taken to be the risk free interest rate r, and σ is a constant volatility parameter, and W is a Wiener process. Note that for $\alpha = 1$ we reduce to simple Brownian motion, and the resulting PDE is the well-known Black-Scholes equation. The parameter α allows for the volatility to be a function of the stock price. For $\alpha < 1$, the volatility of a stock increases as its price decreases, and conversely for $\alpha > 1$ [7].

3.1 Itō's Lemma

Itō's lemma can be stated as follows: Given a stochastic differential equation of the form $dS = adt + bdW$ with $a = a(S,t)$ and $b = b(S,t)$, and a twice differentiable function, $V = f(S,t)$, then V itself is a function of the stochastic variable S, and its process is described by:

$$dV = (V_t + aV_S + \frac{1}{2}b^2 V_{SS})dt + (bV_S)dW. \tag{3.2}$$

3.2 The CEV PDE

Using Itō's lemma, taking V to be the value of an option, substituion of (3.1) into (3.2) yields:

$$dV = (V_t + \mu S V_s + \frac{1}{2}\sigma^2 S^{2\alpha} V_{SS})dt + (\sigma S^{\alpha} V_S)dW. \tag{3.3}$$

We set up a portfolio long one unit of the option and short Δ units of the stock:

$$\Pi = V - \Delta S,$$

the instantaneous change in the value of the portfolio is

$$d\Pi = dV - \Delta dS,$$

which becomes

$$d\Pi = (V_t + \mu(SV_S - \Delta S) + \frac{1}{2}\sigma^2 S^{2\alpha} V_{SS})dt + (\sigma S^{\alpha} V_S - \sigma S^{\alpha}\Delta)dW. \tag{3.4}$$

Setting $\Delta = V_S$, we find

$$d\Pi = (V_t + \frac{1}{2}\sigma^2 S^{2\alpha} V_{SS})dt. \tag{3.5}$$

This is a riskless change in the portfolio, hence its growth must be the same as an equivalent cash deposit at the risk free interest rate (otherwise there would be an arbitrage opportunity).

$$d\Pi = r\Pi dt.$$

Letting $\tau = T - t$, the backward-time CEV equation is then given by:

$$V_{\tau} - rSV_S - \frac{1}{2}\sigma^2 S^{2\alpha} V_{SS} + rV = 0. \tag{3.6}$$

With $\alpha = 1$ we arrive at the backward-time Black-Scholes equation:

$$V_\tau - rSV_S - \frac{1}{2}\sigma^2 S^2 V_{SS} + rV = 0. \tag{3.7}$$

Boundary data for a put option is given by: $V(0,t) = Ee^{-r\tau}$ and $V(\infty,t) = 0$, see [2, 3]. If the stock price goes to 0, the put will then pay the strike value, but discounted until the time to expiry. In approximation methods, a truncation of the boundary at a sufficiently large stock value is neceassry, and the put has value 0 at this artificial boundary. Since the equation is time-backward, the initial condition is given by the payoff function. For a put, we have $V(S,0) = \max(E - S, 0)$.

CHAPTER 4

FINITE DIFFERENCE METHODS

4.1 Time Forward Space Centered

We begin by considering a time forward space centered finite difference scheme of the PDE (3.6) for $t \in [0, T]$, $x \in [0, 2L]$:

$$\frac{V_j^{i+1} - V_j^i}{k} - \frac{1}{2}\sigma^2 (jh)^{2\alpha} \frac{V_{j+1}^i - 2V_j^i + V_{j-1}^i}{h^2} - rjh \frac{V_{j+1}^i - V_{j-1}^i}{2h} + rV_j^i = 0 \tag{4.1}$$

Here, $k = \dfrac{T}{N_T}$ represents the size of the time step, $h = \dfrac{2L}{N_x}$ represents the size of the spacial step, $t_i = ik$, $x_j = jh$ and V_j^i is the value of the option at time level i, spacial point j. Let $\mathbf{V}^i = (V_0^i, V_1^i, \cdots, V_{N_x}^i)^{\mathrm{T}}$ be the vector of options prices at time level i. Since this is an explicit scheme, we can solve for $\mathbf{V}^{i+1}$ in terms of $\mathbf{V}^i$ as follows:

$$\mathbf{V}^{i+1} = \mathrm{F}\mathbf{V}^i + \mathbf{p}^i. \tag{4.2}$$

Where $\mathrm{F} = (1 - rk)\mathrm{I} + \frac{1}{2}kr\mathrm{D}_1\mathrm{T}_1 + \frac{1}{2}k\sigma^2\mathrm{D}_2\mathrm{T}_2$,

$$D_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_x - 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} \frac{(1h)^{2\alpha}}{h^2} & 0 & \cdots & 0 \\ 0 & \frac{(2h)^{2\alpha}}{h^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{((N_x-1)h)^{2\alpha}}{h^2} \end{pmatrix}$$

$$T_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \ddots & 0 \\ 0 & -1 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ -1 & -2 & 1 & \ddots & 0 \\ 0 & -1 & -2 & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & -2 \end{pmatrix}$$

$$\mathbf{p}^i = \begin{pmatrix} \frac{1}{2}k(\sigma^2 \frac{h^{2\alpha}}{h^2} - r)V_0^i \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}k(N_x-1)(\sigma^2(N_x-1)^{2\alpha}\frac{h^{2\alpha}}{h^2} + r)V_{N_x}^i \end{pmatrix}.$$

4.2 Time Backward Space Centered

The formulation for the time backward space centered finite difference scheme is as follows:

$$\frac{V_j^{i+1} - V_j^i}{k} - \frac{1}{2}\sigma^2(jh)^{2\alpha}\frac{V_{j+1}^{i+1} - 2V_j^{i+1} + V_{j-1}^{i+1}}{h^2} - rjh\frac{V_{j+1}^{i+1} - V_{j-1}^{i+1}}{2h} + rV_j^{i+1} = 0. \tag{4.3}$$

This leads to the implicit scheme system of linear equations:

$$\mathrm{B}\mathbf{V}^{i+1} = \mathbf{V}^{i} + \mathbf{q}^{i}. \tag{4.4}$$

Where $\mathrm{B} = (1 + rk)\mathrm{I} - \frac{1}{2}kr\mathrm{D}_1\mathrm{T}_1 - \frac{1}{2}k\sigma^2\mathrm{D}_2\mathrm{T}_2$ and,

$$\mathbf{q}^{i} = \begin{pmatrix} \frac{1}{2}k(\sigma^2\frac{h^{2\alpha}}{h^2} - r)V_0^{i+1} \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2}k(N_x-1)(\sigma^2(N_x-1)^{2\alpha}\frac{h^{2\alpha}}{h^2} + r)V_{N_x}^{i+1} \end{pmatrix}.$$

CHAPTER 5

DG FORMULATION

5.1 Notation

Before discussing our DG method for pricing European and American puts for assets evolving under a CEV process, we introduce notation required for the specification of a DG method. We consider a domain Ω with boundary $\partial\Omega$. We approximate the domain with K nonoverlapping elements, D^k. The local inner product and $L^2(D^k)$ norm are:

$$(u,v)_{D^k} = \textstyle\int_{D^k} uv dx, \quad ||u||^2_{D^k} = (u,u)_{D^k}.$$

For the 1-D case, $D^k = [x_l^k, x_r^k]$. On each element we represent the solution as a polynomial of order $N = N_p - 1$. For $x \in D^k$, we have

$$u_h^k(x,t) = \sum_{n=1}^{N_p} \hat{u}_n^k(t)\psi_n(x) = \sum_{i=1}^{N_p} u_h^k(x_i^k,t)\ell_i^k(x).$$

The first is known as the modal form, where the ψ_n's are the local polynomial basis. The second is known as the nodal form, where the ℓ_i's are the interpolating Lagrange polynomials,

$$\ell_i(x) = \prod_{\substack{j=1 \\ j\neq i}}^{N_p} \frac{x - x_j}{x_i - x_j}.$$

Note: $\ell_i(x_j) = \delta_{i,j}$.

5.2 Reference Element

We map our intervals to the reference element $[-1,1]$ so that for $x \in D^k$ we have:

$$x(r) = x_l^k + \frac{1+r}{2}h^k, \quad h^k = x_r^k - x_l^k, \quad r \in [-1,1].$$

We can now choose our basis functions and gridpoints on the reference element, and map them to the physical grid.

5.3 A Choice of Basis Functions

For the modal basis functions, ψ_n's, a simple choice would be $\psi_n = r^{n-1}$. A better choice [1] would be an orthonormal basis, so that $(\psi_i, \psi_j) = \delta_{i,j}$. The benefit is that the modal mass matrix would then be the identity. Starting with r^n and applying Gram-Schmidt, we obtain:

$$\psi_n(r) = \tilde{P}_{n-1}(r) = \frac{P_{n-1}(r)}{\sqrt{\gamma_{n-1}}},$$

where $P_n(r)$ are the Legendre polynomials of order n and $\gamma_n = \dfrac{2}{2n+1}$.

To determine the modes, $\hat{u}_n$ from the nodal values, we assume our modal representation is interpolatory.

$$u(r_i) = \sum_{n=1}^{N_p} \hat{u}_n \tilde{P}_{n-1}(r_i),$$

for the gridpoints r_i. Then, in matrix form, we have:

$$\mathcal{V}\hat{\mathbf{u}} = \mathbf{u}, \quad \mathcal{V}^T \boldsymbol{\ell}(r) = \tilde{\mathbf{P}}(r),$$

where

$$\mathcal{V}_{i,j} = \tilde{P}_{j-1}(r_i), \quad \hat{\mathbf{u}}_\mathbf{i} = \hat{u}_i, \quad \mathbf{u}_\mathbf{i} = u(r_i),$$

and $\mathcal{V}$ is a generalized Vandermonde matrix.

5.4 Determining the grid points

We see that the choice of grid points will define the Vandermonde matrix. We would like $\mathcal{V}$ to be well-conditioned. This will amount to maximizing $\mathrm{Det}(\mathcal{V})$ (see [1]). The choice of grid points that maximize this determinant are the N_p zeros of:

$$f(r) = (1 - r^2)\tilde{P}'_N(r).$$

These are known as the Legendre-Gauss-Lobatto (LGL) quadrature points.

The local (nodal) mass matrix on the reference element is now given by:

$$\mathcal{M}_{i,j} = (\ell_i(r), \ell_j(r))_{[-1,1]} = (\mathcal{V}\mathcal{V}^T)^{-1}_{i,j}.$$

By transforming back to the element D^k we have:

$$\mathcal{M}^k_{i,j} = \frac{h^k}{2}\mathcal{M}_{i,j}.$$

The local (nodal) stiffness matrix on the reference element is given by:

$$\mathcal{S}_{i,j} = (\ell_i(r), \frac{d\ell_j}{dr})_{[-1,1]} = (\ell_i(x), \frac{d\ell_i}{dx})_{D^k} = \mathcal{S}^k_{i,j}.$$

If we define

$$\mathcal{D}_{r,(i,j)} = \left.\frac{d\ell_j}{dr}\right|_{r_i},$$

then $\mathcal{M}\mathcal{D}_r = \mathcal{S}$ and $\mathcal{D}_r = \mathcal{V}_r\mathcal{V}^{-1}$ where $\mathcal{V}_{r,(i,j)} = \left.\frac{d\tilde{P}_j}{dr}\right|_{r_i}$.

5.5 Timestepping and Flux Term

We use the standard fourth-order explicit Runge-Kutta method for timestepping [17]. Explicit timestepping, in particular, makes American option valuation easy. We simply enforce at each time step that the value of the option is $V(i,j) = \max(V(i,j), \max(E - S(i,j), 0))$. We can also keep track of the free boundary in this way.

DG Methods involve the use of a flux term across element interfaces. The flux can be chosen appropriately to model the physics of the problem, e.g. "upwinding" for wave-like equations. For the heat equation, which is naturally dissipative and has no preferred direction of propagation, one generally uses a central flux, but other fluxes can be considered [1]. Since the Black-Scholes PDE is heatlike, when transferring information across element interfaces, we make use of the central flux:

$$u_h^* = \{\{u_h\}\} = \frac{u^- + u^+}{2},$$

where u^- refers to the interior information of the element and u^+ refers to the exterior information. In the multidimensional setting, u will be a vector.

5.6 A DG Method for the Black-Scholes PDE

We begin by re-writing the Black-Scholes PDE (3.7) as follows:

$$V_\tau = \frac{\partial}{\partial S}(\frac{1}{2}\sigma^2 S^2 V_S) + \frac{\partial}{\partial S}((rS - \sigma^2 S)V) + (\sigma^2 - 2r)V. \tag{5.1}$$

Following Hesthaven and Warburton [1], we rewrite this second order PDE as a system of two first order PDEs. Letting $q = SV_S$ we have:

$$V_\tau = \frac{\partial}{\partial S}(\frac{1}{2}\sigma^2 Sq) + (r - \sigma^2)q - rV. \tag{5.2}$$

We assume V and q can be approximated as:

$$\begin{bmatrix} V(S,t) \\ \\ q(S,t) \end{bmatrix} \simeq \begin{bmatrix} V_h(S,t) \\ \\ q_h(S,t) \end{bmatrix} = \bigoplus_{k=1}^K \begin{bmatrix} V_h^k(S,t) \\ \\ q_h^k(S,t) \end{bmatrix} = \bigoplus_{k=1}^K \sum_{i=1}^{N_p} \begin{bmatrix} V_h^k(S_i,t) \\ \\ q_h^k(S_i,t) \end{bmatrix} \ell_i^k(S).$$

We are representing V and q by $(N_p - 1)$-order piecewise polynomials on K elements. We form the residual for the Black-Scholes equation:

$$R_h(S,t) = \frac{\partial V_h}{\partial \tau} - \frac{\partial}{\partial S}(\frac{1}{2}\sigma^2 Sq_h) - (r - \sigma^2)q_h + rV_h, \tag{5.3}$$

and, following the classical Galerkin approach, require that the residual is orthogonal to our nodal basis functions represented as the interpolating Lagrange polynomials $\ell_i^k(S)$:

$$\int_{D^k} R_h \ell_i dS = 0, \qquad 1 \leq i \leq N_p, \tag{5.4}$$

Integration by parts yields:

$$\int_{D^k} (\frac{\partial V_h^k}{\partial \tau} - (r-\sigma^2)q_h^k + rV_h^k)\ell_i dS + \int_{D^k} \frac{1}{2}\sigma^2 S q_h^k \frac{d\ell_i}{dx} dS = \int_{\partial D^k} \hat{\mathbf{n}} \cdot \frac{1}{2}\sigma^2 S q_h^k \ell_i dS. \tag{5.5}$$

Substitution of the nodal forms yields the "weak form" [1]:

$$\mathbf{M}^k(\frac{d}{d\tau}\mathbf{V}_h^k - (r-\sigma^2)\mathbf{q}_h^k + r\mathbf{V}_h^k) = -\mathbf{S}^T(\frac{1}{2}\sigma^2\mathbf{q}_h^k) + (\frac{1}{2}\sigma^2 q_h)^* \ell_i|_{S_\ell^k}^{S_r^k}, \tag{5.6}$$

$$\mathbf{M}^k \mathbf{q}_h^k = -\tilde{\mathbf{S}}^T \mathbf{V}_h^k + \int_{\partial D^k} \hat{\mathbf{n}} \cdot (SV_h^k)^* \ell^k dS. \tag{5.7}$$

The strong form given by:

$$\mathbf{M}^k(\frac{d}{d\tau}\mathbf{V}_h^k - (r-\sigma^2)\mathbf{q}_h^k + r\mathbf{V}_h^k) = \tilde{\mathbf{S}}(\frac{1}{2}\sigma^2\mathbf{q}_h^k) - \int_{\partial D^k} \hat{\mathbf{n}} \cdot ((\frac{1}{2}\sigma^2 q_h^k) - (\frac{1}{2}\sigma^2 q_h^k)^*)\ell^k dS \tag{5.8}$$

$$\mathbf{M}^k \mathbf{q}_h^k = \mathbf{S}\mathbf{V}_h^k - \int_{\partial D^k} \hat{\mathbf{n}} \cdot ((SV_h^k) - (SV_h^k)^*)\ell^k dS \tag{5.9}$$

Where $\mathbf{M}_{i,j}^k = (\ell_i, \ell_j)$, $\quad \mathbf{S}_{i,j} = (S\ell_i^k, \frac{d\ell_j^k}{dS})$, $\quad \tilde{\mathbf{S}}_{i,j} = (\ell_i^k, \frac{dS\ell_j^k}{dS})$.

5.7 A DG Method for the Generalized (CEV) Black-Scholes PDE

We proceed in a similar fashion to the previous section to derive the DG formulation for the generalized Black-Scholes PDE for puts under the CEV process. We rewrite the PDE (3.6) as follows:

$$V_\tau = \frac{\partial}{\partial S}(\frac{1}{2}\sigma^2 S^{2\alpha} V_S) + \frac{\partial}{\partial S}((rS - \sigma^2 \alpha S^{2\alpha-1})V) + (\sigma^2\alpha(2\alpha-1)S^{2\alpha-2} - 2r)V. \tag{5.10}$$

Again, we rewrite this as a system of two first order PDEs by letting $q = S^\alpha V_S$, and re-write as:

$$V_\tau = \frac{\partial}{\partial S}(\frac{1}{2}\sigma^2 S^\alpha q) + (rS^{1-\alpha} - \sigma^2\alpha S^{\alpha-1})q - rV. \tag{5.11}$$

We form the residual for this PDE as:

$$R_h(S,t) = \frac{\partial V_h}{\partial \tau} - \frac{\partial}{\partial S}(\frac{1}{2}\sigma^2 S^\alpha q_h) - (rS^{1-\alpha} - \sigma^2\alpha S^{\alpha-1})q_h + rV_h. \tag{5.12}$$

The orthogonality condition (5.4) together with integration by parts yields:

$$\int_{D^k}(\frac{\partial V_h^k}{\partial \tau} - (rS^{1-\alpha} - \sigma^2\alpha S^{\alpha-1})q_h^k + rV_h^k)\ell_i dS + \int_{D^k}\frac{1}{2}\sigma^2 S^\alpha q_h^k \frac{d\ell_i}{dx} dS = \int_{\partial D^k} \hat{\mathbf{n}} \cdot \frac{1}{2}\sigma^2 S^\alpha q_h^k \ell_i dS. \tag{5.13}$$

Substitution of the nodal form yields the weak form:

$$\mathbf{M}^k(\frac{d}{d\tau}\mathbf{V}_h^k - \mathbf{A}\mathbf{q}_h^k + r\mathbf{V}_h^k) = -\mathbf{S}^T(\frac{1}{2}\sigma^2\mathbf{q}_h^k) + (\frac{1}{2}\sigma^2 q_h)^*\ell_i|_{S_\ell^k}^{S_r^k}, \tag{5.14}$$

$$\mathbf{M}^k\mathbf{q}_h^k = -\tilde{\mathbf{S}}^T\mathbf{V}_h^k + \int_{\partial D^k} \hat{\mathbf{n}} \cdot (S^\alpha V_h^k)^* \ell^k dS. \tag{5.15}$$

The strong form given by:

$$\mathbf{M}^k(\frac{d}{d\tau}\mathbf{V}_h^k - \mathbf{A}\mathbf{q}_h^k + r\mathbf{V}_h^k) = \tilde{\mathbf{S}}(\frac{1}{2}\sigma^2\mathbf{q}_h^k) - \int_{\partial D^k} \hat{\mathbf{n}} \cdot ((\frac{1}{2}\sigma^2 q_h^k) - (\frac{1}{2}\sigma^2 q_h^k)^*)\ell^k dS, \tag{5.16}$$

$$\mathbf{M}^k\mathbf{q}_h^k = \mathbf{S}\mathbf{V}_h^k - \int_{\partial D^k} \hat{\mathbf{n}} \cdot ((S^\alpha V_h^k) - (S^\alpha V_h^k)^*)\ell^k dS. \tag{5.17}$$

Again $\mathbf{M}_{i,j}^k = (\ell_i, \ell_j)$, $\quad \mathbf{S}_{i,j} = (S^\alpha \ell_i^k, \frac{d\ell_j^k}{dS})$, $\quad \tilde{\mathbf{S}}_{i,j} = (\ell_i^k, \frac{dS^\alpha \ell_j^k}{dS})$ $\quad \mathbf{A}_{i,j} = rS_{i,j}^{1-\alpha} - \sigma^2\alpha S_{i,j}^{\alpha-1}$.

CHAPTER 6

MULTIDIMENSIONAL EQUATIONS

6.1 Basket Options

A basket option is an option with more than one underlying asset. There are many flavors of basket options, even in a "vanilla" setting, and we consider a put option whose payoff is equal to a fixed strike price less a weighted sum of the stock prices. The weights may be considered as the quantity of each stock in the basket:

$$V(\mathbf{S}, T) = \max(K - \sum_{i=1}^{n} w_i S_i, 0).$$

For boundary conditions, if any asset reaches zero value, we discount the remaining payoff function. If any asset approaches infinity, the basket put approaches zero.

6.2 Multidimensional Itō's Lemma

Itō's lemma for processes with constant drift in many dimensions can be stated as follows: Given a system of stochastic differential equations of the form $dS_i = \mu_i S_i dt + g_i(t, S_i) dW_i$ where $\mathbf{W}$ are n-dimensional correlated Brownianian motions so that:

$E[dW_i] = 0, \quad E[dW_i^2] = dt, \quad E[dW_i dW_j] = \rho_{i,j},$

and $V = f(\mathbf{S}, t)$ is a sufficiently differentiable function on $\mathbb{R}^{n+1} \to \mathbb{R}$, then

$$dV = [\frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} \mu_i S_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial V}{\partial S_i S_j} \rho_{i,j} g_i g_j] dt + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} g_i dW_i. \tag{6.1}$$

6.3 Multidimensional Black-Scholes

For valuing options with many stocks as the underlying in a Black-Scholes setting, we have: $dS_i = \mu_i S_i dt + \sigma_i S_i dW_i$. Letting V be the value of the option, substitution into (6.1) yields:

$$dV = [\frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i}\mu_i S_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial V}{\partial S_i S_j}\rho_{i,j}\sigma_i\sigma_j S_i S_j]dt + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i}\sigma_i S_i dW_i. \tag{6.2}$$

We then set up a portfolio Π long one unit of the option and short Δ_i units of the i-th stock, so that $\Pi = V - \sum_{i=1}^{n} \Delta_i S_i$. We now have:

$$d\Pi = dV - \sum_{i=1}^{n} \Delta_i dS_i, \tag{6.3}$$

leading to

$$d\Pi = [\frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i}\mu_i S_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial V}{\partial S_i S_j}\rho_{i,j}\sigma_i\sigma_j S_i S_j - \sum_{i=1}^{n}\mu_i\Delta_i S_i]dt \\ + [\sum_{i=1}^{n} \frac{\partial V}{\partial S_i}\sigma_i S_i - \sum_{i=1}^{n}\sigma_i\Delta_i S_i]dW_i. \tag{6.4}$$

We can eliminate the risk of the portfolio by setting $\Delta_i = \frac{\partial V}{\partial S_i}$. Noting that a risk-free portfolio provides the same rate of return as the risk-free interest rate r, $d\Pi = r\Pi dt$. Hence,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial V}{\partial S_i S_j}\rho_{i,j}\sigma_i\sigma_j S_i S_j = rV - r\sum_{i=1}^{n} \frac{\partial V}{\partial S_i} S_i. \tag{6.5}$$

Taking τ = T - t, we arrive at the backward time multidimensional Black-Scholes equation:

$$\frac{\partial V}{\partial \tau} - r\sum_{i=1}^{n}\frac{\partial V}{\partial S_i}S_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial V}{\partial S_i S_j}\rho_{i,j}\sigma_i\sigma_j S_i S_j + rV = 0. \tag{6.6}$$

6.4 Multidimensional Generalized Black-Scholes under the CEV Process

Consider valuing options derived from many stocks evolving under CEV processes governed by the SDEs:

$$dS_i = \mu_i S_i dt + \sigma_i S_i^{\alpha_i} dW_i.$$

Letting V be the value of the option, substitution into (6.1) yields:

$$dV = [\frac{\partial V}{\partial t} + \sum_{i=1}^{n}\frac{\partial V}{\partial S_i}\mu_i S_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial V}{\partial S_i S_j}\rho_{i,j}\sigma_i\sigma_j S_i^{\alpha_i} S_j^{\alpha_j}]dt + \sum_{i=1}^{n}\frac{\partial V}{\partial S_i}\sigma_i S_i dW_i. \tag{6.7}$$

The multidimensional CEV PDE is then derived in a similar manner as the classical Black-Scholes equation, leading to the time-backward equation:

$$\frac{\partial V}{\partial \tau} - r\sum_{i=1}^{n}\frac{\partial V}{\partial S_i}S_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial V}{\partial S_i S_j}\rho_{i,j}\sigma_i\sigma_j S_i^{\alpha_i} S_j^{\alpha_j} + rV = 0. \tag{6.8}$$

CHAPTER 7

TWO DIMENSIONAL DG FORMULATION

As in Chapter 5, some notation is required to specify our DG method in higher dimensions. We consider a two dimensional spacial domain Ω, which can be triangulated using K elements, so that:

$$\Omega \simeq \Omega_h = \bigcup_{k=1}^{K} D^k.$$

The approximation on each element is given by:

$$u_h^k(\mathbf{x}, t) = \sum_{i=1}^{N_p} u_h^k(\mathbf{x}_i^k, t)\ell_i^k(\mathbf{x}) = \sum_{n=1}^{N_p} \hat{u}_n^k(t)\psi_n(\mathbf{x}),$$

where $\ell_i(\mathbf{x})$ is the multidimensional Lagrange polynomial defined by gridpoints $\mathbf{x}_i$ on the element D^k, and $\{\psi_n(\mathbf{x})\}_{n=1}^{N_p}$ is a two dimensional polynomial basis of order N. Here, N_p is the number of terms in the local expansion for a polynomial of order N in two variables:

$$N_p = \frac{(N+1)(N+2)}{2}.$$

7.1 Reference Element

Calculations are done on a reference triangle,

$$\mathbf{T} = \{\mathbf{r} = (r, s) | (r, s) \geq -1, r + s \leq 0\},$$

and if element D^k has three vertices, $(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3)$, then for $\mathbf{x} \in D^k$ we have:

$$\mathbf{x} = -\frac{r+s}{2}\mathbf{v}^1 + \frac{r+1}{2}\mathbf{v}^2 + \frac{s+1}{s}\mathbf{v}^3.$$

7.2 Basis Functions and Grid Points

The orthonormal basis functions computed by the Gram-Schmidt procedure on the canonical basis $r^i s^j$ are given by:

$$\psi_m(\mathbf{r}) = \sqrt{2}P_i(a)P_j^{(2i+1,0)}(b)(1-b)^i, \ \ (i,j) \geq 0; \, i+j \leq N, \text{ with}$$
$$a = 2\frac{1+r}{1-s} - 1, \quad b = s$$

where $P_n^{(\alpha,\beta)}(x)$ is the n-th order Jacobi Polynomial.

As in the 1-D case, we seek to avoid a mesh of equidistant grid points within our reference triangle. We follow the example of Hesthaven and Warburton [1] in the construction of a warp function that maps an equidistant mesh to a mesh more suited for computation. Consider the function:

$$w(r) = \frac{\sum_{i=1}^{N_p} (r_i^{LGL} - r_i^e)\ell_i^e(r)}{1-r^2}.$$

The numerator transforms equidistant points to the Legendre-Gauss-Lobatto points in 1-D. Starting from an equilateral triangle with vertices $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ and utilizing barycentric coordinates, we establish blending and warping functions b_1, b_2, b_3 and $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ which warp along the corresponding edges $\mathbf{v}_1\mathbf{v}_2$, $\mathbf{v}_2\mathbf{v}_3$, and $\mathbf{v}_3\mathbf{v}_2$, and blend these points along the corresponding inward normals $(1,0), (-1,\sqrt{3})$, and$(-1,-\sqrt{3})$.

$$\mathbf{w}_1 = w(\lambda_3 - \lambda_2)\mathbf{n}_1, \quad \mathbf{w}_2 = \frac{1}{2}w(\lambda_1 - \lambda_3)\mathbf{n}_2, \quad \mathbf{w}_3 = \frac{1}{2}w(\lambda_2 - \lambda_1)\mathbf{n}_3,$$
$$b_1 = 4\lambda_3\lambda_2, \qquad b_2 = 4\lambda_3\lambda_1, \qquad b_3 = 4\lambda_2\lambda_1,$$

where $(\lambda_1, \lambda_3) = (\frac{i}{N}, \frac{j}{N})$, for $(i,j) \geq 0, i+j \leq N$, and $\lambda_2 = 1 - \lambda_1 - \lambda_3$.

We then form the two-dimensional transformation function: $\mathbf{w} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + b_3\mathbf{w}_3$.

7.3 A DG Method for the Black-Scholes PDE

In two dimensions, the backward-time Black-Scholes equation is:

$$V_\tau = \frac{1}{2}(\sigma_1^2 S_1^2 V_{S_1S_1} + 2\sigma_1\sigma_2\rho_{12}S_1S_2V_{S_1S_2} + \sigma_2^2 S_2^2 V_{S_2S_2}) + rS_1V_{S_1} + rS_2V_{S_2} - rV,$$

and we once again rewrite this as a system of PDEs. Letting $q = (S_1V_{S_1}, S_2V_{S_2})^T$,

we set $Q = \text{div}(\begin{pmatrix} \sigma_1^2 S_1 & \sigma_1\sigma_2\rho_{12}S_1 \\ \sigma_1\sigma_2\rho_{12}S_2 & \sigma_2^2 S_2 \end{pmatrix} \cdot q) = \text{div}\ \mathbf{S}.\tilde{Q}$ so that:

$$V_\tau = \frac{1}{2}Q - (\sigma_1^2 + \frac{1}{2}\sigma_1\sigma_2\rho_{12} - r)S_1V_{S_1} - (\sigma_2^2 + \frac{1}{2}\sigma_1\sigma_2\rho_{12} - r)S_2V_{S_2} - rV,$$

$$V_\tau = \frac{1}{2}Q - (\sigma_1^2 + \frac{1}{2}\sigma_1\sigma_2\rho_{12} - r)q_1 - (\sigma_2^2 + \frac{1}{2}\sigma_1\sigma_2\rho_{12} - r)q_2 - rV. \tag{7.1}$$

We may now make use of the divergence theorem to obtain our DG-Formulation:

$$\mathbf{M}^k(\frac{d}{d\tau}\mathbf{V}_h^k - \boldsymbol{\sigma}\cdot\mathbf{q}_h^k + r\mathbf{V}_h^k) = \frac{1}{2}\mathbf{S_1}\tilde{Q_1}_h^k + \frac{1}{2}\mathbf{S_2}\tilde{Q_2}_h^k - \int_{\partial D^k}\hat{\mathbf{n}}\cdot((\tilde{\mathbf{Q}}_h^k) - (\tilde{\mathbf{Q}}_h^k)^*)\boldsymbol{\ell}^k d\mathbf{S} \tag{7.2}$$

$$\mathbf{M}^k\mathbf{q_1}_h^k = \mathbf{S_1}\mathbf{V}_h^k - \int_{\partial D^k}\hat{n}_1\cdot((S_1\mathbf{V}_h^k) - (S_1\mathbf{V}_h^k)^*)\boldsymbol{\ell}^k d\mathbf{S} \tag{7.3}$$

$$\mathbf{M}^k\mathbf{q_2}_h^k = \mathbf{S_2}\mathbf{V}_h^k - \int_{\partial D^k}\hat{n}_2\cdot((S_2\mathbf{V}_h^k) - (S_2\mathbf{V}_h^k)^*)\boldsymbol{\ell}^k d\mathbf{S} \tag{7.4}$$

Where $\mathbf{M}_{i,j}^k = (\ell_i, \ell_j)$, $\mathbf{S_1}_{i,j} = (S_1\ell_i^k, \frac{\partial\ell_j^k}{\partial S_1})$, $\mathbf{S_2}_{i,j} = (S_2\ell_i^k, \frac{\partial\ell_j^k}{\partial S_2})$,

$\tilde{\mathbf{Q}} = (\sigma_1^2 q_1 + \sigma_1\sigma_2\rho_{12}q_2, \sigma_2^2 q_2 + \sigma_1\sigma_2\rho_{12}q_1)$,

$\boldsymbol{\sigma} = (\sigma_1^2 + \frac{1}{2}\sigma_1\sigma_2\rho_{12} - r, \sigma_2^2 + \frac{1}{2}\sigma_1\sigma_2\rho_{12} - r)$.

In the above, all are inner products except $\boldsymbol{\sigma}$ and $\tilde{\mathbf{Q}}$, which are vectors.

7.4 A DG Method for the Generalized (CEV) Black-Scholes PDE

In two dimensions, the backward-time CEV equation is:

$$V_\tau = \frac{1}{2}(\sigma_1^2 S_1^{2\alpha_1} V_{S_1S_1} + 2\sigma_1\sigma_2\rho_{12} S_1^{\alpha_1} S_2^{\alpha_2} V_{S_1S_2} + \sigma_2^2 S_2^{2\alpha_2} V_{S_2S_2}) + rS_1 V_{S_1} + rS_2 V_{S_2} - rV.$$

In a similar manner as before, we let $q = (S_1^{\alpha_1} V_{S_1}, S_2^{\alpha_2} V_{S_2})^T$

with $Q = \mathrm{div}(\begin{pmatrix} \sigma_1^2 S_1^{\alpha_1} & \sigma_1\sigma_2\rho_{12} S_1^{\alpha_1} \\ \sigma_1\sigma_2\rho_{12} S_2^{\alpha_2} & \sigma_2^2 S_2^{\alpha_2} \end{pmatrix} \cdot q) = \mathrm{div}\ \mathbf{S}^{\boldsymbol{\alpha}}.\tilde{Q}$ so that:

$$V_\tau = \frac{1}{2}Q - (\alpha_1\sigma_1^2 S_1^{\alpha_1-1} + \frac{1}{2}\sigma_1\sigma_2\rho_{12}\alpha_2 S_2^{\alpha_2-1} - rS_1^{1-\alpha_1})q_1$$

$$-(\alpha_2\sigma_2^2 S_2^{\alpha_2-1} + \tfrac{1}{2}\sigma_1\sigma_2\rho_{12}\alpha_2 S_1^{\alpha_1-1} - rS_2^{1-\alpha_2})q_2 - rV.$$

The strong form DG-Formulation for the two dimensional generalized Black-Scholes equation under a CEV process is then:

$$\mathbf{M}^k(\frac{d}{d\tau}\mathbf{V}_h^k - \mathbf{A}\cdot\mathbf{q}_h^k + r\mathbf{V}_h^k) = \frac{1}{2}\tilde{\mathbf{S}}_{\alpha_1}\tilde{\mathbf{Q}}_{\mathbf{1}h}^{\ k} + \frac{1}{2}\tilde{\mathbf{S}}_{\alpha_2}\tilde{\mathbf{Q}}_{\mathbf{2}h}^{\ k} - \int_{\partial D^k} \hat{\mathbf{n}}\cdot((\tilde{\mathbf{Q}}_h^k) - (\tilde{\mathbf{Q}}_h^k)^*)\boldsymbol{\ell}^k d\mathbf{S} \quad (7.5)$$

$$\mathbf{M}^k \mathbf{q_1}_h^k = \mathbf{S}_{\alpha_1}\mathbf{V}_h^k - \int_{\partial D^k} \hat{n}_1\cdot((S_1^{\alpha_1}\mathbf{V}_h^k) - (S_1^{\alpha_1}\mathbf{V}_h^k)^*)\boldsymbol{\ell}^k d\mathbf{S} \quad (7.6)$$

$$\mathbf{M}^k \mathbf{q_2}_h^k = \mathbf{S}_{\alpha_2}\mathbf{V}_h^k - \int_{\partial D^k} \hat{n}_2\cdot((S_2^{\alpha_2}\mathbf{V}_h^k) - (S_2^{\alpha_2}\mathbf{V}_h^k)^*)\boldsymbol{\ell}^k d\mathbf{S} \quad (7.7)$$

Where $\mathbf{M}_{i,j}^k = (\ell_i, \ell_j)$, $\mathbf{S}_{\boldsymbol{\alpha_1} i,j} = (S_1^{\alpha_1}\ell_i^k, \frac{\partial \ell_j^k}{\partial S_1})$, $\mathbf{S}_{\boldsymbol{\alpha_2} i,j} = (S_2^{\alpha_2}\ell_i^k, \frac{\partial \ell_j^k}{\partial S_2})$, $\tilde{\mathbf{S}}_{\alpha_1,i,j} = (\ell_i^k, \frac{\partial S_1^{\alpha_1}\ell_j^k}{\partial S_1})$
$\tilde{\mathbf{S}}_{\alpha_2,i,j} = (\ell_i^k, \frac{\partial S_2^{\alpha_2}\ell_j^k}{\partial S_2})$,

$\mathbf{A} = (\alpha_1\sigma_1^2 S_1^{\alpha_1-1} + \frac{1}{2}\sigma_1\sigma_2\rho_{12}\alpha_2 S_2^{\alpha_2-1} - rS_1^{1-\alpha_1}, \alpha_2\sigma_2^2 S_2^{\alpha_2-1} + \frac{1}{2}\sigma_1\sigma_2\rho_{12}\alpha_1 S_1^{\alpha_1-1} - rS_2^{1-\alpha_2})$.

CHAPTER 8

NUMERICAL RESULTS

In this section we present results of numerical experiments of pricing European and American puts for different values of α in the CEV model. In these tables, N is the polynomial order, K is the number of elements used, and E is the strike price. Figures for convergence rates of the DG Method for these same values of α are provided. Simulations were done on a stock space from $[0, 20]$, risk-free interest rate $r = .05$, time to expiry $T = .5$. The choice of parameters was taken from Wong and Zhao [8] where they list tables for $\alpha = 0$ and $\alpha = \frac{2}{3}$ on strike prices of 90, 100, and 110. The grid has been re-scaled in this thesis by a factor of $\frac{1}{10}$, and the case for $\alpha = \frac{1}{2}$ has been added.

Exact solutions for European puts are given in [5] where $\alpha = 1$ has, of course, the classical solution found by Black and Scholes.

$\alpha = 1 : V = -SN(-d_1) + Ee^{-r\tau}N(-d_2)$,
where $d_1 = \dfrac{log(\frac{S}{E}) + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$, $\quad d_2 = \dfrac{log(\frac{S}{E}) + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$, and $N(x)$ is the cumulative distribution function for the standardized Normal distribution.

$\alpha \neq 1 : V = Ee^{-rT}[1 - \chi^2(c, b, a)] - S\chi^2(a, b + 2, c)$,
where $a = \dfrac{[Ee^{-rT}]^{2(1-\alpha)}}{(1-\alpha)^2\nu}$, $\quad b = \dfrac{1}{1-\alpha}$, $\quad c = \dfrac{S^{2(1-\alpha)}}{(1-\alpha)^2\nu}$ $\quad \nu = \dfrac{\sigma^2}{2r(\alpha - 1)}[e^{2r(\alpha-1)T} - 1]$, and $\chi^2(z, k, \nu)$ is the cumulative probability that a variable with a noncentral χ^2 distribution with noncentrality parameter ν and k degrees of freedom is less than z (see [5]).

$\alpha = 0$	European			American		
	E=9	E=10	E=11	E=9	E=10	E=11
N=2, K=40	0.1468	0.4423	0.9955	0.1558	0.4673	1.0733
N=2, K=80	0.1468	0.4424	0.9955	0.1524	0.4650	1.0757
N=2, K=160	0.1468	0.4424	0.9955	0.1515	0.4642	1.0752
N=2, K=320	0.1468	0.4424	0.9955	0.1513	0.4640	1.0752
N=3, K=40	0.1468	0.4423	0.9955	0.1513	0.4640	1.0750
N=3, K=80	0.1468	0.4424	0.9955	0.1513	0.4640	1.0751
N=3, K=160	0.1468	0.4424	0.9955	0.1513	0.4639	1.0751
N=3, K=320	0.1468	0.4424	0.9955	0.1512	0.4639	1.0751
N=4, K=40	0.1468	0.4424	0.9955	0.1516	0.4642	1.0747
N=4, K=80	0.1468	0.4424	0.9955	0.1513	0.4640	1.0751
N=4, K=160	0.1468	0.4424	0.9955	0.1513	0.4639	1.0751
N=4, K=320	0.1468	0.4424	0.9955	0.1512	0.4639	1.0751
Binomial (1000 nodes)	0.1468	0.4423	0.9955	0.1512	0.4638	1.0751
Binomial (5000 nodes)	0.1468	0.4424	0.9955	0.1512	0.4639	1.0751
Exact Solution	0.1468	0.4424	0.9955			

$\alpha = \frac{1}{2}$	European			American		
	E=9	E=10	E=11	E=9	E=10	E=11
N=2, K=40	0.1360	0.4424	1.0065	0.1434	0.4695	1.0907
N=2, K=80	0.1367	0.4421	1.0069	0.1419	0.4657	1.0867
N=2, K=160	0.1369	0.4421	1.0070	0.1417	0.4650	1.0860
N=2, K=320	0.1369	0.4421	1.0070	0.1416	0.4646	1.0858
N=3, K=40	0.1364	0.4416	1.0065	0.1413	0.4643	1.0849
N=3, K=80	0.1368	0.4419	1.0068	0.1415	0.4645	1.0854
N=3, K=160	0.1369	0.4421	1.0069	0.1416	0.4646	1.0857
N=3, K=320	0.1370	0.4421	1.0070	0.1416	0.4646	1.0858
N=4, K=40	0.1370	0.4421	1.0071	0.1419	0.4657	1.0844
N=4, K=80	0.1370	0.4421	1.0070	0.1416	0.4646	1.0856
N=4, K=160	0.1370	0.4421	1.0070	0.1416	0.4647	1.0858
N=4, K=320	0.1370	0.4421	1.0070	0.1416	0.4646	1.0858
Binomial (1000 nodes)	0.1370	0.4419	1.0070	0.1416	0.4646	1.0857
Binomial (5000 nodes)	0.1370	0.4421	1.0071	0.1416	0.4646	1.0858
Exact Solution	0.1370	0.4421	1.0070			

$\alpha = \frac{2}{3}$	European			American		
	E=9	E=10	E=11	E=9	E=10	E=11
N=2, K=40	0.1322	0.4424	1.0101	0.1410	0.4720	1.0986
N=2, K=80	0.1334	0.4421	1.0107	0.1391	0.4666	1.0909
N=2, K=160	0.1337	0.4420	1.0109	0.1386	0.4654	1.0898
N=2, K=320	0.1338	0.4420	1.0110	0.1385	0.4649	1.0887
N=3, K=40	0.1329	0.4417	1.0105	0.1381	0.4621	1.0885
N=3, K=80	0.1335	0.4419	1.0109	0.1384	0.4637	1.0893
N=3, K=160	0.1337	0.4420	1.0110	0.1386	0.4648	1.0894
N=3, K=320	0.1338	0.4420	1.0110	0.1385	0.4649	1.0894
N=4, K=40	0.1340	0.4420	1.0111	0.1398	0.4672	1.0877
N=4, K=80	0.1338	0.4420	1.0110	0.1390	0.4647	1.0893
N=4, K=160	0.1338	0.4420	1.0110	0.1386	0.4649	1.0895
N=4, K=320	0.1338	0.4420	1.0110	0.1385	0.4649	1.0894
Binomial (1000 nodes)	0.1339	0.4419	1.0110	0.1385	0.4649	1.0894
Binomial (5000 nodes)	0.1338	0.4420	1.0110	0.1385	0.4649	1.0894
Exact Solution	0.1338	0.4420	1.0110			

$\alpha = 1$	European			American		
	E=9	E=10	E=11	E=9	E=10	E=11
N=2, K=40	0.1159	0.4503	1.0108	0.1232	0.4899	1.1186
N=2, K=80	0.1266	0.4413	1.0207	0.1352	0.4689	1.1009
N=2, K=160	0.1274	0.4420	1.0189	0.1329	0.4667	1.0979
N=2, K=320	0.1276	0.4420	1.0191	0.1325	0.4659	1.0971
N=3, K=40	0.1250	0.4401	1.0186	0.1301	0.4649	1.0958
N=3, K=80	0.1270	0.4415	1.0190	0.1315	0.4654	1.0965
N=3, K=160	0.1275	0.4419	1.0191	0.1321	0.4655	1.0968
N=3, K=320	0.1276	0.4419	1.0191	0.1324	0.4656	1.0969
N=4, K=40	0.1280	0.4419	1.0192	0.1371	0.4729	1.0952
N=4, K=80	0.1276	0.4420	1.0191	0.1333	0.4674	1.0975
N=4, K=160	0.1276	0.4420	1.0191	0.1326	0.4659	1.0972
N=4, K=320	0.1276	0.4420	1.0191	0.1324	0.4656	1.0970
Binomial (1000 nodes)	0.1277	0.4418	1.0191	0.1324	0.4655	1.0970
Binomial (5000 nodes)	0.1276	0.4419	1.0190	0.1324	0.4655	1.0970
Exact Solution	0.1276	0.4420	1.0191			

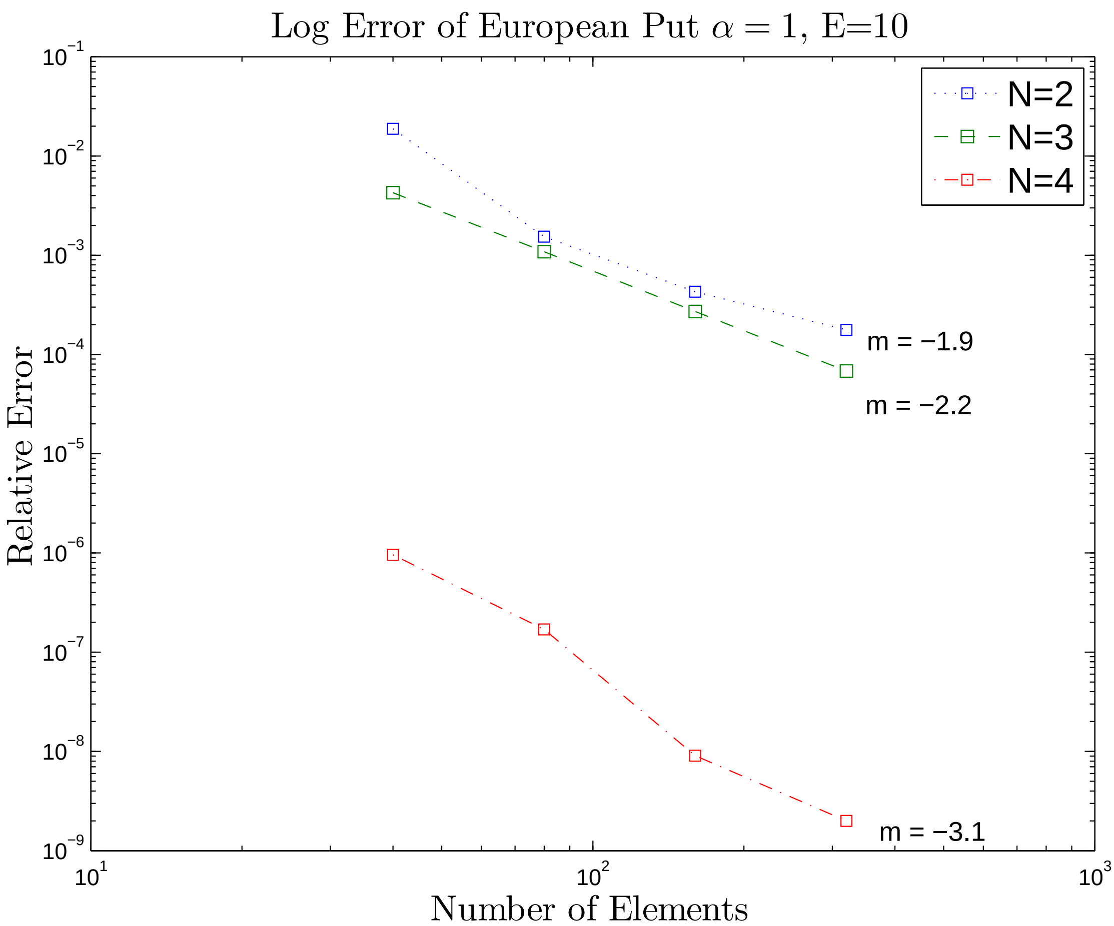

Figure 1. Convergence rates for DG under the Black-Scholes model.

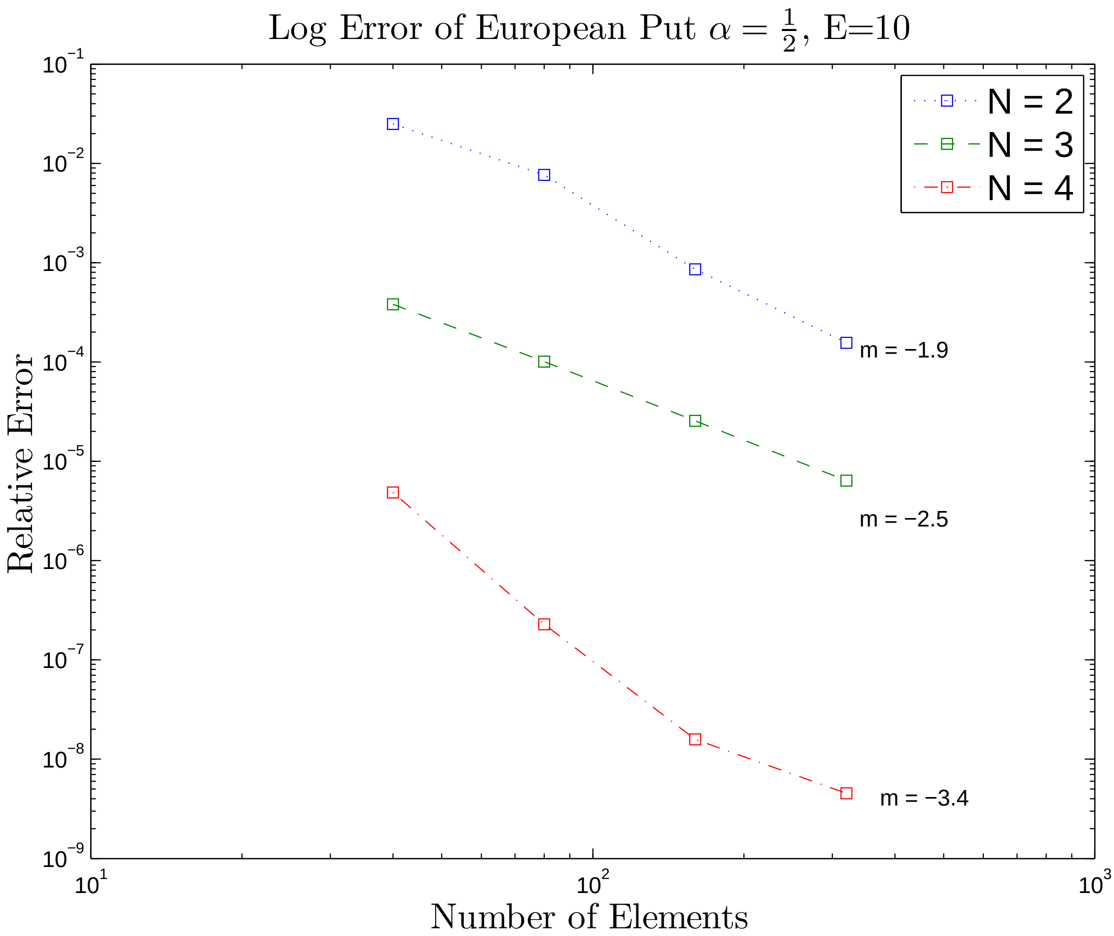

Figure 2. Convergence rates for DG with $\alpha = \frac{1}{2}$.

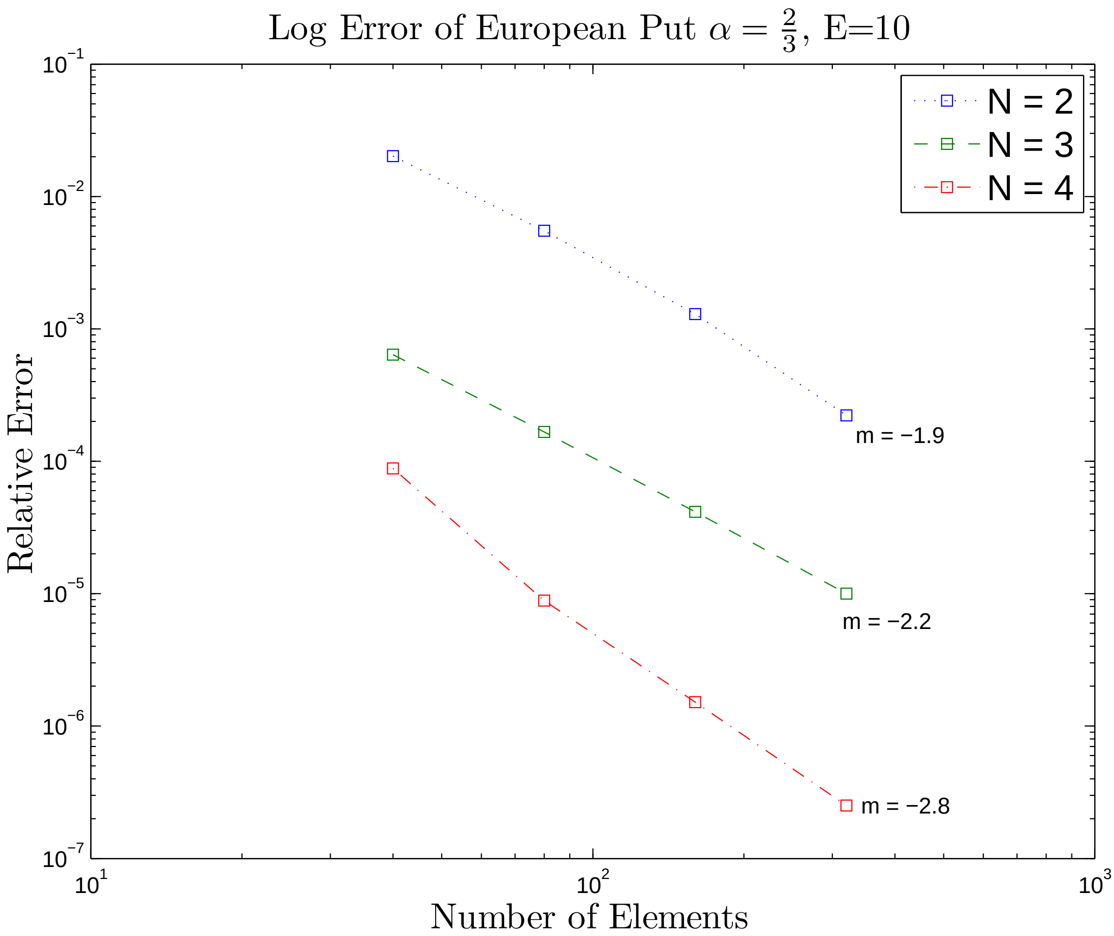

Figure 3. Convergence rates for DG with $\alpha = \frac{2}{3}$.

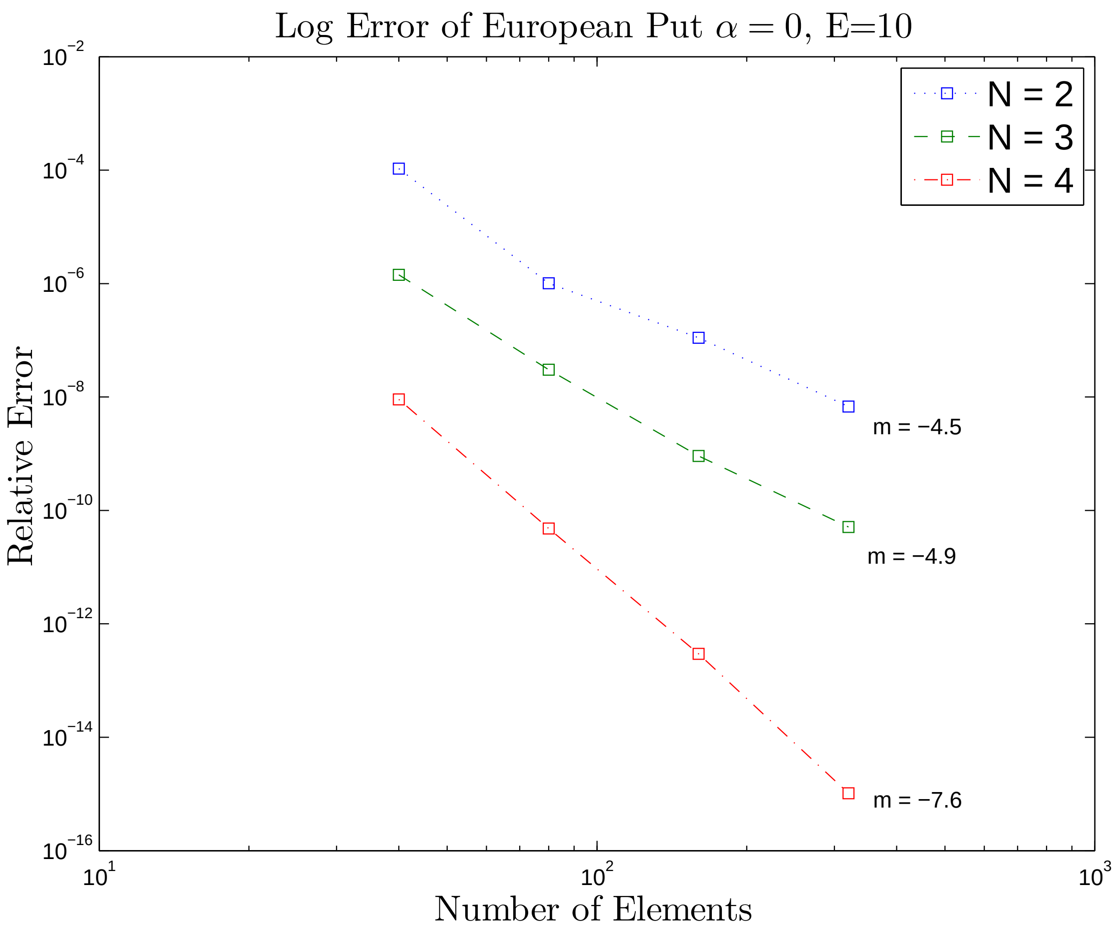

Figure 4. Convergence rates for DG with $\alpha = 0$.

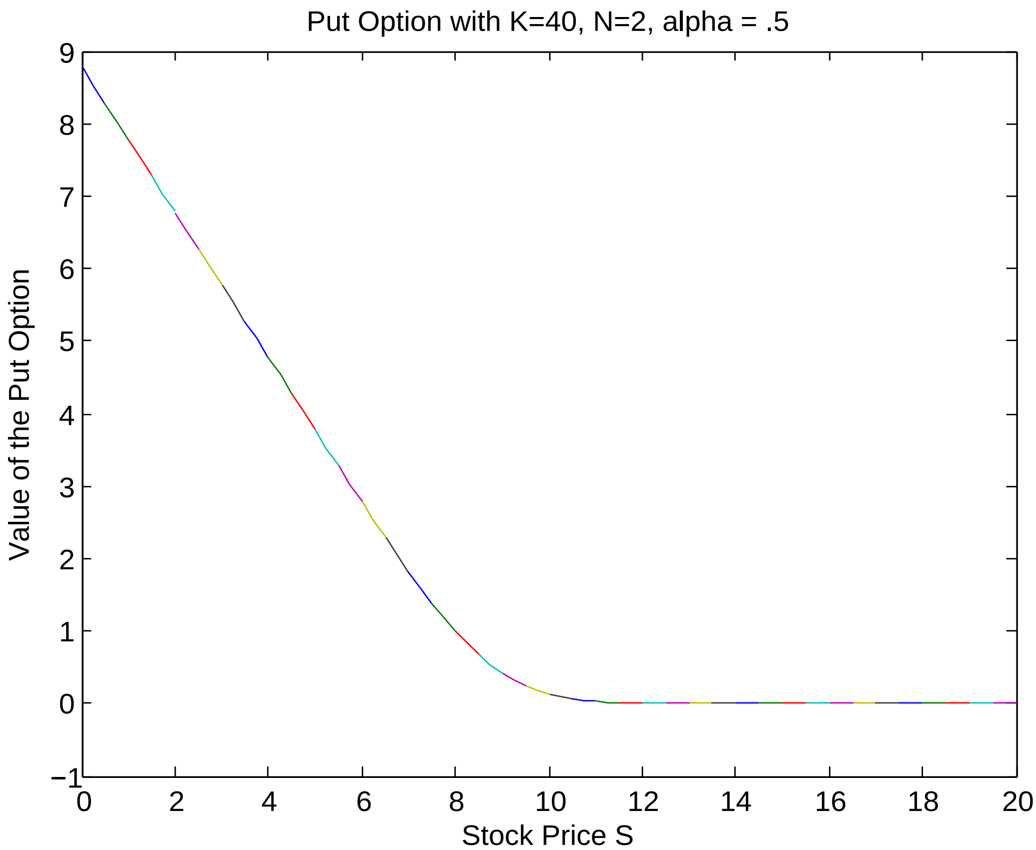

Figure 5. Graph of a Put Option, E = 10, $\sigma = .2$, $r = .05$, $T = .5$.

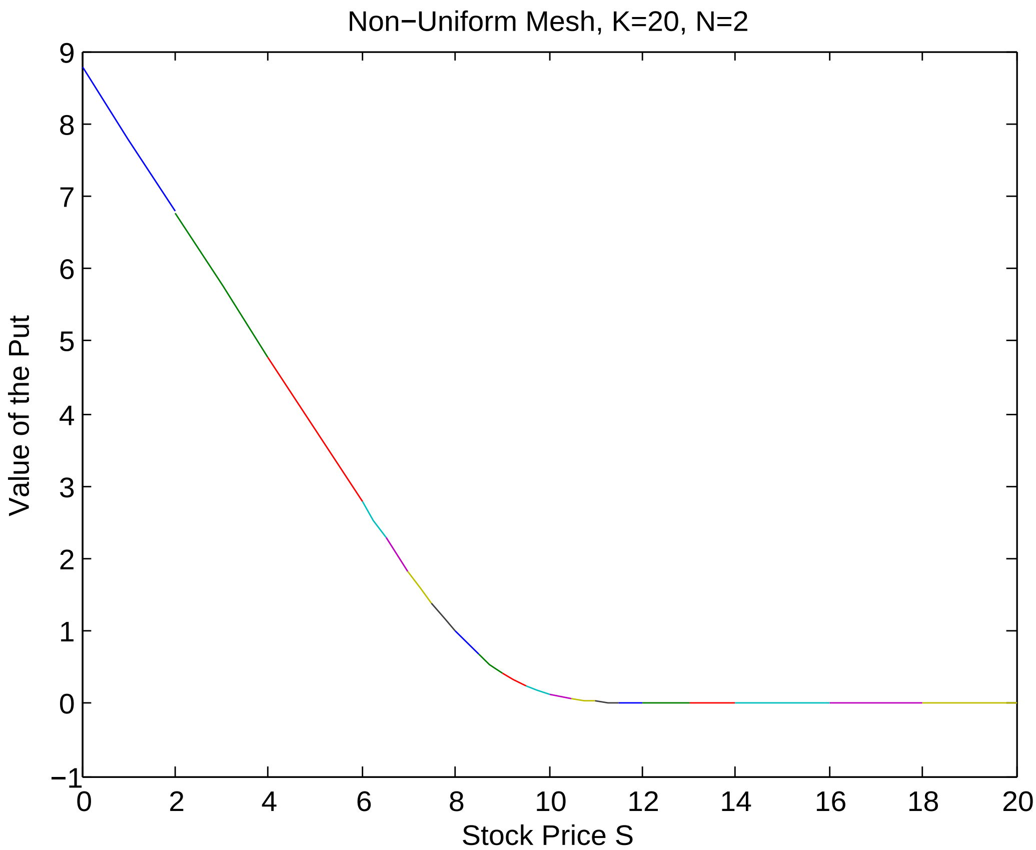

Figure 6. Graph of a Put Option with a Non-Uniform Mesh, E = 10, $\sigma = .2$, $r = .05$, $T = .5$.

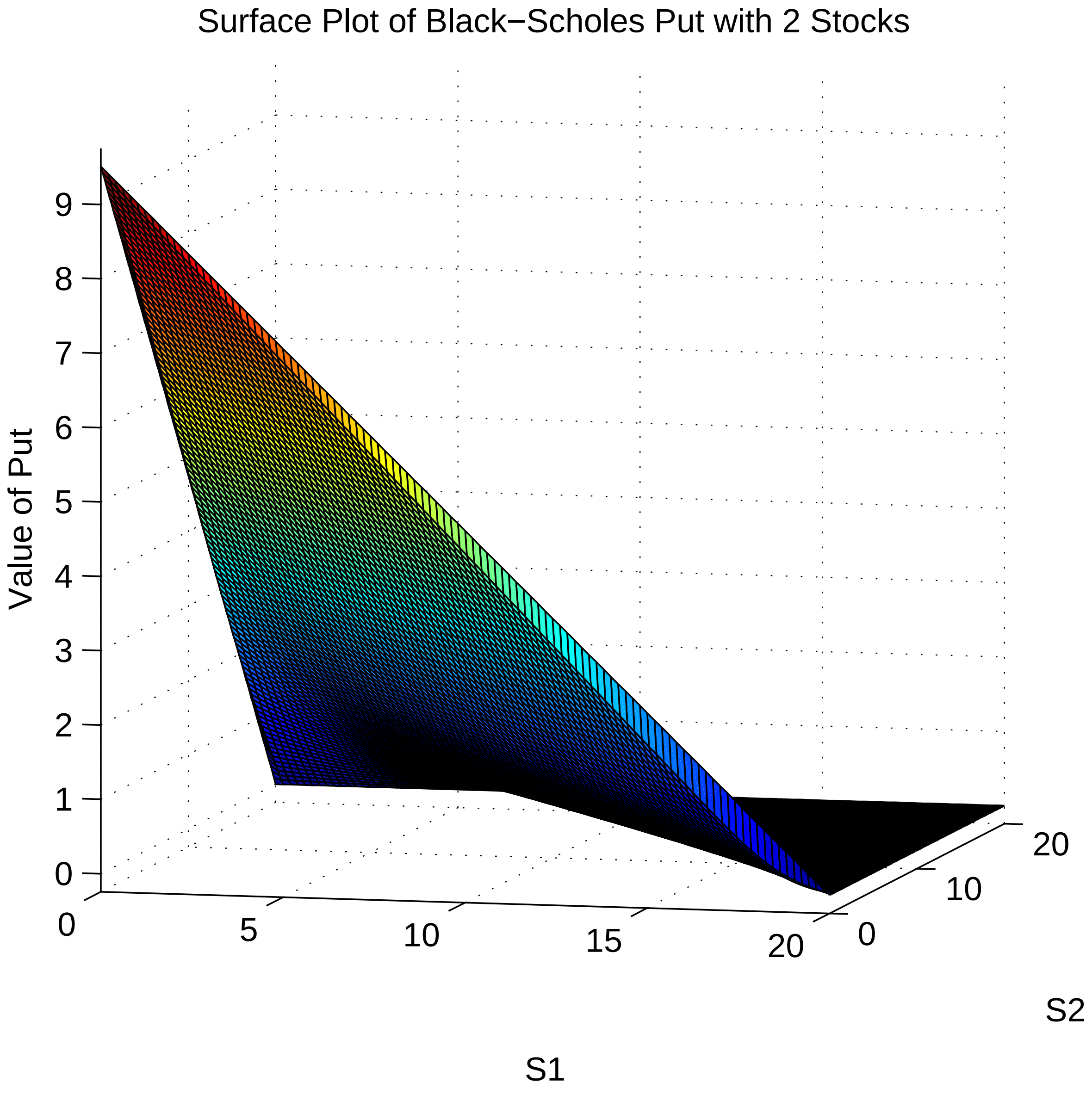

Figure 7. Graph of a Basket Put Option, E = 10, $\sigma_1 = .3$, $\sigma_2 = .2$, $\rho_{12} = .5$, $r = .05$, $T = 1$. The payoff function is $max(E - \frac{1}{2}S_1 - \frac{1}{2}S_2, 0)$

CHAPTER 9

CONCLUSION

The Discontinuous-Galerkin Method developed in this thesis converges to the correct option value for both European and American puts under the CEV process as verified by exact solutions (in the case of Europeans) and the Binomial Method (in the case of Americans). Since the DG Method allows for valuing options on the entire stock space, this method can also be extended to path-dependent options. With a non-uniform mesh, this method converges faster than the Finite Difference Methods developed in Chapter 4, for stock prices at or near the strike price. This method can be parallelized to further enhance computational speeds. This method also outpaces the Binomial Method for computing valuations on the entire stock-space to high degrees of accuracy. We are free to increase either the polynomial order or the number of elements to achieve higher accuracy.

APPENDICES

Appendix A

AMERICAN CALLS AND EARLY EXERCISE

Proposition: On a stock with no dividends, early exercise of a call option is never optimal.

Proof: Let E be the strike price, T be the time to expiry, and r the risk free interest rate. Consider the portfolio containing 1 call option, short 1 share of stock, and having a savings account at interest r with balance Ee^{-rT}, which is the present value of E dollars to be received at time T. The value of this portfolio at T is max(S - E, 0) - S + E = max(E - S, 0), which is the value of a put option, and thus is non-negative for all time before expiry.

So for $0 \leq t < T$,

$C_t + Ee^{-r(T-t)} - S_t \geq 0.$

$\Rightarrow C_t \geq S_t - Ee^{-r(T-t)}$

$\Rightarrow C_t > S_t - E$

Thus, the value for a call option at any time before expiry is greater than the payoff one would receive from exercising the option early. $\square$

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Experience

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- ⋆ Ran discussion sessions for different mathematics courses at the university, including Differential Equations, Business Calculus, and Finite Mathematics.
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- ⋆ Graded, written, and proctored student exams and quizzes.

2006 **Supplemental Instructor**, *Co-op at Northeastern University*, Boston, MA.

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2004-2005 **Junior Software Developer**, *Co-op at Empirix*, Bedford, MA.

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- ⋆ Engaged in writing code in C++ to fix defects and documented these fixes.
- ⋆ Actively contributed to the software development process by reviewing technical and design specifications.

PhD Thesis

Title *A Discontinuous-Galerkin Method for the CEV process*

Advisor David Nicholls

Description Numerical Analysis techniques for valuing European as well as American options. Code written in Matlab.

Awards

2009 Outstanding TA, received from the Mathematics Department at UIC.

2007 Top 100 most influential seniors at Northeastern University.

2006 Wenzinger Scholarship recipient.

2002-2007 Dean's Scholarship recipient.

Talks

2012 *A DG-Method for the CEV process.* Applied Math Seminar, UIC

2010 *Sums of Reciprocals of Squares using Complex Analysis.* Undergrad Math Club, UIC

2009 *Recurrence Relations and the Fibonacci Formula.* Undergrad Math Club, UIC

2007 *RSA Encryption.* Senior Computer Science Seminar, Northeastern University

2006	*Advanced Sudoku Strategies.* Tutoring Club, Northeastern University

Other Projects

Graduate Projects

Matlab	Implementation of various numerical methods for pricing options, including Finite Difference schemes, binomial method and Monte-Carlo.
Python	Implementation of the cutting algorithm in Linear Programming.
Matlab	Implementation of a scheduling algorithm for airline traffic.
Matlab	Implementation of a minimum containing disk for an arbitrary set of points.

Bachelor projects

JAVA	Design and implementation of the tile board game Carcassonne.
JAVA	Ray-tracing computer graphics simulator, including reflections and Phong shading with highlights.
JAVA	Sudoku solver (non brute-force techniques).
Scheme	Fractal simulator

Computer skills

Basic	XML, HTML, PHP
Intermediate	C, C++, Scheme, Python
Expert	JAVA, Matlab
Tools	Eclipse, CGAL, Sage, SourceSafe
Miscellaneous	Windows, Office, Linux