Thurston's Metric satisfies the Triangle Inequality

Nathan Lopez
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In his unpublished 1998 manuscript *Minimal Stretch Maps between Hyperbolic Surfaces*, Thurston considers the following question,

> "Given any two hyperbolic surfaces S and T, what is the least possible value of the global Lipschitz constant
> $$L(\phi) = \sup_{x \neq y} \frac{d(\phi(x), \phi(y))}{d(x, y)}$$
> for a homeomorphism $\phi : S \to T$ in a given homotopy class?"

If we consider a single surface M, then we can examine the function $L : \mathcal{T}(M) \times \mathcal{T}(M) \to \mathbb{R}$ defined by

$$L(g, h) = \inf_{\phi \sim \mathrm{id}} \log L(\phi) = \inf_{\phi \sim \mathrm{id}} \log \sup_{x \neq y} \frac{d_h(\phi(x), \phi(y))}{d_g(x, y)}$$

In the same paper, Thurston proves that $L(g, h) \geq 0$ and $L(g, h) = 0$ if and only if $g = h$ ([1] p. 5), and that, in general, L is not symmetric. He leaves the remaining proposition to the reader,

Proposition 0.1. *L satisfies the triangle inequality. More precisely, we have*

$$L(f, g) + L(g, h) \geq L(f, h).$$

To prove Proposition 0.1, we use the following characterization of the infimum.

Lemma 0.2. *Let $E \subset \mathbb{R}$ and $C \in \mathbb{R}$. Then $\inf E \leq C$ if and only if there is some $e \in E$ so that $e \leq C$.*

Proof. If every $e \in E$ is larger that C, then C is a lower bound for E, and it follows that $C \leq \inf E$. On the other hand, if there is some $e \in E$ satisfying $e \leq C$, then $\inf E$ is no larger than C because $\inf E$ is a lower bound of E. ☐

Now we prove Proposition 0.1.

Proof. By the Lemma, we need to find $\phi : (S, f) \to (S, h)$ so that

$$\log L(\phi) \leq L(f, g) + L(g, h).$$

Let $\epsilon > 0$. By the definition of the infimum, there are maps $\phi_1 : (S, f) \to (S, g)$ and $\phi_2 : (S, g) \to (S, h)$ so that

$$\log L(\phi_1) \leq L(f, g) + \epsilon$$
$$\log L(\phi_2) \leq L(g, h) + \epsilon.$$

Consider the composition $\phi := \phi_2 \circ \phi_1 : (S, f) \to (S, h)$. We have,

$$\begin{aligned} \log L(\phi) &= \log L(\phi_2 \circ \phi_1) \\ &= \log L(\phi_2) L(\phi_1) \qquad (*) \\ &= \log L(\phi_2) + \log L(\phi_1) \\ &\leq L(f, g) + L(g, h) + 2\epsilon. \end{aligned}$$

Since ϵ was arbitrary, this establishes the proposition modulo the equality labeled $(*)$, which requires justification. Using the definition, we compute

$$L(\phi_2 \circ \phi_1) = \sup_{x \neq y} \frac{d_h(\phi_2(\phi_1(x)), \phi_2(\phi_1(y)))}{d_f(x, y)}$$

$$
\begin{aligned}
&= \sup_{x \neq y} \left(\frac{d_h(\phi_2(\phi_1(x)), \phi_2(\phi_1(y)))}{d_g(\phi_1(x), \phi_1(y))} \cdot \frac{d_g(\phi_1(x), \phi_1(y))}{d_f(x,y)} \right) \\
&= \sup_{x \neq y} \frac{d_h(\phi_2(\phi_1(x)), \phi_2(\phi_1(y)))}{d_g(\phi_1(x), \phi_1(y))} \cdot \sup_{x \neq y} \frac{d_g(\phi_1(x), \phi_1(y))}{d_f(x,y)} \\
&= L(\phi_2) L(\phi_1).
\end{aligned}
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This completes the proof of Proposition 0.1. □

References

[1] William Thurston. Minimal stretch maps between hyperbolic surfaces. arXiv:math/9801039