Dp-Rank in Randomizations

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Overview

Randomizations

- Keisler (1999): Introduced randomizations of first order structures, the idea being to form a new structure whose elements are random elements of the original structure.
- Keisler and Ben Yaacov (2009): Introduced viewing randomizations as continuous structures.

Definition

For a complete (first order or continuous) theory T, its randomization T^R is the continuous theory whose models are the randomizations of models of T.

Fact

T^R admits quantifier elimination.

Properties Preserved By Randomizations

Theorem (Ben Yaacov 2009)

A classical theory T is ________ if and only if T^R is ________ .

- *ω-categorical*
- *ω-stable*
- *stable*
- *NIP*

Theorem (Ben Yaacov 2011)

If a theory T is simple and unstable, then T^R is not simple.

Dp-Rank and NIP

Dp-rank was originally introduced by Usvyatsov in 2008 as an attempt to capture how far a certain type (or theory) is from having the independence property.

Definition (Recall)

A formula $\phi(x; y)$ is NIP if we cannot find an infinite set A of $|x|$-tuples such that for all $A_0 \subset A$, there is some b_{A_0} of size $|y|$ such that

$$\phi(A; b_{A_0}) = A_0.$$

A theory is NIP if every formula is.

Dp-Rank

Definition

For a theory T, $dp-rk(T) < \kappa$ if the following does NOT exist:

- formulas $\phi_\alpha(x_\alpha; y)$ for $\alpha < \kappa$
- an array $(a^i_\alpha : i < \omega, \alpha < \kappa)$ of tuples with $|a^i_\alpha| = |x_\alpha|$
- for every $\eta : \kappa \to \omega$, a tuple b_η with $|b_\eta| = |y|$ such that

$$\vDash \phi_\alpha(a^i_\alpha; b_\eta) \Leftrightarrow \eta(\alpha) = i$$

This is called an *ICT-pattern* of length κ.
We say that $dp-rk(T) = \kappa$ if κ is the least such that this pattern does not exist.

Continuous Dp-Rank

Definition

A collection of continuous formulas $\{\phi_j(x; y) : j < n\}$ is an ICT-pattern of length n if there exist the following

- $((a^i_0, \ldots, a^i_{n-1}) : i < \omega)$, an array of tuples
- $0 \leq r_j < s_j \leq 1$ for $j < n$
- For all $\bar{i} = (i_0, \ldots, i_{n-1}) \in \omega^n$, $b_{\bar{i}}$ such that for all $j < n$

$$\phi_j(a^{i_j}_j, b_{\bar{i}}) \geq s_j$$

and for all $k < n, k \neq i_j$,

$$\phi_j(a^k_j, b_{\bar{i}}) \leq r_j.$$

We will focus on theories with dp-rank $= 1$. We call these *dp-minimal*.

Geometric Interpretation

- Let $(\phi_0(x; y), \phi_1(x; y))$ be an ICT-pattern of length 2. Let $\epsilon_j = s_j - r_j$ for $j = 0, 1$.
- For a fixed $m < \omega$, consider the set $C_m := \{(\phi_0(a_0^i, b), \phi_1(a_1^i, b))_{i<m} : b \in \mathcal{M}\} \subset ([0,1]^2)^m$.
- Fix $i_1 < m$ and pick $b_0, \ldots, b_{m-1}$ corresponding to $\bar{i} = (0, i_1), \ldots, (m-1, i_1)$ respectively. This gives us an m element subset of C_m
- Looking at the first column of each of these gives us m elements of $\mathbb{R}^m$ whose convex hull contains an m-simplex with side length $\sqrt{2}\epsilon_0$.
- Do the analogous thing with the second column.

Goal

To show that for a collection of n formulas $Q = (\phi_0(x; y), \ldots, \phi_{n-1}(x; y))$, the following are equivalent:

- Q is an ICT pattern of length n
- Some combinatorial characterization
- The ratio of the width of the convex hulls of the columns (as described on the previous slide) to the (asymptotic) width of m-simplices goes to 0 as $m \to \infty$.

Why is this what we want?

- By width, we mean Gaussian Mean Width (the average distance between two parallel planes which bound the set)
- This behaves nicely (in other words, preserves our geometric interpretation of dp-rank) when we crush convex compacts in $\mathbb{R}^n$.
- Crushing convex compacts is what happens to our sets of interest when we put two formulas together using truncated subtraction.
- Since $\{0, 1, \dot{-}, \frac{x}{2}\}$ is a complete set of connectives in continuous logic, and we have quantifier elimination, this is what we need to induct on formulas.

Thank You!