General Properties of Definable and Algebraic Closure

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December 10, 2012

10.7 Exercise

Let $\mathcal{M}$ be an L-structure, $A \subset M$, and $a \in M^n$. Statements (1), (2), and (3) are equivalent:

(1) a is definable in $\mathcal{M}$ over A.

(2) For any $\mathcal{N} \succeq \mathcal{M}$ the only realization of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N}$ is a.

(3) For any $\epsilon > 0$ there is an $L(A)$-formula $\phi(x)$ and $\delta > 0$ such that $\phi^{\mathcal{M}}(a) = 0$ and the diameter of $\{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$ is $\leq \epsilon$.

If $\mathcal{N}$ is any fixed ω_1-saturated elementary extension of $\mathcal{M}$, then (1) is equivalent to:

(4) The only realization of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N}$ is a.

(1)$\Rightarrow$(2) Suppose a is definable in $\mathcal{M}$ over A. Let $(\phi_k(x))_{k<\omega}$ be $L(A)$-formulas such that

$$\forall \epsilon > 0 \exists K \forall k \geq K \forall \overline{x} \in M^n |\phi_k^{\mathcal{M}}(\overline{x}) - d(\overline{x}, a)| \leq \epsilon.$$

Let $\epsilon > 0$ be given and let K witness this for $\frac{\epsilon}{2}$. Let $\mathcal{N} \succeq \mathcal{M}$ and let $b \in N^n$ realize $tp_{\mathcal{M}}(a/A)$. For $k \geq K$, $|\phi_k^{\mathcal{M}}(a) - d(a,a)| \leq \frac{\epsilon}{2}$, so $\phi_k^{\mathcal{M}}(a) \leq \frac{\epsilon}{2}$, thus $\phi_k(x) \leq \frac{\epsilon}{2}$ is in $tp_{\mathcal{M}}(a/A)$. Hence, $\mathcal{N} \vDash \phi_k(b) \leq \frac{\epsilon}{2}$. $\mathcal{M} \vDash \sup_x |\phi_k(x) - d(x,a)| \leq \dfrac{\epsilon}{2}$, so by elementarity, $\mathcal{N} \vDash \sup_x |\phi_k(x) - d(x,a)| \leq \dfrac{\epsilon}{2}$, so in particular, $\mathcal{N} \vDash |\phi_k(b) - d(b,a)| \leq \frac{\epsilon}{2}$. Thus, $d(a,b) \leq |d(a,b) - \phi_k^{\mathcal{N}}(b)| + |\phi_k^{\mathcal{N}}(b)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Hence, since $\epsilon > 0$ was arbitrary, $a = b$. So a is the only realization of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N} \succeq \mathcal{M}$.

(2)$\Rightarrow$(4) Clear.

(4)$\Rightarrow$(3) We will prove the contrapositive: suppose (3) fails, then $\exists \epsilon > 0$ such that for all $L(A)$-formulas $\phi(x)$ and $\delta > 0$, if $\phi^{\mathcal{M}}(a) = 0$, then the set $\{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$ has diameter $\geq \epsilon$. Let such an $\epsilon > 0$ be given. Let $p(x)$ be the following type over $A \cup \{a\}$: $tp_{\mathcal{M}}(a/A) \cup \{d(x,a) \geq \frac{\epsilon}{2}\}$.

Claim 1. *$p(x)$ is finitely satisfiable.*

Proof. Let $\Gamma(x)$ be a finite subset of $p(x)$, and let $\{\phi_1(x) = 0, \ldots, \phi_m(x) = 0, d(x, a) \geq \frac{\epsilon}{2}\} \supset \Gamma(x)$ where $\phi_i(x) = 0 \in tp_{\mathcal{M}}(a/A)$. Then for $\phi(x) = \max\{\phi_1(x), \ldots, \phi_m(x)\}$, $\phi(x) = 0 \in tp_{\mathcal{M}}(a/A)$, so $\phi(x) = 0 \in p(x)$. Thus, it is enough to show that $\{\phi(x) = 0, d(x, a) \geq \frac{\epsilon}{2}\}$ is satisfiable. By $\neg(3)$, $\{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$ has diameter $\geq \epsilon$ for every $\delta > 0$, so $\{b \in M^n | \phi^{\mathcal{M}}(b) = 0\} = \bigcup_{\delta > 0} \{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$ has diameter $\geq \epsilon$. So choose b such that $d(b, a) \geq \frac{\epsilon}{2}$ and $\phi^{\mathcal{M}}(b) = 0$. This satisfies $\Gamma(x)$. $\square$

Now, let $\mathcal{N} \succeq \mathcal{M}$ be ω_1-saturated. Since $p(x)$ is finitely satisfiable, and A (without loss of generality, by 10.10(3)) is countable, $p(x)$ is realized by some $b \in N^n$. But $d(a, b) \geq \frac{\epsilon}{2} > 0$, so $a \neq b$. Thus, a is not the only realization of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N}$.

(3)⇒(1) Let $\phi_k(x)$ an $L(A)$-formula and $\delta_k > 0$ be witnesses of (3) for $\epsilon = \frac{1}{k}$. Then $\phi_k^{\mathcal{M}}(a) = 0$ and $\phi_k^{\mathcal{M}}(b) < \delta_k \Rightarrow d(a, b) \leq \frac{1}{k}$ since $a \in \{b \in M^n | \phi_k^{\mathcal{M}}(b) < \delta_k\}$ and it has diameter $\leq \frac{1}{k}$. So by Proposition 9.19, $\{a\}$ is definable.

10.8 Exercise

Let $\mathcal{M}$ be an L-structure, $A \subset M$, and $a \in M^n$. Statements (1), (2), (3), and (4) are equivalent:

(1) a is algebraic in $\mathcal{M}$ over A.

(2) For any $\mathcal{N} \succeq \mathcal{M}$, every realization of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N}$ is in M^n.

(3) For any $\epsilon > 0$ there is an $L(A)$-forumla $\phi(x)$ and $\delta > 0$ such that $\phi^{\mathcal{M}}(a) = 0$ and the set $\{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$ has a finite ϵ-net.

(4) For any $\mathcal{N} \succeq \mathcal{M}$, the set of realizations of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N}$ is compact.

If $\mathcal{N}$ is any fixed ω_1-saturated extension of $\mathcal{M}$, then (1) is equivalent to:

(5) The set of realizations of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N}$ is compact.

If $\mathcal{N}$ is any fixed κ-saturated elementary extension of $\mathcal{M}$, with κ uncountable, then (1) is equivalent to:

(6) The set of realizations of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N}$ has density character $< \kappa$.

(1)⇒(3) Let $C \subset M^n$ be a compact set containing a and let $(\phi_k)_{k<\omega}$ be $L(A)$-formulas such that $\forall x \in M^n$, $|\phi_k^{\mathcal{M}}(x) - d(x, C)| \leq \frac{1}{k}$. Let $\epsilon > 0$ be given. Choose k so that $\frac{3}{k} < \frac{\epsilon}{3}$. Let $\delta = \frac{1}{k}$ and $\phi(x) = \phi_k(x) \dot{-} \frac{1}{k}$ (where $x \dot{-} y = \max(x - y, 0)$).

Then, since $|\phi_k^{\mathcal{M}}(a) - d(a, C)| = \phi_k^{\mathcal{M}}(a) \leq \frac{1}{k}$ because $a \in C$, $\phi^{\mathcal{M}}(a) = \phi_k^{\mathcal{M}}(a) \dot{-} \frac{1}{k} = 0$. Let $b \in M^n$ with $\phi^{\mathcal{M}}(b) < \delta$. $\phi_k^{\mathcal{M}}(b) \dot{-} \frac{1}{k} < \frac{1}{k}$, so $\phi_k^{\mathcal{M}}(b) < \frac{2}{k}$. Thus $d(b, C) \leq |\phi_k^{\mathcal{M}}(b) - d(b, C)| + |\phi_k^{\mathcal{M}}(b)| \leq \frac{1}{k} + \frac{2}{k} = \frac{3}{k} < \frac{\epsilon}{3}$. Now consider the open cover $\{B(c, \frac{\epsilon}{2}) : c \in C\}$ of C. Since C is compact, there are $c_1, \ldots, c_m \in C$ such that $B(c_1, \frac{\epsilon}{2}), \ldots, B(c_m, \frac{\epsilon}{2})$ is a finite subcover of C. Let $b \in \{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$. Since $d(b, C) < \frac{\epsilon}{3}$, we can choose $x \in C$ such that $d(b, x) < \frac{\epsilon}{2}$. Since $\{B(c_i, \frac{\epsilon}{2}) | 1 \leq i \leq m\}$ is a cover of C, there is c_i such that $x \in B(c_i, \frac{\epsilon}{2})$. So $d(b, c_i) \leq d(b, x) + d(x, c_i) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $\{B(c_i, \epsilon) | 1 \leq i \leq m\}$ is a finite ϵ-net of $\{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$.

(3)$\Rightarrow$(1) Let $\mathcal{N} \succeq \mathcal{M}$ be ω_1-saturated.

Claim 2. *For any $\epsilon > 0$, there is an $L(A)$-formula $\phi(x)$ and $\delta > 0$ such that $\phi^{\mathcal{N}}(a) = 0$ and $\{b \in N^n | \phi^{\mathcal{N}}(b) < \delta\}$ has a finite ϵ-net.*

Proof. Let $\epsilon > 0$ be given. Choose an $L(A)$-formula $\phi(x)$ and $\delta > 0$ such that there are $b_1, \dots, b_m$ which give a finite ϵ-net of $\{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$. So

$\mathcal{M} \vDash \sup_b \min(\delta \dot{-} \phi(b), \min_{1 \le i \le m} (d(b, b_i) \dot{-} \epsilon)) = 0$. Since $\mathcal{M} \preceq \mathcal{N}$,

$\mathcal{N} \vDash \sup_b \min(\delta \dot{-} \phi(b), \min_{1 \le i \le m} (d(b, b_i) \dot{-} \epsilon)) = 0$. So for $b \in \{b \in N^n | \phi^{\mathcal{N}}(b) < \delta\}$, $\delta \dot{-} \phi^{\mathcal{N}}(b) > 0$, so $\min_{1 \le i \le m} (b, b_i) \dot{-} \epsilon = 0$ for some i, so $d(b, b_i) \le \epsilon$. Thus, $b_1, \dots, b_m$ give a finite ϵ-net of $\{b \in N^n | \phi^{\mathcal{N}}(b) < \delta\}$.

□

For $k \ge 1$, let $\phi_k(x)$ an $L(A)$-formula and $\delta_k > 0$ be witnesses of this in $\mathcal{N}$ for $\epsilon = \frac{1}{k}$. Let $C = \bigcap_{k < \omega} Z(\phi_k(x) \dot{-} \frac{\delta_k}{2})$.

Claim 3. *C is compact.*

Proof. Let $\epsilon > 0$ be given. Let k be such that $\frac{1}{k} < \epsilon$. Then, since $C \subset Z(\phi_k(x) \dot{-} \frac{\delta_k}{2}) \subset \{b \in N^n | \phi_k^{\mathcal{N}}(b) < \delta_k\}$, and there is a finite ϵ-net of $\{b \in N^n | \phi_k^{\mathcal{N}}(b) < \delta_k\}$, we get a finite ϵ-net of C.

Let $(b_k)_{k<\omega}$ be a sequence in C. We will define $C_1 \supset C_2 \supset \dots$ subsets of C with $diam(C_k) \le \frac{1}{k}$ and $(a_k)_{k<\omega}$ a subsequence of (b_k) such that $\{k \in \omega | b_k \in C_i\}$ is infinite and (a_k) is Cauchy.

Let $C_1 = C$, $a_1 = b_1$. Given $a_1, \dots, a_k$ and $C_1 \supset \dots \supset C_k$, let $B_1, \dots, B_m$ be a finite $\frac{1}{k+1}$-net of C. Then $B_1 \cap C_k, \dots, B_m \cap C_k$ is a finite $\frac{1}{k+1}$-net of C_k. Since there are infinitely many j such that $b_j \in C_k$, for at least one B_i, $B_i \cap C_k$ is such that for infinitely many j, $b_j \in B_i \cap C_k$. Let $C_{k+1} = B_i \cap C_k$ and let $a_{k+1} = b_j$ where j is the least such that $b_j \in C_{k+1}$ and if $a_1 = b_{i_1}, \dots, a_k = b_{i_k}$, $j > i_1, \dots, i_k$.

So for any $\epsilon > 0$, choose N such that $\frac{1}{N} < \epsilon$. Then for $k, j > N$, $a_k, a_j \in C_N$, so $d(a_k, a_j) < \frac{1}{N} = \mathrm{diam}(C_N) < \epsilon$. So (a_k) is Cauchy.

Since $\mathcal{N}$ is complete, (a_k) converges to some c. Now consider $\phi_k(x) \dot{-} \frac{\delta_k}{2}$. Let $\epsilon > 0$ be given, and let Δ be the modulus of uniform continuity for $\phi_k(x) \dot{-} \frac{\delta_k}{2}$. Choose j such that $d(a_j, c) < \Delta(\epsilon)$. Then $|(\phi_k^{\mathcal{N}}(a_j) \dot{-} \frac{\delta_k}{2}) - (\phi_k^{\mathcal{N}}(c) \dot{-} \frac{\delta_k}{2})| \le \epsilon$. But since $a_j \in C$, $\phi_k^{\mathcal{N}}(a_j) \dot{-} \delta_k = 0$, so $\phi_k^{\mathcal{N}}(c) \dot{-} \delta_k \le \epsilon$. Thus, since $\epsilon > 0$ was arbitrary, $\phi_k^{\mathcal{N}}(c) \dot{-} \frac{\delta_k}{2} = 0$. So since k was arbitrary, $c \in C$. Hence, C is sequentially compact.

Thus, since $\mathcal{N}$ is a metric space, C is compact. □

By Proposition 9.14, $C = Z(P)$ for some predicate P which is definable over A. So by Proposition 10.6, C is definable over A since $\mathcal{N}$ is ω_1-saturated.

Since $\phi_k^{\mathcal{N}}(a) = 0$ for all k, $\phi_k^{\mathcal{N}}(a) \dot{-} \frac{\delta_k}{2} = 0$, so $a \in C$.

Thus, $a \in acl_{\mathcal{N}}(A) = acl_{\mathcal{M}}(A)$ by Corollary 10.5. So a is algebraic over A in $\mathcal{M}$, as required.

(3)⇒(4) Let $\mathcal{N} \succeq \mathcal{M}$ and let $D \subset N^n$ be the set of realizations of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N}$.

Claim 4. *For every $\epsilon > 0$, there is a finite ϵ-net of D.*

Proof. Let $\epsilon > 0$ be given. Let $\phi(x)$ be an $L(A)$ formula and $\delta > 0$ be such that $\{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$ has a finite ϵ-net. Let $c_1, \ldots, c_m$ be such that $B(c_1, \epsilon), \ldots, B(c_m, \epsilon)$ is that net. So $\mathcal{M} \vDash \sup_b(\min(\delta \dot{-} \phi(b), \min_{1 \le i \le m}(d(c_i, b) \dot{-} \epsilon))) = 0$. So since $\mathcal{M} \preceq \mathcal{N}$, $\mathcal{N} \vDash \sup_b(\min(\delta \dot{-} \phi(b), \min_{1 \le i \le m}(d(c_i, b) \dot{-} \epsilon))) = 0$. Let $b \in D$. Since $\phi^{\mathcal{M}}(a) = 0$ and $b \vDash tp_{\mathcal{M}}(a/A)$, $\phi^{\mathcal{N}}(b) = 0$. $\delta \dot{-} \phi(b) > 0$, so for some i, $d(c_i, b) \dot{-} \epsilon = 0$, so $\epsilon \ge d(c_i, b)$. Thus, $B(c_1, \epsilon), \ldots, B(c_m, \epsilon)$ is a finite ϵ-net of D. □

Claim 5. *D is sequentially compact.*

Proof. Let $(b_k)_{k<\omega}$ be a sequence in D. We will define $C_1 \supset C_2 \supset \ldots$ subsets of D with $diam(C_k) \le \frac{1}{k}$ and $(a_k)_{k<\omega}$ a subsequence of (b_k) such that $\{k \in \omega | b_k \in C_i\}$ is infinite and (a_k) is Cauchy.

Let $C_1 = D$, $a_1 = b_1$. Given $a_1, \ldots, a_k$ and $C_1 \supset \ldots \supset C_k$, let $B_1, \ldots, B_m$ be a finite $\frac{1}{k}$-net of D. Then $B_1 \cap C_k, \ldots, B_m \cap C_k$ is a finite $\frac{1}{k+1}$-net of C_k. Since there are infinitely many j such that $b_j \in C_k$, for at least one B_i, $B_i \cap C_k$ is such that for infinitely many j, $b_j \in B_i \cap C_k$. Let $C_{k+1} = B_i \cap C_k$ and let $a_{k+1} = b_j$ where j is the least such that $b_j \in C_{k+1}$ and if $a_1 = b_{i_1}, \ldots, a_k = b_{i_k}$, $j > i_1, \ldots, i_k$.

So for any $\epsilon > 0$, choose N such that $\frac{1}{N} < \epsilon$. Then for $k, j > N$, $a_k, a_j \in C_N$, so $d(a_k, a_j) < \frac{1}{N} = \mathrm{diam}(C_N) < \epsilon$. So (a_k) is Cauchy.

Since $\mathcal{M}$ is complete, (a_k) converges to some c. Now let ϕ be an $L(A)$-formula such that $\phi = 0 \in tp_{\mathcal{M}}(a/A)$. Let $\epsilon > 0$ be given, and let Δ be the modulus of uniform continuity for ϕ. Choose k such that $d(a_k, c) < \Delta(\epsilon)$. Then $|\phi^{\mathcal{N}}(a_k) - \phi^{\mathcal{N}}(c)| \le \epsilon$. But since $a_k \in D$, $\phi^{\mathcal{N}}(a_k) = 0$, so $\phi^{\mathcal{N}}(c) \le \epsilon$. Thus, since $\epsilon > 0$ was arbitrary, $\phi^{\mathcal{N}}(c) = 0$. So since ϕ was arbitrary, $c \vDash tp_{\mathcal{M}}(a/A)$, so $c \in D$. Hence, D is sequentially compact.

□

Thus, since $\mathcal{N}$ is a metric space, D is compact, as required.

(4)⇒(3) Since $\mathcal{M} \succeq \mathcal{M}$, the set of realizations of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{M}$ is compact. Let $\epsilon > 0$ be given. Let $b_1, \ldots, b_n \vDash tp_{\mathcal{M}}(a/A)$ be such that $B(b_1, \epsilon), \ldots, B(b_n, \epsilon)$ is a finite ϵ-net of the realizations of $tp_{\mathcal{M}}(a/A)$. Let $\phi(x) = \min_{1 \le i \le m}(d(x, b_i) \dot{-} \frac{\epsilon}{2})$. Let $\delta = \frac{\epsilon}{2}$.

Then $\phi^{\mathcal{M}}(x) < \delta \Leftrightarrow d(x, b_i) \dot{-} \frac{\epsilon}{2} < \frac{\epsilon}{2}$ for some $1 \le i \le m \Leftrightarrow d(x, b_i) < \epsilon$ for some $1 \le i \le m$. That is, $x \in \{b \in M^n | \phi^{\mathcal{M}}(x) < \delta\}$ if and only if x is in one of the $B(b_i, \epsilon)$, so this set has a finite ϵ-net.

(3)⇒(2) Let $\epsilon > 0$ be given. Let $\phi(x) = 0 \in tp_{\mathcal{M}}(a/A)$ and $\delta > 0$ be such that $\{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$ has a finite ϵ-net.

Let $B(b_1, \epsilon), \ldots, B(b_m, \epsilon)$ with $b_1, \ldots, b_m \in M^n$ be such a net.

$\mathcal{M} \vDash \sup_c(\min(\delta \dot{-} \phi(c), \min_{1 \le i \le m}(d(c, b_i) \dot{-} \epsilon))) = 0$, so since $\mathcal{M} \preceq \mathcal{N}$,

$\mathcal{N} \models \sup_c(\min(\delta \dot{-} \phi(c), \min_{1\leq i\leq m}(d(c,b_i)\dot{-}\epsilon))) = 0$. So for $c \in N^n$ with $\phi^{\mathcal{N}}(c) = 0$, for some $1 \leq i \leq m$, $d(c, b_i) \leq \epsilon$.

So let $c \in N^n$ and suppose $c \models tp_{\mathcal{M}}(a/A)$. Then for each $\frac{1}{k}$, since $\phi^{\mathcal{N}}(c) = 0$ for all $\phi = 0 \in tp_{\mathcal{M}}(a/A)$, we can find some $b_k \in \mathcal{M}$ such that $d(c, b_k) \leq \frac{1}{k}$. Thus, c is the limit of the sequence $(b_k)_{k<\omega}$ in $\mathcal{M}$ and $\mathcal{M}$ is complete, so $c \in M^n$.

(2)⇒(3) Suppose $\neg(3)$. That is, $\exists\epsilon > 0$ such that for every $L(A)$-formula ϕ and $\delta > 0$ such that $\phi^{\mathcal{M}}(a) = 0$, there is no finite ϵ-net of $\{b \in M | \phi^{\mathcal{M}}(b) < \delta\}$. Let B be the set of all realizations of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{M}$. For $b \in B$, let $\psi_b(x)$ be $\epsilon \dot{-} d(x, b)$. Let p be the following type over $A \cup B$: $\{\phi(x) \leq \frac{1}{n} | \phi \text{ is an } L(A)\text{-forumla}, \phi^{\mathcal{M}}(a) = 0, n \in \mathbb{N}\} \cup \{\psi_b(x) : b \in B\}$.

Claim 6. *p is finitely satisfiable.*

Proof. Let $\Gamma \subset p$ be finite. Then there are $b_1, \ldots, b_n \models tp_{\mathcal{M}}(a/A)$ and $\phi_1, \ldots, \phi_m$ $L(A)$-formulas such that $\phi_i^{\mathcal{M}}(a) = 0$ so that $\Gamma \subset \{\psi_{b_1} = 0, \ldots, \psi_{b_n} = 0, \phi_1(x) < \frac{1}{n_1}, \ldots, \phi_m(x) < \frac{1}{n_m}\}$. Let $\phi = \max(\phi_1, \ldots, \phi_m)$ and $\delta = \min(\frac{1}{n_1}, \ldots, \frac{1}{n_m})$. We know $\phi^{\mathcal{M}}(a) = 0$, so $B(b_1, \epsilon), \ldots, B(b_n, \epsilon)$ is not a cover of $\{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\}$. So choose $x \in \{b \in M^n | \phi^{\mathcal{M}}(b) < \delta\} \setminus (B(b_1, \epsilon) \cup \ldots \cup B(b_n, \epsilon))$. Thus, $x \models \Gamma$. □

Let $\mathcal{N} \succeq \mathcal{M}$ be such that there is $y \in N^n$ realizing p. So $y \models tp_{\mathcal{M}}(a/A)$ since for all $L(A)$-formulas with $\phi(x) = 0 \in tp_{\mathcal{M}}(a/A)$, and all $n \in \mathbb{N}$, $\phi^{\mathcal{N}}(y) \leq \frac{1}{n}$, so $\phi^{\mathcal{N}}(y) = 0$. $d(y, b) \geq \epsilon > 0$ for all $b \in B$, so $y \notin B$. Thus, $y \notin M^n$.

(4)⇒(5) Clear.

(5)⇒(4) Let $\mathcal{N} \succeq \mathcal{M}$ and let $\mathcal{N}' \succeq \mathcal{N}$ be an ω_1-saturated elementary extension. Then $\{b \in \mathcal{N} | b \models tp_{\mathcal{M}}(a/A)\} = \{b \in \mathcal{N}' | b \models tp_{\mathcal{M}}(a/A)\} \cap \mathcal{N}$. $\{b \in \mathcal{N}' | b \models tp_{\mathcal{M}}(a/A)\}$ is compact by assumption and $\mathcal{N}$ is complete, and thus closed, so $\{b \in \mathcal{N} | b \models tp_{\mathcal{M}}(a/A)\}$ is compact.

(4)⇒(6) The set of realizations of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N}$ is compact, and thus, separable. That is, it has a countable dense subset, so its density character is $< \kappa$, since κ is is uncountable.

(6)⇒(4) Suppose there is $\epsilon > 0$ such that there is no finite ϵ-net of the realizations in $\mathcal{N}$ of $p(x) = tp_{\mathcal{M}}(a/A)$. Note: in the proof of (3)⇒(4), we saw that it is enough to show that if for every $\epsilon > 0$, the set of realizations of $tp_{\mathcal{M}}(a/A)$ has a finite ϵ-net, to see that it is compact. So if we assume the set of realizations in not compact, there must be some such $\epsilon > 0$.

Consider the following type in κ variables:

$$\Gamma((x_i)_{i<\kappa}) = \bigcup_{i<\kappa} p(x_i) \cup \{d(x_i, x_j) \geq \epsilon : i \neq j\}.$$

Claim 7. *Γ is finitely satisfiable.*

Proof. Suppose not. Let n be such that $p(x_1) \cup \ldots \cup p(x_n) \cup \{d(x_i, x_j) \geq \epsilon | i, j \leq n\}$ is realized by some $a_1, \ldots, a_n$ but $p(x_1) \cup \ldots \cup p(x_n) \cup p(x_{n+1}) \cup \{d(x_i, x_j) \geq \epsilon | i, j \leq n+1\}$ is not satisfiable. Then, for every $b \models p(x)$, $d(a_i, b) < \epsilon$ for some $1 \leq i \leq n$. So these give a finite ϵ-net of the set of realizations. ⇒⇐. □

Thus, by compactness, $\Gamma((x_i)_{i<\kappa})$ is satisfiable.

Fact 1. If $\mathcal{M}$ is ω-saturated and $(\inf_y \phi(\overline{x}, y))^{\mathcal{M}} = 0$, then there is $y \in \mathcal{M}$ such that $\phi(\overline{x}, y) = 0$.

Proof. Let $q(y)$ be the type over $\overline{x}$ $\{\phi(\overline{x}, y) \leq \frac{1}{k} | k < \omega\}$. This is finitely satisfiable, since for any $k < \omega$, there is y such that $\phi(\overline{x}, y) \leq \frac{1}{k}$. Thus, by ω-saturation, since $\overline{x}$ is finite, there is y realizing q, and for such a y, we must have $\phi(\overline{x}, y) = 0$. $\square$

Lemma 2. *If $q((x_i)_{i<\kappa})$, a type over A with $|A| < \kappa$, is finitely satisfiable and $\mathcal{M}$ is κ-saturated, then q is realized in $\mathcal{M}$*

Proof. Let $p((x_i)_{i<\kappa})$ be a completion of q. For $\gamma < \kappa$, let $\phi((x_i)_{i\leq\gamma})$ denote a formula whose variables are among $(x_i)_{i\leq\gamma}$. Let $p|^{\gamma}((x_i)_{i\leq\gamma})$ be the set of formulas from p who variables are among $(x_i)_{i\leq\gamma}$.

We will inductively build a sequence $(a_i)_{i<\kappa}$ which realizes p. By κ-saturation, since $|A| < \kappa$, we can find $a_0 \vDash p|^0(x_0)$.

Then, suppose we have $(a_i)_{i<\gamma}$. Let $\phi((x_i)_{i<\gamma}, x_\gamma) = 0$ be from $p|^{\gamma}((x_i)_{i\leq\gamma})$. Then, since p is complete, $\inf_y \phi((x_i)_{i<\gamma}, y) = 0$ is in p, and thus, in $p|^{\beta}$ for some $\beta < \gamma$. Thus, $\inf_y \phi((a_i)_{i<\gamma}, y) = 0$. So by the previous fact, there is some $a_\gamma \in \mathcal{M}$ such that $\phi((a_i)_{i<\gamma}, a_\gamma) = 0$. Hence, $p|^{\gamma}((a_i)_{i<\gamma}, x_\gamma)$ is finitely satisfiable.

So, if we consider $p|^{\gamma}((a_i)_{i<\gamma}, x_\gamma)$ as a 1-type over $A \cup \{a_i | i < \gamma\}$, since $\gamma < \kappa$, $|A \cup \{a_i | i < \gamma\}| < \kappa$, so by κ-saturation, there is a_γ realizing it.

Thus, we get $(a_i)_{i<\kappa}$ realizing $p((x_i)_{i<\kappa})$, and hence, $q((x_i)_{i<\kappa})$.

$\square$

So by κ-saturation, since A is, without loss of generality, countable by 10.11(3), $\Gamma((x_i)_{i<\kappa})$ is realized by some $(a_i)_{i<\kappa}$ in $\mathcal{N}$.

Now suppose the set of realizations of p in $\mathcal{N}$ has density character $\lambda < \kappa$. Let S be a size λ dense subset of the set of realizations.

We know from Γ that there are at least κ many distinct realizations and that for every a_i there is $s \in S$ such that $d(a_i, s) < \frac{\epsilon}{2}$. So since $|S| < \kappa$, there must be some $s \in S$ with $i \neq j$ such that $d(a_i, s) < \frac{\epsilon}{2}$ and $d(a_j, s) < \frac{\epsilon}{2}$ (or else there could only be at most λ many a_i's). But then $d(a_i, a_j) \leq d(a_i, s) + d(a_j, s) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \Rightarrow\Leftarrow$.

Thus, the set of realizations of $tp_{\mathcal{M}}(a/A)$ in $\mathcal{N}$ has density character $\geq \kappa$.

10.11 Exercise

Let $\mathcal{M}$ be an L-structure and A, B be subsets of $\mathcal{M}$. We write dcl instead of $dcl_{\mathcal{M}}$.
Properties of dcl:

(1) $A \subset dcl(A)$.

Proof. Let $a \in A$. Then $d(x, a)$ is an $L(A)$-formula, so $a \in dcl(A)$. $\square$

(2) If $A \subset dcl(B)$ then $dcl(A) \subset dcl(B)$.

Proof.

Lemma 3. *For $X \subset \mathcal{M}$, if $\sigma \in Aut(\mathcal{M}/X)$, then $\sigma(x) = x$ for all $x \in dcl(X)$.*

Proof. Let $x \in dcl(X)$. By 10.7(2), x is the only realization of $tp_{\mathcal{M}}(x/X)$. Let $\sigma \in Aut(\mathcal{M}/X)$. Then $\sigma(x) \vDash tp_{\mathcal{M}}(x/X)$, so $\sigma(x) = x$. □

Let $a \in dcl(A)$. By 10.7(2), it is enough to show that the only realization of $tp_{\mathcal{M}}(a/B)$ is a. Let $b \vDash tp_{\mathcal{M}}(a/B)$. Let $\sigma \in Aut(\mathcal{M}/B)$ be such that $\sigma(a) = b$. Since σ fixes B point wise, by the lemma, it fixes $dcl(B)$ point wise, and thus, A point wise. $\sigma \in Aut(\mathcal{M}/A)$, so by the lemma, $\sigma(a) = a$. Thus, $b = a$.

Hence, $a \in dcl(B)$. □

(3) If $a \in dcl(A)$ then there exists a countable set $A_0 \subset A$ such that $a \in dcl(A_0)$.

Proof. Let $a \in dcl(A)$. Let $\phi_k(x, \overline{y})$ and $\overline{a_k} \in A^{|\overline{y}|}$ be such that $|d(x, a) - \phi_k(x, \overline{a_k})| \leq \frac{1}{k}$. Let $A_0 = \bigcup_{k<\omega} \overline{a_k}$. A_0 is countable, since each $|y|$ is finite. Thus, $d(x, a)$ is definable over A_0, so $a \in dcl(A_0)$. □

(4) If A is a dense subset of B, then $dcl(A) = dcl(B)$.

Proof. $A \subset B \Rightarrow A \subset dcl(B)$ by (1), so $dcl(A) \subset dcl(B)$ by (2).

Let $b \in dcl(B)$. By 10.7(2), it is enough to show that b is the only realization of $tp_{\mathcal{M}}(b/A)$.

Claim 8. *For $A \subset B$ dense, if $c \vDash tp_{\mathcal{M}}(b/A)$ then $c \vDash tp_{\mathcal{M}}(b/B)$.*

Proof. Let $\phi(x, b_1, \ldots, b_m) = 0 \in tp_{\mathcal{M}}(b/B)$ with $b_1, \ldots, b_m \in B$. Let $\epsilon > 0$ be given. We will show that $\phi(c, b_1, \ldots, b_m) \leq \epsilon$, and thus, since $\epsilon > 0$ was arbitrary, $\phi(c, b_1, \ldots, b_m) = 0$.

Let δ be such that $\max(d(a_1, b_1), \ldots, d(a_m, b_m)) < \delta \Rightarrow |\phi(x, b_1, \ldots, b_m) - \phi(x, a_1, \ldots, a_m)| \leq \epsilon$ ($\delta = \Delta(\epsilon)$ where Δ is the modulus of uniform continuity for ϕ). We can choose such an $a_1, \ldots, a_m \in A$ since $A \subset B$ is dense. So since $\phi(b, b_1, \ldots, b_m) = 0$, $\phi(b, a_1, \ldots, a_m) \leq \epsilon$. Thus, since $c \vDash tp_{\mathcal{M}}(b/A)$, $\phi(c, a_1, \ldots, a_m) \leq \epsilon$, as required.

Hence $c \vDash tp_{\mathcal{M}}(b/B)$.

□

So suppose $a \vDash tp_{\mathcal{M}}(b/A)$. Then by the claim, $a \vDash tp_{\mathcal{M}}(b/B)$. So since $b \in dcl(B)$, by 10.7(2), we must have $b = a$.

Hence, b is the only realization of $tp_{\mathcal{M}}(b/A)$, so $b \in dcl(A)$.

□

10.12 Exercise

Let $\mathcal{M}$ be an L-structure and A, B be subsets of $\mathcal{M}$. We write acl instead of $acl_{\mathcal{M}}$.
Properties of acl:

(1) $A \subset acl(A)$.

Proof. Let $a \in A$. Then $\{a\}$ is definable by $d(x, a)$, an $L(A)$-formula, and it is compact, so $a \in acl(A)$.

□

(2) If $A \subset acl(B)$ then $acl(A) \subset acl(B)$.

Proof.

Lemma 4. *For any $X \subset \mathcal{M}$, if $\sigma \in Aut(\mathcal{M}/X)$ and $x \in acl(X)$, then $\sigma(x) \in acl(X)$.*

Proof. Let $x \in acl(X)$ and $\sigma \in Aut(\mathcal{M}/X)$. Then $tp_{\mathcal{M}}(\sigma(x)/X) = tp_{\mathcal{M}}(x/X)$. So by 10.8(2), since $x \in acl(X)$, the only realizations of this type are in $\mathcal{M}$. Thus, by 10.8(2), $\sigma(x) \in acl(X)$. □

Let $a \in acl(A)$. Let $\mathcal{N} \succeq \mathcal{M}$ and $b \in \mathcal{N}$ be such that $b \vDash tp_{\mathcal{M}}(a/B)$. So, by 10.8(2), it is enough to show that $b \in \mathcal{M}$ to see that $a \in acl(B)$.

Let $\sigma \in Aut(\mathcal{N}/B)$ be such that $\sigma(a) = b$. σ fixes B, so it fixes $acl(B)$ set wise. So $\sigma(A) \subset acl(B) \subset \mathcal{M}$.

Claim 9. $\sigma(a) \in acl_{\mathcal{N}}(\sigma(A))$

Proof. Let $C \subset \mathcal{N}$ be compact with $a \in C$ which is definable over A (in fact, since $a \in acl(A)$, there is such a C in $\mathcal{M}$). $\sigma(C)$ is compact since σ is an isometry, and definable over $\sigma(A)$, so since $\sigma(a) \in \sigma(C)$, $\sigma(a) \in acl_{\mathcal{N}}(\sigma(A))$.

□

Since $\sigma(A) \subset \mathcal{M}$, by Proposition 10.5, $acl_{\mathcal{N}}(\sigma(A)) = acl_{\mathcal{M}}(\sigma(A))$, so $\sigma(a) \in \mathcal{M}$, and thus, $b \in \mathcal{M}$ as required.

□

(3) If $a \in acl(A)$ then there exists a countable set $A_0 \subset A$ such that $a \in acl(A_0)$.

Proof. Let C be a compact set definable over A with $a \in C$.

Let $\phi_k(x, \overline{y})$ and $\overline{a_k} \in A^{|\overline{y}|}$ be such that $|d(x, C) - \phi_k(x, \overline{a_k})| \leq \frac{1}{k}$. Let $A_0 = \bigcup_{k<\omega} \overline{a_k}$. A_0 is countable, since each $|y|$ is finite. Thus, C is definable over A_0, so $a \in acl(A_0)$.

□

(4) If A is a dense subset of B, then $acl(A) = acl(B)$.

Proof. $A \subset B \Rightarrow A \subset acl(B)$ by (1) $\Rightarrow acl(A) \subset acl(B)$ by (2).

So let $b \in acl(B)$. Consider $tp_{\mathcal{M}}(b/A)$. By Claim 8, if $a \models tp_{\mathcal{M}}(b/A)$, then $a \models tp_{\mathcal{M}}(b/B)$, so since $b \in acl(B)$, by 10.8(2), $a \in \mathcal{M}$. Thus, by 10.8(2), $b \in acl(A)$.

Hence, $acl(A) = acl(B)$.

$\square$