Weak Theories of Arithmetic

Louise Hay Logic Seminar, UIC

Victoria Noquez

September 29, 2011

Abstract

We will use a forcing argument to show that certain statements provable in a nonstandard extension of primitive recursive arithmetic are also provable in primitive recursive arithmetic.

1 PRA^ω

1.1 Finite Types

- N is a type, meant to denote the natural numbers

For types σ and τ,

- $\sigma \to \tau$ is a type, denoting functions from things of type of σ to things of type τ
- $\sigma \times \tau$ is a type, denoting the cross product of the set of things of type σ and the set of things of type τ

We use $\sigma, \tau \to \rho$ to abbreviate $\sigma \to (\tau \to \rho)$.

1.2 Language of PRA^ω (L)

L has variables of all finite types and the following constants

- 0 of type N (zero)
- S of type $N \to N$ (successor)

For types σ and τ

- $\langle\ ,\ \rangle$ of type $\sigma, \tau \to \sigma \times \tau$ (pairing)
- $(\ )_0$ and $(\ )_1$ of type $\sigma \times \tau \to \sigma$ and $\sigma \times \tau \to \tau$ respectively (projections)
- R of type $N, (N, N \to N), N \to N$ (primtive recursion)
- $Cond_\sigma$ of type $N, \sigma, \sigma \to \sigma$ (indicator)

1.3 Axioms of PRA^ω

For $r[z]$ of type N, z of appropriate type

- **Application** For s, t terms, x a variable,

$$r[(\lambda x t)(s)] = r[t[s/x]]$$

- **Projection** For x, y terms

$$r[(\langle x, y\rangle)_0] = r[x]$$
$$r[(\langle x, y\rangle)_1] = r[y]$$

- **Successor** For x, y of type N

$$\neg S(x) = 0$$
$$S(x) = S(y) \to x = y$$

- **Primitive Recursion** For a, x of type N, f of type $N, N \to N$

$$R(a, f, 0) = a$$
$$R(a, f, (S(x))) = f(x, R(a, f, x))$$

- **Indicator** For n of type N, x, y of type σ

$$r[Cond_\sigma(0, x, y)] = r[x]$$
$$r[Cond_\sigma(S(n), x, y)] = r[y]$$

2 Σ_1-induction

For every Σ_1-formula ϕ in L,

$$\forall x(\phi(0) \wedge \forall y < x(\phi(y) \to \phi(y+1)) \to \phi(x))$$

Fact: Over PRA^ω, this is equivalent to saying that every bounded function on N has a least upper bound, and attains it. That is, for all f of type $N \to N$,

$$\exists z \forall y(f(y) \leq z) \to \exists x \forall y(f(y) \leq f(x))$$

3 $NPRA^\omega$

3.1 Language of $NPRA^\omega$ (L^{st})

- Symbols of L
- $st(t)$, a unary predicate over N (standard)
- ω, a constant of type N (infinity)

3.2 Axioms of $NPRA^\omega$

- Axioms of PRA^ω
- $\neg st(\omega)$ (ω is non-standard)
- For x, y of type N,

$$st(x) \wedge y < x \to st(y)$$

(everything below a standard element is standard)

- For $x_1, \ldots, x_k$ of type N and f of type $N^k \to N$

$$st(x_1) \wedge \ldots \wedge st(x_k) \to st(f(x_1, \ldots, x_k))$$

(the standard part of the universe is closed under primitive recursion)

- For $\psi(\vec{x})$ quantifier free, internal, and not involving ω, with free variables shown,

$$\forall^{st} \vec{x} \psi(\vec{x}) \to \forall \vec{x} \psi(\vec{x})$$

4 The Interpretation

4.1 Translating the terms of L^{st} to terms of L

- Let ω be a type N variable in L, corresponding to the constant ω in L^{st}
- For each variable x in L^{st} of type σ, let $\widetilde{x}$ be of type $N \to \sigma$ in L
- If $t[x_1, \ldots, x_n]$ is a term of L^{st} with free variables shown, let $\widehat{t}$ denote $t[\widetilde{x_1}(\omega), \ldots \widetilde{x_k}(\omega)]$ where the constant ω is replacted with the variable ω

4.2 The Forcing Relation $\Vdash$

For a unvary predicate p, let $Cond(p)$ denote $\forall z \exists \omega \geq z p(\omega)$. For predicate p, q, let $q \preceq p$ denote $\forall u(q(u) \to p(u)) \wedge Cond(q)$. We define $p \Vdash \phi$ for formulas ϕ of L^{st} inductively as follows:

- $p \Vdash t_1 = t_2 \equiv \exists z \forall \omega \geq z(p(\omega) \to \widehat{t_1}(\omega) = \widehat{t_2}(\omega))$
- $p \Vdash t_1 < t_2 \equiv \exists z \forall \omega \geq z(p(\omega) \to \widehat{t_1}(\omega) < \widehat{t_2}(\omega))$
- $p \Vdash st(t) \equiv \exists z \forall \omega \geq z(p(\omega) \to \widehat{t}(\omega) < z)$
- $p \Vdash \phi \to \psi \equiv \forall q \preceq p(q \Vdash \phi \to q \Vdash \psi)$
- $p \Vdash \phi \wedge \psi \equiv (p \Vdash \phi) \wedge (p \Vdash \psi)$
- $p \Vdash \neg\phi \equiv \forall q \preceq p(q \nVdash \phi)$
- $p \Vdash \forall x \phi \equiv \forall \widetilde{x}(p \Vdash \phi)$

Facts:

- $p \Vdash \phi \vee \psi \equiv \forall q \preceq p \exists r \preceq q(r \Vdash \phi \vee r \Vdash \psi)$
- $p \Vdash \exists x \phi \equiv \forall q \preceq p \exists r \preceq q \exists \widetilde{x}(r \Vdash \phi)$
- $Cond(p) \to \neg(p \Vdash \phi \wedge p \Vdash \neg\phi)$

Let $\Vdash \phi$ denote $\forall p(Cond(p) \to p \Vdash \phi)$.

5 The Theorem

Theorem 1. *Suppose $NPRA^\omega$ proves $\forall^{st} x \exists y \phi(x, y)$ where ϕ is a quantifier free formula of L with free variables shown. Then $PRA^\omega + \Sigma_1$-induction proves $\forall x \exists y \phi(x, y)$.*

5.1 Outline of Proof

1. For ϕ in the language L^{st}, if ϕ is provable classically, then PRA^ω proves $\Vdash \phi$.
   (a) For each formula ϕ in the language of L^{st}, if ϕ is provable in intuitionistic logic, and has free variables $\vec{x}$, then PRA^ω proves $\Vdash \forall \vec{x} \phi$.
   (b) For each formula ϕ of L^{st}, PRA^ω proves $\Vdash \neg\neg\phi \to \phi$.
2. If ϕ is an axiom of PRA^ω, then PRA^ω proves $\Vdash \phi$
3. PRA^ω proves $\Vdash (\phi(0) \wedge \forall k < x(\phi(k) \to \phi(k+1)) \to \phi(x)$ for any x of type N and Σ_1-formula ϕ of L.
4. Suppose ϕ is any formula of L^{st} and $NPRA^\omega$ proves ϕ. Then PRA^ω proves $\Vdash \phi$.
5. Cleverly apply this to prove the theorem.

You can find the whole proof at **www.math.uic.edu/∼noquez/research.html**