Strong Minimality in Continuous Logic

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Overview

Continuous Logic

Idea: Instead of being true or false, formulas have a value in $[0, 1]$.

- The signature (or language) is the same as in classical logic: functions, constants, and predicates (relations), but now the predicates are functions from M^n to $[0, 1]$.
- Logical symbols:
  - d, the metric on the underlying space
  - variables, constants
  - a symbol for each continuous function $u : [0, 1]^n \to [0, 1]$ (these are connectives)
  - sup and inf (these are quantifiers)
- Convention: we use $\phi(\overline{x}) = 0$ to mean $\phi(\overline{x})$ is "true".

Observations

- This is a "positive" language. There is no negation, and inf acts as $\exists$, but only gives approximate witnesses.
- $\{0, 1, \frac{x}{2}, \dot{-}\}$ is a *full* set of connectives, meaning that any formula can be approximated by formulas only using these connectives.
- From now on, "formula" refers to formulas only using these connectives.
- A *definable predicate* $P(\overline{x})$ is a function from $\mathcal{M}^n \to [0, 1]$ which can be uniformly approximated by formulas.

Structures and Theories

- $\mathcal{L}$-structures are complete metric spaces $\mathcal{M}$.
- For an $\mathcal{L}$-formula $\phi(\overline{x})$ and $\overline{a} \in \mathcal{M}$, $\mathcal{M} \models \phi(\overline{a}) = 0$ if $\phi^{\mathcal{M}}(\overline{a}) = 0$.
- $\phi(\overline{x}) = 0$ is called an $\mathcal{L}$-condition.
- Theories are collections of $\mathcal{L}$-conditions with no free variables.

Example

$\mathcal{L} = \emptyset$, $\mathcal{M}$ an infinite set with the discrete metric.

$$\mathcal{M} \models \sup_x d(x,x) = 0$$

$$\mathcal{M} \models \inf_{x_1} \dots \inf_{x_n} \max_{1 \leq i < j \leq n} (1 - d(x_i, x_j)) = 0$$

Viewing Classical Structures as Continuous Structures

- Let $\mathcal{L}$ be a classical language. Let $\mathcal{L}'$ be a continuous language with all of the same symbols as $\mathcal{L}$.
- For $\mathcal{M}$ a (classical) $\mathcal{L}$-structure, let $\mathcal{M}'$ be a continuous $\mathcal{L}'$-structure with the same universe as $\mathcal{M}$, equipped with the discrete metric. Note that $\mathcal{M}'$ is complete.
- View $\mathcal{M}'$ as an $\mathcal{L}'$-structure as follows:
    - For a constant symbol c, $c^{\mathcal{M}'} = c^{\mathcal{M}}$
    - For a function symbol f, $\bar{a} \in \mathcal{M}$, $f^{\mathcal{M}'}(\bar{a}) = f^{\mathcal{M}}(\bar{a})$
    - For a relation symbol R, $\bar{a} \in \mathcal{M}'$,

$$R^{\mathcal{M}'}(\bar{a}) = \begin{cases} 0 & \mathcal{M} \models R(\bar{a}) \\ 1 & \mathcal{M} \models \neg R(\bar{a}) \end{cases}$$

- Note that the $\mathcal{L}'$-terms are just $\mathcal{L}$-terms.

Viewing Classical Formulas as Continuous Formulas

For a classical $\mathcal{L}$-formula $\theta(\overline{x})$, define the continuous $\mathcal{L}'$-formula $\theta'(x)$ inductively as follows:

$\theta(\overline{x})$	$\theta'(\overline{x})$
$t_1(\overline{x}) = t_2(\overline{x})$	$d(t_1(\overline{x}), t_2(\overline{x}))$
$R(t_1(\overline{x}), \ldots, t_n(\overline{x}))$	$R(t_1(\overline{x}), \ldots, t_n(\overline{x}))$
$\phi(\overline{x}) \wedge \psi(\overline{x})$	$\max(\phi'(\overline{x}), \psi'(\overline{x}))$
$\phi(\overline{x}) \vee \psi(\overline{x})$	$\min(\phi'(\overline{x}), \psi'(\overline{x}))$
$\neg\phi(\overline{x})$	$1 - \phi'(\overline{x})$
$\exists y \phi(y, (\overline{x}))$	$\inf_y \phi'(y, \overline{x})$
$\forall y \phi(y, \overline{x})$	$\sup_y \phi'(y, \overline{x})$

Note: $\min(\phi'(\overline{x}), \psi'(\overline{x})) = \phi'(\overline{x}) \dot{-} (\phi'(\overline{x}) \dot{-} \psi'(\overline{x}))$ and $\max(\phi'(\overline{x}), \psi'(\overline{x})) = 1 \dot{-} ((1 \dot{-} (\phi'(\overline{x}) \dot{-} \psi'(\overline{x}))) \dot{-} \psi'(\overline{x}))$ are $\mathcal{L}'$-formulas.

Viewing Classical Theories as Continuous Theories

Fact

For an $\mathcal{L}$-formula $\theta(\overline{x})$ and $\mathcal{L}$-structure $\mathcal{M}$, for all $\overline{x} \in \mathcal{M}'$, $\mathcal{M}' \vDash \theta'(\overline{x}) = 0$ or $\mathcal{M}' \vDash \theta'(\overline{x}) = 1$, and

$$\mathcal{M} \vDash \theta(\overline{x}) \Leftrightarrow \mathcal{M}' \vDash \theta'(\overline{x}) = 0$$

$$\mathcal{M} \vDash \neg\theta(\overline{x}) \Leftrightarrow \mathcal{M}' \vDash \theta'(\overline{x}) = 1$$

- Let T be a classical $\mathcal{L}$-theory.
- Let T' be the continuous $\mathcal{L}'$-theory $\{\theta' = 0 | T \vdash \theta\} \cup \{1 \dot{-} \theta' = 0 | T \vdash \neg\theta\}$.

Fact

For an $\mathcal{L}$-structure $\mathcal{M}$, $\mathcal{M} \vDash T \Leftrightarrow \mathcal{M}' \vDash T'$

A Proposed Characterization of Strong Minimality

The following characterization of strong minimality for continuous logic was suggested by Isaac Goldbring:

"Definition"

A continuous theory T is "strongly minimal" if for any $\mathcal{M} \models T$, and any definable predicate $P(x)$, $Z(P) = \{x \in \mathcal{M} | \mathcal{M} \models P(x) = 0\}$ is totally bounded, or $\mathcal{M} \setminus Z(P)$ is totally bounded.

Pros

Theorem

(Exchange Principle) Let $\mathcal{M} \models T$, assume T is "strongly minimal". For $a, b \in \mathcal{M}$ and $A \subset \mathcal{M}$, if $a \in acl(Ab) \setminus acl(A)$, then $b \in acl(Aa)$.

Theorem

For a classical theory T, if T' is "strongly minimal", then T is strongly minimal (in the classical sense).

Cons

Theorem

If T is "strongly minimal" and $\mathcal{M} \models T$, then $\mathcal{M}$ is locally compact at a point.

So the theory of infinite dimensional Hilbert spaces is not "strongly minimal".

More Cons

A classical theory T being strongly minimal does not guarantee that its corresponding continuous theory T' is "strongly minimal".

Example

T, the theory of infinite sets in the empty language, is strongly minimal, but its corresponding continuous theory T' is not "strongly minimal".

- Let $A = \{a_i : i < \omega\}$ be an infinite set with an infinite complement.
- $\phi_k(x) = \max(1 \dot{-} d(x, a_0), \frac{1}{2}(1 \dot{-} d(x, a_1)), \ldots, \frac{1}{2^k}(1 \dot{-} d(x, a_k)))$
- $\phi_k(x)$ converges uniformly to

$$P(x) = \begin{cases} 0 & x \notin A \\ \frac{1}{2^k} & x = a_k \end{cases}$$

- P is a definable prediate, and $Z(P) = \mathcal{M} \setminus A$ and $\mathcal{M} \setminus Z(P) = A$ are both infinite, so not totally bounded.

A Revision

Definition

A continuous theory T is *strongly minimal* if for any $\mathcal{M} \vDash T$, and any definable predicate $P(x)$, $Z(P) = \{x \in \mathcal{M} | \mathcal{M} \vDash P(x) = 0\}$ is totally bounded, or for every $\delta > 0$, $\mathcal{M} \setminus \{x \in \mathcal{M} | \mathcal{M} \vDash P(x) \leq \delta\}$ is totally bounded.

Pros

Theorem (N.)

(Exchange Principle) Let $\mathcal{M} \vDash T$, assume T is strongly minimal. For $a, b \in \mathcal{M}$ and $A \subset \mathcal{M}$, if $a \in acl(Ab) \setminus acl(A)$, then $b \in acl(Aa)$.

More Pros

Theorem (N.)

For a classical theory T, T is strongly minimal if and only if T', its corresponding continuous theory, is strongly minimal.

Proof sketch:

- P a definable predicate, $\phi_k \to P$
- For each ϕ_k there are finitely many $r_0, \ldots, r_n \in [0,1]$ such that for all x, $\phi_k(x) = r_i$ for some i.
- There is $0 \leq i \leq n$ such that $\{x | \phi_k(x) = r_i\}$ is cofinite. Let $r^k := r_i$.
- $(r^k : k < \omega)$ is Cauchy, so converges to some $r^* \in [0,1]$
- If $r^* \neq 0$, $Z(P)$ is finite (totally bounded).
- If $r^* = 0$, for any $\delta > 0$, for k sufficiently large, for cofinitely many x, $P(x) \leq |P(x) - \phi_k(x)| + |\phi_k(x)| = |P(x) - \phi_k(x)| + |\phi_k(x) - r^*| \leq \delta$, so $\{x | P(x) \leq \delta\}$ is cofinite, so $Z(P) \setminus \{x | P(x) \leq \delta\}$ is finite.

Cons

None yet!

Conjecture

The theory of infinite dimensional Hilbert spaces is strongly minimal.

Thank You!