Proof Theory Project

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1 PRA^ω

To define PRA^ω, we start with a many sorted version of first-order predicate logic with a sort for each finite type, and an equality relation $=$ at type N only. Finite types are defined inductively as follows: N is a type, denoting the natural numbers in the intended interpretation. For types σ and τ, $\sigma \times \tau$ and $\sigma \to \tau$ are types denoting the cross product of σ and τ and the set of functions from σ to τ respectively. We use $\sigma, \tau \to \rho$ to abbreviate $\sigma \to (\tau \to \rho)$.

We have variables for all finite types and the following constants:

- 0 of type N
- S of type $N \to N$
- For types σ, τ, a constant of type $\sigma, \tau \to \sigma \times \tau$ for paring, $\langle x, y\rangle$
- For types σ, τ constants of type $\sigma \times \tau \to \sigma$ and $\sigma \times \tau \to \tau$ for the projections $(z)_0$ and $(z)_1$
- R of type $N, (N, N \to N), N \to N$
- For each type σ, $Cond_\sigma$ of type $N, \sigma, \sigma \to \sigma$.

The set of lambda terms is closed under lambda abstraction, denoted λxt, and application, denoted $t(s)$. If t and s are terms and x is a variable of the appropriate type, then $t[s/x]$ denotes the result of substituting s for x in t, renaming bound variables if necessary.

1.1 Axioms of PRA^ω

For $r[z]$ a term of type N, z a variale of appropriate type, s and t terms and x a variable,

$$r[(\lambda x.t)(s)] = r[t[s/x]]$$

For x, y terms of types σ, τ respectively,

$$r[(\langle x, y\rangle)_0] = r[x]$$

$$r[(\langle x, y\rangle)_1] = r[y]$$

For x, y of type N,

$$\neg S(x) = 0$$

$$S(x) = S(y) \rightarrow x = y$$

For a, x of type N and f of type $N, N \rightarrow N$,

$$R(a, f, 0) = a$$

$$R(a, f, S(x)) = f(x, R(a, f, x))$$

For $r[z]$ a term of type N and z of type σ, n of type N, and x, y of type σ

$$r[Cond_\sigma(0, x, y)] = r[x]$$

$$r[Cond(S(n), x, y)] = r[y]$$

For ϕ Σ_1, and induction scheme equivalent to

$$\forall x(\phi(0) \wedge \forall y < x(\phi(y) \rightarrow \phi(y+1)) \rightarrow \phi(x))$$

Note that since we have projections and successor, with the axioms for R we can define all primitive recursive functions, and thus, by identifying relations with their characteristic functions, we can use primitive recursion to define the relation $x < y$. For y of type N, we use $y + 1$ to abbreviate $S(y)$.

We will use the following in our proof of the main theorem:

Lemma 1. *Over PRA^ω, Σ_1-induction is equivalent to the following principle:*

$$\exists z \forall y(f(y) \leq z) \rightarrow \exists x \forall y(f(y) \leq f(x)) \tag{1}$$

which says that every bounded function on N has a least upper bounded, and it attains it.

Proof. Note that for a Σ_1 formula $\phi(x)$, Σ_1-induction is equivalent to

$$\forall x(\phi(0) \wedge \forall k < x(\phi(k) \rightarrow \phi(k+1)) \rightarrow \phi(x)).$$

The contrapositive of (1) is

$$\forall x \exists y(f(y) > f(x)) \rightarrow \forall z \exists y(f(y) > z).$$

$\Rightarrow$: Suppose $\forall x \exists y(f(y) > f(x))$. Let $\phi(z)$ be $\exists y f(y) > z$. Let z be given. $\phi(0) \equiv \exists y(f(y) > 0)$ holds since $\exists y(f(y) > f(0))$ by assumption and $f(0) \geq 0$. Let $k \leq z$ and assume $\exists y(f(y) > k)$. By assumption there exists y_2 such that $f(y_2) > f(y)$, so since $f(y_2) > f(y) > k$, $f(y_2) > k + 1$, so $\exists y(f(y) > k + 1)$, so $\phi(k + 1)$.

Hence, by Σ_1-induction, since $\forall z \phi(z) \equiv \forall z \exists y f(y) > z$, the claim holds.

$\Leftarrow$: Let $\phi(u, v)$ be a Δ_0 formula satisfying $\exists v \phi(0, v) \wedge \forall u(\exists v \phi(u, v) \rightarrow \exists v \phi(u + 1, v))$. Define $f(x)$ to be the greatest $w \leq x$ such that $\forall u < w \exists v \leq x \phi(u, v)$.

Claim 1. *$\forall x \exists y(f(y) > f(x))$.*

Proof of claim: Let x be given. If $f(x) = 0$, let y be given (by assupmtion) such that $\phi(0, y)$. Then $f(y + 1) \geq 1$ since $(\exists v \leq y + 1)\phi(0, v)$. So $\forall u < 1$, $(\exists v \leq y + 1)\phi(u, v)$ holds.

If $f(x) = w > 0$, then $\forall u < w \exists v \leq x \phi(u, v)$. Let $w = s + 1$. Then $\forall u \leq s \exists v \leq x \phi(u, v)$. Let $u \leq s$. $\exists v \leq x \phi(u, v)$, so by assumption, $\exists v_u \phi(u+1, v_u)$. Let v_z be such that $\phi(0, v_z)$ and let $y = \max\{v_u | u \leq s\} \cup \{v_z, x\}$. Then $f(y) > w = s + 1$, since $(\forall u \leq s + 1)(\exists v \leq y)\phi(u, v)$. □(Claim)

So by assumption, $\forall z \exists y(f(y) > z)$, that is, for all z, there is y such that the greatest w such that $\forall u < w \exists v \leq y \phi(u, v)$ is greater than z.

Let x be given. Then there is y such that $(\forall u < x + 1)(\exists v \leq y)\phi(u, v)$, so for x, there is some v such that $\phi(x, v)$. So $\exists v \phi(x, v)$.

□

2 $NPRA^\omega$

Now we look at a nonstandard version of PRA^ω, which we will call $NPRA^\omega$.

We start by adding a relation symbol $st(t)$ ranging over N, and a new constant ω of type N.

We use $\forall^{st} x \phi$ and $\exists^{st} x \phi$ to abbreviate $\forall x(st(x) \to \phi)$ and $\exists x(st(x) \to \phi)$ respectively. A formula ϕ is said to be internal if it does not involve st, and external otherwise.

2.1 Axioms of $NPRA^\omega$

We add to the axioms of PRA^ω the following:

$$\neg st(\omega)$$

For x, y of type N,

$$st(x) \wedge y < x \to st(y)$$

For $x_1, \ldots, x_k$ of type N and f of type $N, \ldots, N \to N$,

$$st(x_1) \wedge \ldots st(x_k) \to st(f(x_1, \ldots, x_k))$$

For $\psi(\vec{x})$ quantifier-free, internal and not involving ω, with only the free variables shown

$$\forall^{st} \vec{x} \psi(\vec{x}) \to \forall \vec{x} \psi(\vec{x})$$

3 Proving the Theorem

The interpretation and lemmas of this section will be used to prove the following theorem

Theorem 2. *Suppose $NPRA^\omega$ proves $\forall^{st} x \exists y \phi(x, y)$, where ϕ is quantifier-free in the language of PRA^ω with the free variables shown. Then $PRA^\omega + \Sigma_1$-induction proves $\forall x \exists y \phi(x, y)$.*

The interpretation of $NPRA^\omega$ in PRA^ω uses a forcing argument, described entirely in the language of PRA^ω. Let L denote the language of PRA^ω and L^{st} denote the languae of $NPRA^\omega$.

First, we need to translate terms of L^{st} to terms of L. Let ω be a type N variable in L corresponding to the constnat ω in L^{st}. For each variable x of type σ in L^{st}, let $\tilde{x}$ of type $N \to \sigma$ in L. Finally, if $t[x_1, \ldots, x_n]$ is a term of L^{st} with free variables shown, let $\hat{t}$ denote the term $t[\tilde{x}_1(\omega), \ldots, \tilde{x}_k(\omega)]$ of L where the constant ω of L^{st} is also replaced by the corresponding variable of L.

For a unary predicate p on N in L, define $Cond(p) \equiv \forall z \exists w \geq z p(w)$. For a predicates q and a condition p, let $q \preceq p$ be defined by $\forall u(q(u) \to p(u)) \wedge Cond(q)$.

Now for a predicate p and a formula ϕ of L^{st}, we define the forcing relation $p \Vdash \phi$ as follows:

- $p \Vdash t_1 = t_2 \equiv \exists z \forall w \geq z(p(w) \to \hat{t_1} = \hat{t_2})$
- $p \Vdash t_1 < t_2 \equiv \exists z \forall w \geq z(p(w) \to \hat{t_1} < \hat{t_2})$
- $p \Vdash st(t) \equiv \exists z \forall w \geq z(p(w) \to \hat{t} < z)$
- $p \Vdash \phi \to \psi \equiv \forall q \preceq p(q \Vdash \phi \to q \Vdash \psi)$
- $p \Vdash \phi \Vdash \phi \wedge \psi \equiv (p \Vdash \phi) \wedge (p \Vdash \psi)$
- $p \Vdash \forall x \phi \equiv \forall \tilde{x}(p \Vdash \phi)$

Lemma 3. *For a predicate p, $Cond(p) \Leftrightarrow p \nVdash \perp$.*

Proof. $p \Vdash \perp$
$\Leftrightarrow \exists z \forall w \geq z(p(w) \to \perp)$
$\Leftrightarrow \exists z \forall w \geq z(\neg p(w))$
$\Leftrightarrow \neg \forall z \exists w \geq z(p(w))$
$\Leftrightarrow \neg Cond(p)$

□

Let $\Vdash \phi$ denote $\forall p(Cond(p) \to p \Vdash \phi)$.

Lemma 4. *Suppose t and s are terms of L^{st}, $r[z]$ is a type N term of PRA^ω, and z has the same type as t. Then PRA^ω proves*

$$r[\widehat{t}[\lambda\omega\widehat{s}/\tilde{x}]] = r[\widehat{t[s/x]}].$$

Proof. The proof is by induction on terms. If $t = x$, then
$r[\widehat{t}[\lambda\omega\widehat{s}/\tilde{x}]]$
$= r[\widehat{x}[\lambda\omega\widehat{s}/\tilde{x}]]$
$= r[\tilde{x}(\omega)[\lambda\omega\widehat{s}/\tilde{x}]]$ by the definition of $\hat{}$
$= r[\lambda\omega\widehat{s}(\omega)]$
$= r[\widehat{s}]$
$= r[\widehat{x[s/x]}]$
$= r[\widehat{t[s/x]}]$

If $t = y$ is a variable or constant other than x, then
$r[\widehat{t}[\lambda\omega\widehat{s}/\tilde{x}]]$
$= r[\widehat{y}[\lambda\omega\widehat{s}/\tilde{x}]]$
$= r[\widehat{y}]$ since $\tilde{x}$ does not appear in $\widehat{y}$

$= r[\widehat{y[s/x]}]$.

If $t = f(t_1, \ldots, t_n)$ where $f, t_1, \ldots, t_n$ are terms for which the claim holds, then $r[\widehat{t}[\lambda\omega\widehat{s}/\tilde{x}]]$

$= r[\widehat{f(t_1, \ldots, t_n)}[\lambda\omega\widehat{s}/\tilde{x}]]$

$= r[\widehat{f}(\widehat{t_1}, \ldots, \widehat{t_n})[\lambda\omega\widehat{s}/\tilde{x}]]$

$= r[\widehat{f}[\lambda\omega\widehat{s}/\tilde{x}](\widehat{t_1}[\lambda\omega\widehat{s}/\tilde{x}], \ldots, \widehat{t_n}[\lambda\omega\widehat{s}/\tilde{x}])]$

$= r[\widehat{f[s/x]}(\widehat{t_1[s/x]}, \ldots, \widehat{t_n[s/x]})]$ by induction

$= r[\widehat{f(t_1, \ldots, t_n)[s/x]}]$

$= r[\widehat{t[s/x]}]$.

$\square$

Lemma 5 (Substitution)**.** *For each forrmula ϕ and terms s in the language L^{st}, PRA^ω proves $p \Vdash \phi[s/x] \leftrightarrow (p \Vdash \phi)[\lambda\omega\widehat{s}/\tilde{x}]$.*

Proof. By induction on formula.s Suppose ϕ is $t_1 = t_2$ for some terms t_1, t_2 of type N.

$p \Vdash \phi[s/x] \Leftrightarrow p \Vdash (t_1 = t_2)[s/x]$

$\Leftrightarrow p \Vdash t_1[s/x] = t_2[s/x]$

$\Leftrightarrow \exists z \forall w \geq z(p(w) \rightarrow \widehat{t_1[s/x]} = \widehat{t_2[s/x]})$

$\Leftrightarrow \exists z \forall w \geq z(p(w) \rightarrow \widehat{t_1}[\lambda\omega\widehat{s}/\tilde{x}] = \widehat{t_2}[\lambda\omega\widehat{s}/\tilde{x}])$

$\Leftrightarrow (\exists z \forall w \geq z(p(w) \rightarrow \widehat{t_1} = \widehat{t_2}))[\lambda\omega\widehat{s}/\tilde{x}]$

$\Leftrightarrow (p \Vdash t_1 = t_2)[\lambda\omega\hat{s}/\tilde{x}]$.

Now let ϕ be $t_1 < t_2$ for terms t_1, t_2 of type N.

$p \Vdash \phi[s/x] \Leftrightarrow p \Vdash (t_1 < t_2)[s/x]$

$\Leftrightarrow p \Vdash t_1[s/x] < t_2[s/x]$

$\Leftrightarrow \exists z \forall w \geq z(p(w) \rightarrow \widehat{t_1[s/x]} < \widehat{t_2[s/x]})$

$\Leftrightarrow \exists z \forall w \geq z(p(w) \rightarrow \widehat{t_1}[\lambda\omega\widehat{s}/\tilde{x}] < \widehat{t_2}[\lambda\omega\widehat{s}/\tilde{x}])$

$\Leftrightarrow (\exists z \forall w \geq z(p(w) \rightarrow \widehat{t_1} < \widehat{t_2}))[\lambda\omega\widehat{s}/\tilde{x}]$

$\Leftrightarrow (p \Vdash t_1 < t_2)[\lambda\omega\hat{s}/\tilde{x}]$.

Let ϕ be $st(t)$ for a term t of type N.

$p \Vdash \phi[s/x] \Leftrightarrow p \Vdash st[s/x](t[s/x])$

$\Leftrightarrow p \Vdash st(t[s/x])$

$\Leftrightarrow \exists z \forall w \geq z(p(w) \rightarrow \widehat{t[s/x]} < z)$

$\Leftrightarrow \exists z \forall w \geq z(p(w) \rightarrow \widehat{t}[\lambda\omega\widehat{s}/\tilde{x}] < z)$

$\Leftrightarrow (\exists z \forall w \geq z(p(w) \rightarrow \widehat{t} < z))[\lambda\omega\widehat{s}/\tilde{x}]$

$\Leftrightarrow (p \Vdash st(t))[\lambda\omega\widehat{s}/\tilde{x}]$.

Finally, suppose the claim holds for ϕ and ψ.

$p \Vdash (\phi \rightarrow \psi)[s/x] \Leftrightarrow p \Vdash \phi[s/x] \rightarrow \psi[s/x]$

$\Leftrightarrow \forall q \preceq p(q \Vdash \phi[s/x] \rightarrow q \Vdash \psi[s/x])$

$\Leftrightarrow \forall q \preceq p((q \Vdash \phi)[\lambda\omega\widehat{s}/\tilde{x}] \rightarrow (q \Vdash \psi)[\lambda\omega\widehat{s}/\tilde{x}])$

$\Leftrightarrow (\forall q \preceq p(q \Vdash \phi \rightarrow q \Vdash \psi))[\lambda\omega\widehat{s}/\tilde{x}]$

$\Leftrightarrow (p \Vdash \phi \rightarrow \psi)[\lambda\omega\widehat{s}/\tilde{x}]$.

$p \Vdash (\phi \wedge \psi)[s/x] \Leftrightarrow p \Vdash \phi[s/x] \wedge \psi[s/x]$

$\Leftrightarrow (p \Vdash \phi[s/x]) \wedge (p \Vdash \psi[s/x])$

$\Leftrightarrow (p \Vdash \phi)[\lambda\omega\widehat{s}/\tilde{x}] \wedge (p \Vdash \psi)[\lambda\omega\widehat{s}/\tilde{x}]$

$\Leftrightarrow (p \Vdash \phi \wedge p \Vdash \psi)[\lambda\omega\widehat{s}/\tilde{x}]$

$\Leftrightarrow (p \Vdash \phi \wedge \psi)[\lambda\omega\widehat{s}/\tilde{x}]$.

$p \Vdash (\forall y\phi)[s/x] \Leftrightarrow p \Vdash \forall y\phi[s/x]$
$\Leftrightarrow \forall\tilde{y}(p \Vdash \phi[s/x])$
$\Leftrightarrow \forall\tilde{y}((p \Vdash \phi)[\lambda\omega\widehat{s}/\tilde{x}])$
$\Leftrightarrow (\forall\tilde{y}(p \Vdash \phi))[\lambda\omega\widehat{s}/\tilde{x}]$
$\Leftrightarrow (p \Vdash \forall y\phi)[\lambda\omega\widehat{s}/\tilde{x}]$. □

Lemma 6. *For each formula ϕ of L^{st}, PRA^{ω} proves $p \Vdash \phi \wedge q \preceq p \to q \Vdash \phi$ for conditions p.*

Proof. By induction on formulas.

Suppose ϕ is $t_1 = t_2$ for terms t_1, t_2 of type N. Assume $p \Vdash \phi$ and $q \preceq p$. Then $p \Vdash t_1 = t_2$ and $\forall x(q(x) \to p(x)) \wedge Cond(q)$. $p \Vdash t_1 = t_2 \to \exists z \forall w \geq z(p(w) \to \widehat{t_1} = \widehat{t_2})$. Choose z such that $\forall w \geq z(p(w) \to \widehat{t_1} = \widehat{t_2})$. Then, since $\forall w \geq z(q(w) \to p(w)$, $\forall w \geq z(q(w) \to \widehat{t_1} = \widehat{t_2})$, so $q \Vdash t_1 = t_2$, that is, $q \Vdash \phi$.

If ϕ is $t_1 < t_2$ for terms t_1, t_2 of type N. Again, assume $p \Vdash \phi$ and $q \preceq p$. Then $p \Vdash t_1 < t_2$ and $\forall x(q(x) \to p(x)) \wedge Cond(q)$. $p \Vdash t_1 < t_2 \to \exists z \forall w \geq z(p(w) \to \widehat{t_1} < \widehat{t_2})$. Choose z such that $\forall w \geq z(p(w) \to \widehat{t_1} < \widehat{t_2})$. Then, since $\forall w \geq z(q(w) \to p(w)$, $\forall w \geq z(q(w) \to \widehat{t_1} < \widehat{t_2})$, so $q \Vdash t_1 < t_2$, that is, $q \Vdash \phi$.

Now suppose ϕ is $st(t)$. Assume $p \Vdash \phi$ and $q \preceq p$, so $p \Vdash st(t)$ and $\forall x(q(x) \to p(x)) \wedge Cond(q)$. $p \Vdash st(t) \to \exists z \forall w \geq z(p(w) \to \widehat{t} < z)$. Choose z such that $\forall w \geq z(p(w) \to \widehat{t} < z)$. Then, since $\forall w \geq z(q(w) \to p(w)$, $\forall w \geq z(q(w) \to \widehat{t} < z)$, so $q \Vdash st(t)$, that is, $q \Vdash \phi$.

Suppose the claim holds for formulas ϕ and ψ.

If $p \Vdash \phi \wedge \psi \wedge q \preceq p$, then $p \Vdash \phi$ and $p \Vdash \psi$, so by induction, $q \Vdash \phi$ and $q \Vdash \psi$, so $q \Vdash \phi \wedge \psi$.

If $p \Vdash \phi \to \psi \wedge q \preceq p$, then $\forall r \preceq p(r \Vdash \phi \to r \Vdash \psi)$. So, if $r \preceq q$, since $q \preceq p$, $r \preceq p$, so $\forall r \preceq q(r \Vdash \phi \to r \Vdash \psi)$. That is, $r \Vdash \phi \to \psi$.

If $p \Vdash \forall x\phi \wedge q \preceq p$, then $\forall\tilde{x}(p \Vdash \phi)$, so by induction, $\forall\tilde{x}(q \Vdash \phi)$. Thus, $q \Vdash \forall x\phi$.

□

Lemma 7. *For each formula ϕ in the language of L^{st}, PRA^{ω} proves $\Vdash (\bot \to \phi)$.*

Proof. Let p be a condition.

$p \Vdash \bot \to \phi \Leftrightarrow \forall q \preceq p(q \Vdash \bot \to q \Vdash \phi)$
$\Leftrightarrow \forall q \preceq p(\neg Cond(p) \to q \Vdash (\phi))$
$\Leftrightarrow \forall q \preceq p(Cond(p) \vee q \Vdash (\phi))$
which is true since $\forall q \preceq p(Cond(p))$ by definition of $\preceq$.

□

Lemma 8. *For each formula ϕ in the language of L^{st}, if ϕ is provable in intuitionistic logic, then PRA^{ω} proves $\Vdash \phi$.*

Proof. □

Lemma 9. *Let t be any term. $PRA^\omega + \Sigma_1$-induction proves the following: Let p be any condition and let q be the predicate defined by*

$$q(w) \equiv p(w) \wedge \forall u < w(p(u) \to \hat{t}(u) < \hat{t}(w)).$$

Then, if q is a condition, $q \Vdash \neg st(t)$.

Proof. Suppose q is a condition and let r be a predicate such that $\forall u(r(u) \to q(u))$. It suffices to show that if $r \Vdash st(t)$ then r is not a condition. This is because $q \Vdash \neg st(t) \Leftrightarrow q \Vdash st(t) \to \perp \Leftrightarrow \forall r \preceq q(r \Vdash st(t) \to r \Vdash \perp) \Leftrightarrow \forall r \preceq q(r \Vdash st(t) \to \neg Cond(r))$.

Suppose $r \Vdash st(t)$, i.e.,

$$\exists z \forall w \geq z(r(w) \to \hat{t}(w) < z). \tag{2}$$

Since $r \preceq q$, we know $\forall u(r(u) \to q(u))$, so for all w, $r(w) \to q(w) \to p(w) \wedge \forall u < w(p(u) \to \hat{t}(u) < \hat{t}(w))$. Thus, since $r(u) \to p(u)$ for all u,

$$\forall u \forall v(r(u) \wedge r(v) \wedge u < v \to \hat{t}(u) < \hat{t}(v)). \tag{3}$$

Define f by $f(v) = \max_{u \leq v \wedge r(u)} \hat{t}(u)$. By (2) f is bounded by some z. Since we are assuming Σ_1 induction, by Lemma 1, $\exists z \forall y(f(y) \leq z) \to \exists x \forall y(f(y) \leq f(x))$. So $\exists x \forall y(f(x) \geq f(y))$. Let x witness this and u be such that $f(x) = \hat{t}(u)$ (note that $r(u)$ holds). Then for any v with $r(v)$, take $y > v$ and note that $f(y) = (\max_{u \leq y \wedge r(u)} \hat{t}(u)) \geq \hat{t}(v)$. So $\hat{t}(v) \leq \hat{t}(u)$.

Let $w > u$ be given. By (3), $r(w) \wedge u < w \to \hat{t}(u) < \hat{t}(w)$. Thus, $\forall w > u \neg r(w)$, so r is not a condition.

□

Lemma 10. *$PRA^\omega + \Sigma_1$-induction proves that $\neg\neg st(t) \to st(t)$ is forced.*

Proof. Let p be a predicate and suppose $p \Vdash \neg\neg st(t)$. Then $\forall q \preceq p(q \Vdash \neg st(t) \to q \Vdash \perp) \Leftrightarrow \forall q \preceq p(q \Vdash \neg st(t) \to \neg Cond(q))$, so for all $q \preceq p$, since q is a condition, then $q \nVdash \neg st(t)$. Let q be as in the previous lemma. Clearly, $\forall u(q(u) \to p(u))$, so if q is a condition, $q \Vdash \neg st(t)$. Thus, q is not a condition.

So $\exists z \forall w \geq z \neg q(w)$, i.e., for some z, $\forall w \geq z(p(w) \to \exists u < w(p(u) \wedge \widehat{t}(w) \leq \widehat{t}(u)))$. Since p is a condition, pick $w \geq z$ such that $p(w)$ holds. Let $v = \max_{u \leq w \wedge p(u)} \hat{t}(u)$. Then $\forall w \geq z(p(w) \to \hat{t}(w) \leq v)$. So $\forall w \geq z(p(w) \to \hat{t}(w) \leq v)$. Thus, $p \Vdash st(t)$.

Hence, for any condition r, for any $p \preceq r$, $p \Vdash \neg\neg st(t) \to p \Vdash st(t)$, so $r \Vdash \neg\neg st(t) \to st(t)$. Thus, since r is arbitrary, $\neg\neg st(t) \to st(t)$ is forced. □

Lemma 11. *For each formula ϕ of L^{st}, PRA^ω proves $\Vdash \neg\neg\phi \to \phi$.*

Proof. □

Lemma 12. *For each formula ϕ in the language of L^{st}, if ϕ is provable classically, then PRA^ω proves $\Vdash \phi$.*

Proof. Follows from Lemma 8 and Lemma 11. □