Randomizations

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Abstract

Keisler first introduced the notion of the randomization of a structure M: a new structure whose universe is a set of "random elements" of M. It was later discovered that it is natural to consider Keisler's randomizations in the framework of continuous logic, and in this way, a class of continuous structures arises naturally from existing first order structures. As such, for a classical first order theory T, we get a continuous theory T^R, whose models are essentially spaces of M-valued random variables, where M is a model of T.

In this talk we will discuss the technical considerations required to consider randomizations as continuous structures and some of the properties of theories which are preserved. If time permits, we will discuss Ben Yaacov's proof that randomizing a theory preserves (the continuous analog of) NIP.

1 Atomless Probability Algebras

We let $(\Omega, \mathcal{B}, \mu)$ denote a probability space, where Ω is the sample space (possible outcomes), $\mathcal{B}$ is a collection of subsets of outcomes (events), and μ assigns probabilities to events, so $\mu : \mathcal{B} \to [0,1]$.

We say that $B \in \mathcal{B}$ is an *atom* if $\mu(B) > 0$ but there is no $C \in \mathcal{B}$ with $C \subset B$ such that $0 < \mu(C) < \mu(B)$. B is *atomless* if it contains no atoms. $(\Omega, \mathcal{B}, \mu)$ is atomless if Ω is.

Axioms of APA (Atomless Probability Algebras):

1. Boolean Algebra Axioms (e.g. $\sup_x \sup_y d(x \cup y, y \cup x) = 0$).
2. Measure Axioms: $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$, $\mu(X \cup Y) + \mu(X \cap Y) = \mu(X) + \mu(Y)$. Technical note: we don't actually have addition, it's truncated addition, so we would actually need to write this out in terms of truncated subtraction.
3. μ and d agree, that is, the metric is the one given by the measure: $\sup_x \sup_y |d(x, y) - \mu(x \Delta y)| = 0$.
4. Atomless

Here are some facts about APA:

Fact 1 (Proposition 16.9 in [2]). APA is ω-stable.

Corollary 1. *APA is complete.*

Corollary 2. *APA is NIP.*

2 Continuous Logic

We will not go into much detail here, other than that the signature of continuous structures is similar to that of classical structures, except that predicates become functions from M^n to $[0,1]$, and functions and predicates have certain continuity requirements.

Logical symbols are d, for the metric on the underlying space, variables, a symbol for each continuous function $u : [0,1]^n \to [0,1]$ of finitely many variables $n \geq 1$, and sup and inf, which play the role of quantifiers. Note that this is a positive language, in that we don't have negation. Further note that inf may not necessarily provide a witness, just an approximation.

Fact 2 (Proposition 6.6 in [1]). $\{0, 1, \frac{x}{2}, \dot{-}\}$ is a *full* set of connectives, meaning that any formula can be approximated by a formula only using these connectives.

3 Randomizations

Let T be a classical $\mathcal{L}$-theory. We are interested in considering T^R, the randomization of T. Note: we can actually define randomizations more generally for T a continuous theory, then consider classical theories as continuous ones with the discrete metric. For the purposes of this talk though, we will restrict our attention to the case when T is classical.

3.1 Language

T^R is an $\mathcal{L}^R$ theory, where $\mathcal{L}^R$ has two sorts:

- K, the "random variable" sort
- $\mathcal{B}$, the "event" sort

$\mathcal{L}^R$ has the following signature:

- Constants $\top, \bot$ of sort $\mathcal{B}$
- $\bigcap, \bigcup : \mathcal{B}^2 \to \mathcal{B}$
- $\neg : \mathcal{B} \to \mathcal{B}$
- $[\![\phi(\cdot)]\!] : K^n \to \mathcal{B}$ where $\phi(x_1, \ldots, x_n) \in \mathcal{L}$
- $\mu : \mathcal{B} \to [0,1]$
- $d_{\mathcal{B}} : \mathcal{B}^2 \to [0,1]$
- $d_K : K^2 \to [0,1]$

For $\mathcal{M} \models T$, a *randomization* of $\mathcal{M}$ is $(K, \mathcal{B})$, the $\mathcal{L}^R$ structure resulting from a pre-structure $(K', \mathcal{B}')$ where $(\Omega, \mathcal{B}', \mu)$ is an atomless, finitely additive probability space, and $K' \subset M^\Omega$.

Note that $(K, \mathcal{B}) \equiv (K', \mathcal{B}')$ and $(\Omega, \mathcal{B}', \mu) \models APA$.

View $(K, \mathcal{B})$ as an $\mathcal{L}^R$ structure as follows:

- For $\phi(\overline{x}) \in \mathcal{L}$ and $\overline{f} \in K$,

$$[\![\phi(\overline{f})]\!] = \{\omega \in \Omega | \mathcal{M} \models \phi(f_1(\omega), \ldots, f_n(\omega))\} \in \mathcal{B}$$

- For all $B \in \mathcal{B}$, $\epsilon > 0$, there are $f, g \in K$ such that

$$\mu(B \Delta [\![f = g]\!]) < \epsilon$$

- For all $\theta(x, \overline{y}) \in \mathcal{L}$ and $\overline{g} \in K$, for all $\epsilon > 0$ there exists $f \in K$ such that

$$\mu([\![(\exists x)\theta(x, \overline{g})]\!] \Delta [\![\theta(f, g)]\!]) < \epsilon$$

- $d_K(f, g) = \mu[\![f \neq g]\!]$
- $d_{\mathcal{B}}(A, B) = \mu(A \Delta B)$

Fact 3 (Proposition 2.2 in [1])**.** If T is the complete theory of $\mathcal{M}$, every randomization $(\mathcal{K}, \mathcal{B})$ of $\mathcal{M}$ is a pre-model of T^R.

Fact 4 (Theorem 2.3 in [1])**.** If T is the complete theory of $\mathcal{M}$, every pre-model of T^R is represented by some randomization of $\mathcal{M}$.

3.2 Terms

Note that the terms of $\mathcal{L}^R$ are

- $\bot, \top$ in $\mathcal{B}$
- B, variable in $\mathcal{B}$
- $A \cap B$, $A \cup B$, $\neg A$ in $\mathcal{B}$, where A, B are terms in $\mathcal{B}$
- $[\![\phi(f_1, \ldots, f_k)]\!]$ in $\mathcal{B}$ where $f_1, \ldots, f_k$ are variables in K and $\phi(x_1, \ldots, x_k) \in \mathcal{L}$
- f, variable in K

3.3 Atomic Formulas

So the atomic formulas are

- $d_k(f, g)$ where f, g are variables in K
- $\mu(B)$ where B is a term in $\mathcal{B}$
- $d_{\mathcal{B}}(A, B)$ where A, B are terms in $\mathcal{B}$

4 Facts about T^R

The following fact is important in proofs about T^R which use induction on formulas:

Fact 5 (Theorem 2.9 in [1]). T^R admits strong quantifier elimination. That is, for $\mathcal{N} \models T^R$, for every formula ϕ there is a quantifier free formula ψ such that $\mathcal{N} \models$ "$|\phi - \psi| = 0$".

Fact 6 (Theorem 2.1 in [1]). If T is complete, T^R is complete.

Theorem 3. *If T, a complete theory, then T is _____ if and only if T^R is _____.*

- *ω-categorical*
- *ω-stable*
- *stable*
- *NIP*

Theorem 4. *If T is simple and unstable, T^R is not simple.*

References

[1] I. Ben Yaacov and H. Jerome Keisler. Randomizations of models as metric structures. *ArXiv e-prints*, January 2009.

[2] C. Ward Henson, Itay Ben Yaacov, Alexander Berenstein, and Alexander Usvyatsov. *Model Theory with Applications to Algebra and Analysis*. Cambridge University Press, 2008.