Math 210: Surface Area of a Sphere

Victoria Noquez

August 1, 2015

We will use a surface integral to calculate the surface are of a sphere of radius a.

Recall from section 13.5: spherical coordinates are (ρ, ϕ, θ) where ρ is the radius, ϕ is the angle from the positive z-axis, and θ is the angle on the xy-plane from the positive x-axis going counterclockwise (same as in polar coordinates).

The conversion from spherical to rectangular coordinates is

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

Consider the sphere of radius a. This can be described in spherical coordinates by $S = \{(\rho, \phi, \theta) : \rho = a, 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$.

So we may parameterize the sphere by letting $u = \phi$ and $v = \theta$ via

$$t(u, v) = \langle a \sin u \cos v, a \sin u \sin v, a \cos u \rangle$$

with $0 \leq u \leq \pi$ and $0 \leq v \leq 2\pi$.

The surface area of S can be calculated by

$$\int\int_S 1 dS = \int\int_R |t_u \times t_v| dA$$

where R is the region in the uv-plane $\{(u, v) : 0 \leq u \leq \pi, 0 \leq v \leq 2\pi\}$.

$$\begin{aligned} t_u &= \langle \tfrac{\partial}{\partial u}(a \sin u \cos v), \tfrac{\partial}{\partial u}(a \sin u \sin v), \tfrac{\partial}{\partial u}(a \cos u) \rangle \\ &= \langle a \cos u \cos v, a \cos u \sin v, -a \sin u \rangle \end{aligned}$$

$$\begin{aligned} t_v &= \langle \tfrac{\partial}{\partial v}(a \sin u \cos v), \tfrac{\partial}{\partial v}(a \sin u \sin v), \tfrac{\partial}{\partial v}(a \cos u) \rangle \\ &= \langle -a \sin u \sin v, a \sin u \cos v, 0 \rangle \end{aligned}$$

$$t_u \times t_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a \cos u \cos v & a \cos u \sin v & -a \sin u \\ -a \sin u \sin v & a \sin u \cos v & 0 \end{vmatrix}$$

$= \mathbf{i}(0-(-a\sin u)(a\sin u\cos v)) - \mathbf{j}(0-(-a\sin u)(-a\sin u\sin v))$
$\quad +\mathbf{k}((a\cos u\cos v)(a\sin u\cos v) - (a\cos u\sin v)(-a\sin u\sin v))$
$= (a^2\sin^2 u\cos v)\mathbf{i} + (a^2\sin^2 u\sin v)\mathbf{j} + (a^2\sin u\cos u\cos^2 v + a^2\sin u\cos u\sin^2 v)\mathbf{k}$
$= (a^2\sin^2 u\cos v)\mathbf{i} + (a^2\sin^2 u\sin v)\mathbf{j} + (a^2\sin u\cos u(\sin^2 v+\cos^2 v))\mathbf{k}$
$= (a^2\sin^2 u\cos v)\mathbf{i} + (a^2\sin^2 u\sin v)\mathbf{j} + (a^2\sin u\cos u)\mathbf{k}$ since $\sin^2 v+\cos^2 v = 1$.

Now we need the length of this vector.

$|\langle a^2\sin^2 u\cos v, a^2\sin^2 u\sin v, a^2\sin u\cos u)\rangle|$
$= \sqrt{(a^2\sin^2 u\cos v)^2 + (a^2\sin^2 u\sin v)^2 + (a^2\sin u\cos u)^2}$
$= \sqrt{a^4\sin^4 u\cos^2 v + a^4\sin^4 u\sin^2 v + a^2\sin^2 u\cos^2 u}$
$= a^2\sqrt{\sin^4 u\cos^2 v + \sin^4 u\sin^2 v + \sin^2 u\cos^2 u}$
$= a^2\sqrt{\sin^4 u(\cos^2 v + \sin^2 v) + \sin^2 u\cos^2 u}$
$= a^2\sqrt{\sin^4 u + \sin^2 u\cos^2 u}$ since $\cos^2 v + \sin^2 v = 1$
$= a^2\sqrt{\sin^2 u(\sin^2 u + \cos^2 u)}$
$= a^2\sqrt{\sin^2 u}$ since $\sin^2 u + \cos^2 u = 1$
$= a^2\sin u$

Note that this coincides with the correction factor given in section 13.5 for integrals over spherical coordinates. This is where it comes from.

So the surface are is

$\int\int_R a^2\sin u dA$
$= \int_0^{2\pi}\int_0^{\pi} a^2\sin u du dv$
$= \int_0^{2\pi} a^2(-\cos u)|_0^{\pi} dv$
$= \int_0^{2\pi} a^2(-\cos\pi - (-\cos(0)))dv$
$= \int_0^{2\pi} a^2(-(-1) - (-1))dv$
$= \int_0^{2\pi} 2a^2 dv$
$= 2a^2 v|_0^{2\pi}$
$= 2a^2(2\pi - 0)$
$= 4\pi a^2$