Louise Hay Logic Seminar notes

McKinley Meyer, speaker
Noah Schoem, scribe

September 1, 2016

Today, McKinley will give the first of a 2-part talk on forcing: An Overview of the Forcing Apparatus.

Definition 1. A *poset* is a set $\mathbb{P}$ with a partial order $\leq$.
We call elements of $\mathbb{P}$ "conditions" and think of $\leq$ as "is stronger than".

Note: in the context of forcing, $\mathbb{P}$ will always have a maximum element denoted 1.
Note: We say p and q are incompatible and write $p \perp q$ if $\nexists r \in \mathbb{P} r \leq p \wedge r \leq q$.

Examples

Example 2. $Add(\omega, \lambda) = \{f : \lambda \times \omega \rightharpoonup \{0,1\} \mid |dom(f)| < \omega\}$, and we say $f \leq g$ if $f \supseteq g$. Here, 1 is $\emptyset$. This poset will add λ-many reals. More on that in the next talk.

Example 3. $Col(\omega, \delta) = \{f : \omega \rightharpoonup \delta \mid |dom(f)| < \omega\}$ with $\leq$, 1 same as above. This will "collapse" δ. More on that in the next talk.

Definition 4. A set $\mathcal{A} \subseteq \mathbb{P}$ is an *antichain* if any two elements of $\mathcal{A}$ are incompatible.
An antichain $\mathcal{A}$ is *maximal* if $\forall p \in \mathbb{P} \exists q \in \mathcal{A} p \not\perp q$.

Here are some examples of antichains:
In $Add(\omega, 1)$, say $|f| = |g| = 1$ where $f(4) = 0$ and $g(4) = 1$. Then $f \perp g$, and additionally any $p \in \mathbb{P}$ is compatible with one of f and g. Thus $\{f, g\}$ is a maximal antichain.
However, if we take $f(4) = 0$, $f(8) = 0$, and $g(4) = 1$, this is not maximal since the partial function h given by $h(4) = 0$ and $h(8) = 1$ is incompatible with f and g. Observe that $\{f, g, h\}$ is a maximal antichain.

Definition 5. A $D \subseteq \mathbb{P}$ is *dense* if $\forall p \in \mathbb{P} \exists q \in D q \leq p$.

As an example in $Add(\omega, 1)$, say $D = \{f : \omega \rightharpoonup \{0,1\} \mid 56 \in dom(f)\}$. Clearly D is dense: given any $g \in \mathbb{P}$, if $56 \notin dom(g)$ then $g \cup \{(56, 1)\} \in D$.
Another dense set is $\{f : \omega \rightharpoonup \{0,1\} \mid |f| > 99\}$.
Think of dense sets as being weak restrictions on the conditions.

The Generic

Now we come to the most important definition of the day:

Definition 6 (Generic filters)**.** A $G \subseteq \mathbb{P}$ is an *generic filter* if

- (Filter 1) $\forall p \in \mathbb{P} \forall q \in \mathbb{G} q \leq p \rightarrow p \in G$
- (Filter 2) $\forall p, q \in G$, there is a $r \in G$ such that $r \leq p, q$
- (Genericity) For any $D \subseteq \mathbb{P}$ dense, $D \cap G \neq \emptyset$

Generics are big things, in the following sense:

Proposition 7. *Suppose $\mathbb{P} \in M \models ZFC$, G is generic, and $\mathbb{P}$ is separative, i.e. $\forall p \in \mathbb{P} \exists q, r \in \mathbb{P}(q, r \leq p \wedge q \perp r)$. Then $G \notin M$.*

Proof. Suppose $G \in M$. Then $\mathbb{P} \setminus G \in M$. Let p, q, r be as above. Then at most one of q, r is in G. So at least one of $q, r \in \mathbb{P} \setminus G$. Since p was arbitrary, we have that $\mathbb{P} \setminus G$ is dense. But $(\mathbb{P} \setminus G) \cap G = \emptyset$ and hence G cannot be generic. This is a contradiction. □

So do generic filters even exist?

Proposition 8. *Let $\mathbb{P} \in M \models ZFC$ and M be countable. Then there is a $\mathbb{P}$-generic filter G over M.*

Proof. G exists because we can build it. Since M is countable, we can enumerate the dense subsets of $\mathbb{P}$ by $\{D_n \mid n < \omega\}$.

We construct G inductively. Say $p_0 \in D_0$, and take $p_{n+1} \leq p_n$ such that $p_{n+1} \in D_{n+1}$. We know we can do this because D_{n+1} is dense.

Now take $G = \{p \in \mathbb{P} \mid \exists n < \omega \; p_n \leq p\}$. Clearly G is a filter:

If $p \in G$, $q \in \mathbb{P}$ such that $p \leq q$, then there is some p_n such that $p_n \leq p \leq q$. Hence $q \in G$.

Let $p, q \in G$. Then there is $p_n \leq p$, $p_m \leq q$. WLOG, $n < m$, hence $p_m \leq p_n$. Hence $p_m \in G$ is a common extension of p, q.

Lastly, by construction $D \cap G \neq \emptyset$ since $D = D_n$ for some n and $p_n \in D_n \cap G$. □

The next question is, how do we extend M to include G in a sensible manner?

Names

Definition 9. A $\mathbb{P}$-name is a set whose elements are of the form (σ, p) where σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

This might look ill-founded, but it's not: clearly $\emptyset$ is a $\mathbb{P}$-name and you build up from there.

Theorem 10. *We can construct a model $M[G]$ such that:*

1. $M \subseteq M[G]$
2. $G \in M[G]$ *and* $\bigcup G \in M[G]$
3. $M[G] \models ZFC$

We're not going to prove this. What we *will* do is talk about what formulae $M[G]$ will satisfy:

Theorem 11. *We can define a relation $\Vdash$ entirely within M such that:*

1. $p \Vdash \phi$ *iff* $\forall G$ *generic with* $p \in G$, $M[G] \models \phi$;
2. *For any* G, $M[G] \models \phi$ *iff* $\exists p \in G p \Vdash \phi$.

So all of the information we have about $M[G]$ is encoded in M. Hence, if we can prove that $Th(M)$ is consistent, then M and $\Vdash$ gives us the consistency of $Th(M[G])$.

As for what $M[G]$ actually looks like:

$M[G]$ is the collection of all $\mathbb{P}$-names interpreted by G, i.e. given τ a name, we say

$$\tau_G = \{\sigma_G \mid (\sigma, p) \in \tau, p \in G\}$$

and then $M[G] = \{\tau_G \mid \tau \text{ is a } \mathbb{P}\text{-name}\}$.

If $x \in M$, we have *canonical names* $\check{x} = \{(\check{y}, 1) \mid y \in x\}$. Then $\check{x}_G = x$.

There's also a name for the generic filter, denoted Γ or $\dot{G}$, and it's just $\{(\check{p}, p) \mid p \in \mathbb{P}\}$.