Louise Hay Logic Seminar notes

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1 A Blazing-Fast Recall of the First Lecture, and Names

We begin by recalling the essential components of forcing: posets, antichains, maximal antichains, dense sets, and generic filters. We also recall that throughout, we have that M is a countable transitive model of ZFC and that we have something called the *generic extension* $M[G]$ which also is a countable transitive model of ZFC.

We also had the relation $\Vdash$ and mentioned that for any $p \in \mathbb{P}$ and any formula ϕ, we have the statement $p \Vdash \phi$ and the Forcing and Generic Model Theorems:

Theorem 1.

1. *(Forcing Theorem)* $p \Vdash \phi \iff \forall G$ *$\mathbb{P}$-generic over M with $p \in G$,* $M[G] \models \phi$.
2. *(Generic Model Theorem) If G is $\mathbb{P}$-generic over M, then* $M[G] \models \phi \iff \exists p \in G\ p \Vdash \phi$.

We haven't really talked about what $M[G]$ looks like (and we will need to in order to do applications of forcing). We do so now.

Definition 2. $M^{\mathbb{P}}$, the set of $\mathbb{P}$-*names* over M, can be thought of as being built inductively:

1. $\emptyset$ is a $\mathbb{P}$-name.
2. A $\mathbb{P}$-name is a set in M whose elements are of the form (σ, p) where σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$.

For each $X \in M$, there is a *canonical* name for X, denoted

$$\check{X} = \{(\check{y}, 1_{\mathbb{P}}) \mid y \in X\}$$

and there is a canonical name for the generic filter:

$$\dot{G} = \{(\check{p}, p) \mid p \in P\}$$

Definition 3.

1. (Another inductive definition) For each name τ, we define $\tau_G = \{\sigma_G \mid \exists p \in G (\sigma, p) \in \tau\}$.
2. $M[G]$ is defined to be $\{\tau_G \mid \tau \in M^{\mathbb{P}}\}$.

With this, we have that every $X \in M$ is in $M[G]$; for $\check{X}_G = X$. This can be proved using well-founded induction.

Furthermore, $\dot{G}_G = G$; this is a simple corollary. Hence $G \in M[G]$.

Remark 4. M and $M[G]$ have the exact same ordinals. Showing this is left as an exercise.

2 Application 1: Cardinal Collapse

Recall that we defined

Definition 5. $Col(\omega, \delta) := \{f : \omega \rightharpoonup \delta \mid |dom(f)| < \omega\}$ with partial ordering $\supseteq$.

This poset gets its name from the following:

Theorem 6. *If G is $Col(\omega, \delta)$-generic over M, then $M[G]$ believes that $\bigcup G : \omega \to \delta$ is a surjection. In particular, if $|\delta|^M > \aleph_0$, that is, if δ is uncountable in M, then $|\delta|^{M[G]} = \aleph_0$.*

Proof. Most of these are density arguments.

First, since G is a filter, any two $p, q \in G$ are compatible, hence agree on their common domain. Thus $\bigcup G$ is a function.

Obtaining that $\bigcup G$ is total is a density argument. For any $n \in \omega$, we have that $D_n := \{p \in Col(\omega, \delta) \mid n \in dom(p)\}$ is dense; for if $q \in Col(\omega, \delta)$ and $n \notin dom(q)$, then $q \subseteq q \cup \{(n, 0)\} \in D_n$. Hence there is a $p \in D_n \cap G$; such p has that $n \in dom(p)$, hence $n \in dom\left(\bigcup G\right)$.

Likewise, the surjectivity of G is a density argument. For each $\alpha < \delta$, let $Q_\alpha = \{p \in Col(\omega, \delta) \mid \alpha \in ran(p)\}$ and observe this is dense; for if $q \in Col(\omega, \delta)$ and $\alpha \notin ran(q)$, then since $|dom(q)|$ is finite, there is some $n \notin dom(q)$. Hence $q \subseteq q \cup \{(n, \alpha)\} \in Q_\alpha$. Thus there is a $p \in Q_n \cap G$, so $\alpha \in ran\left(\bigcup G\right)$. ☐

3 Application 2: Independence of CH

Recall that we defined

Definition 7. $Add(\omega, \omega_2) := \{f : \omega_2 \times \omega \rightharpoonup \{0, 1\} \mid |dom(f)| < \omega\}$ with partial ordering $\supseteq$.

By work done by Gödel's L, we may assume that $M \models GCH$; that is, we are going to black-box $Con(GCH)$.

Remark 8. You can also get $Con(CH)$ using forcing; we will not describe how.

Theorem 9. *Let G be $Add(\omega, \omega_2)$-generic over M. Then $M[G] \models (2^{\aleph_0})^{M[G]} \geq |{\aleph_2}^M|$.*

Proof. To obtain $(2^{\aleph_0})^{M[G]} \geq |{\aleph_2}^M|$, we count in $M[G]$. As with $Col(\omega, \delta)$, $\bigcup G$ is a function; in this case $\bigcup G : \omega \times \omega_2^M \to \{0, 1\}$ and we define a collection of functions $f_\alpha : \omega \to \{0, 1\}$ by $f_\alpha(n) = f(\alpha, n)$. When $\alpha \neq \beta$, we have that $f_\alpha \neq f_\beta$; this is because $\{p \in Add(\omega, \omega_2) \mid \exists n\; p(\alpha, n) \neq p(\beta, n)\}$ is dense, hence meets G. Thus there is a $p \in G$ such that $p \Vdash \dot{f}_\alpha(n) \neq \dot{f}_\beta(n)$.

Hence $M[G]$ has ω_2^M-many subsets of ω. ☐

Remark 10. Carefully counting a large enough collection of names, and use of the yet-to-be-defined countable chain condition, will show that $M[G] \models (2^{\aleph_0})^{M[G]} = \left|{\aleph_2}^M\right|$.

This is not yet enough to violate CH; we saw that generic extensions can collapse cardinals. We need to ensure that ${\aleph_2}^M = {\aleph_2}^{M[G]}$. It turns out that $Add(\omega, \omega_2)$ has a nice combinatorial property that guarantees this:

Definition 11. We say that a poset $\mathbb{P}$ has the *countable chain condition* (ccc) if every antichain (equivalently maximal antichain) is countable.

Countable chain condition posets are really important:

Theorem 12. *Countable chain condition posets preserve cardinals; that is, if $\mathbb{P}$ satisfies the countable chain condition and G is generic, then $M \models$ "κ is a cardinal" $\iff$ $M[G] \models$ "κ is a cardinal".*

Proof. The reverse direction is clear: the sentence "κ is a cardinal" is just saying $\forall f \forall \alpha < \kappa \neg(f : \alpha \twoheadrightarrow \kappa)$. This is a Π_1 statement, and thus is downwards absolute for transitive models of set theory.

Conversely, suppose that $M \models$ "κ is a cardinal" but $M[G] \models \neg$"κ is a cardinal". Without loss of generality we may take κ to be a successor in M, and hence regular in M; this is because in ZFC, a limit of cardinals is a cardinal.

There is a name $\dot{f}$ and an ordinal $\alpha < \kappa$ such that $M[G] \models \dot{f}_G : \alpha \twoheadrightarrow \kappa$. By the Forcing Theorem, there is a $p_0 \in \mathbb{P}$ such that $p_0 \Vdash \dot{f}_G : \check{\alpha} \twoheadrightarrow \check{\kappa}$.

Let $\beta < \alpha$. Since $\dot{f}(\beta)$ is a name for an ordinal and p_0 forces $\dot{f}$ to be a function, we have that $D_\beta := \{p \in \mathbb{P} \mid \exists \alpha\ p \Vdash \dot{f}(\beta) = \alpha\}$ is dense below p_0; this is just one of the properties of forcing (and in fact, is often taken to be the inductive definition of what it means to force an existential statement).

Choose $A_\beta \subseteq D_\beta$ an antichain maximal below p_0; then by the ccc, there is a countable set of ordinals X_β such that $p \in X_\beta \implies \exists \gamma \in A_\beta\ p \Vdash \dot{f}(\beta) = \gamma$.

Thus $p_0 \Vdash \dot{f}(\beta) < \sup\ X_\beta$. But then $p_0 \Vdash \sup\ \dot{f}[\alpha] < \sup_{\beta<\alpha}(\sup\ X_\beta)$. But this all happened in M; since κ is regular in M, $\sup_{\beta<\alpha}(\sup\ X_\beta) < \kappa$; and thus $\dot{f}$ cannot be surjective in $M[G]$.

This is a contradiction and thus $M[G] \models$ "κ is a cardinal". □

Theorem 13. *$Add(\omega, \omega_2)$ has the countable chain condition.*

We will be invoking the following result:

Lemma 14 (The Δ-system lemma)**.** *Let $|X| = \aleph_1$ and let $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ be a collection of (distinct) finite subsets of X. Then there is an $I \subseteq \omega_1$ of size $\aleph_1$ and an $r \subseteq X$ such that $\alpha \neq \beta \in I \implies x_\alpha \cap x_\beta = r$.*

A set $\langle x_\alpha \mid \alpha < \kappa \rangle$ is called a Δ-system if there is a kernel r, i.e. for all $\alpha \neq \beta$, $x_\alpha \cap x_\beta = r$. The lemma just says that $[\omega_1]^{<\omega}$ has a Δ-system of size ω_1.

We will not be proving this; the proof can be found in Jech.

Proof that $Add(\omega, \omega_2)$ has the ccc. We show that every subset of $Add(\omega, \omega_2)$ of size ω_1 is not an antichain. Let $\langle p_i \mid i < \omega_1 \rangle$ be a family of elements of $Add(\omega, \omega_2)$. Then each $dom(p_i)$ is finite, and hence there is an uncountable $J \subseteq \omega_1$ and an r such that for every $i \neq j \in J$, $dom(p_i) \cap dom(p_j) = r$. But $\{p_i \upharpoonright r \mid i \in J\}$ has size at most $2^{|r|} < \omega$.

Thus, by the ω_1-size Pigeonhole Principle, there are uncountably many p_i that agree on their common domain r, and hence are all mutually compatible. □

Remark 15. We showed something a little stronger: we say that a poset $\mathbb{P}$ is $\aleph_1$-Knaster if every uncountable subset has uncountably many mutually compatible elements. The above argument shows that $Add(\omega, \omega_2)$ is $\aleph_1$-Knaster, and clearly $\aleph_1$-Knaster $\implies$ ccc. The converse is not true.

Putting all of this together, we obtain that $M[G] \models 2^{\aleph_0} \geq \aleph_2$ and thus $M[G] \models \neg CH$.

Remark 16. ccc and $\aleph_1$-Knaster are specimens of a veritable zoo of nice properties of posets. There's $< \kappa$-cc, κ-Knaster, something called $< \kappa$-closed, $< \kappa$-distributive, and scores of others that all give nice properties of generic extensions.

Remark 17. One might ask what else we can force to be the value of $2^{\aleph_0}$, or more generally 2^κ for any κ. Easton's Theorem is a partial result for regular cardinals; as far as I'm aware (and Maxwell may very well correct me here), the question for singular cardinals remains open.

References

These were put together from my notes from Dima's Set Theory class last year, and Spencer Unger's Forcing notes from the 2013 UCLA Summer School in Logic for undergraduates.

As often, *Set Theory* by Thomas Jech covers many of the finer details exhaustively.