LHLS Notes: Class Forcing and Easton's Theorem

Noah Schoem

To do Easton's Theorem, follow Jech's formulation. Do on a set first, then talk about on the class of regular cardinals.

It turns out that ZFC says very little about how the continuum function behaves about regular cardinals, in the following sense:

Theorem 1 (Easton's Theorem). *Let $M \models ZFC + GCH$ be transitive, and let $F : Reg \to Card$ be a (class) function such that for all $\kappa, \lambda \in Reg$:*

1. *$F(\kappa) > \kappa$ (Cantor's Theorem)*
2. *$F(\kappa) \leq F(\lambda)$ when $\kappa \leq \lambda$ (Weak Monotonicity)*
3. *$cf\ F(\kappa) > \kappa$ (König's Lemma)*

Then there is a generic extension $M[G]$ of M such that $M[G] \models \forall \kappa \in Reg\ 2^\kappa = F(\kappa)$.

This involves two steps of argument: first, we do this for the simpler case of when F is defined on a proper initial segment of regular cardinals, and then we describe the necessary changes to obtain class forcing.

Easton's Theorem on an Initial Segment of Reg

We define our forcing poset as follows:

Definition 2. For $\kappa \in dom(F)$, let $(P_\kappa, \leq)$ be the forcing of conditions $p : \kappa \times F(\kappa) \rightharpoonup \{0, 1\}$ with $|dom(p)| < \kappa$, and $p \leq q$ iff $p \supseteq q$.

You may recognize P_κ as $Add(\kappa, F(\kappa))$.

Definition 3. $(P, <)$, the *Easton product of P_κ*, is the set of all $p \in \prod_{\kappa \in dom(F)} P_\kappa$, such that for every $\gamma \in Reg$, $|s(p) \cap \gamma| < \gamma$, where $s(p) = \{\kappa \in dom(F) \mid p_\kappa \neq 1_{P_\kappa}\}$.

For each such p and each κ, we denote $p(\kappa, \alpha, \beta) := p_\kappa(\alpha, \beta)$.

Given a P-generic G, if we take the projection $\pi_\kappa(p)$ of p onto the κth coordinate, we get a P_κ-generic $G_\kappa := \pi_\kappa{}''G$. Furthermore, (since P_κ is the Cohen forcing to add $F(\kappa)$-many subsets of κ), we obtain, for each κ, $F(\kappa)$ many new subsets of κ of the form

$$a^\kappa_\beta := \{\alpha < \kappa \mid (\exists p \in G) p(\kappa, \alpha, \beta) = 1\}$$

Hence $M[G] \models (2^\kappa)^M \geq (F(\kappa))^M$. As with Cohen forcing, we now need to verify that Easton forcing preserves cardinals and doesn't add more than $F(\kappa)$-many subsets of κ.

(decomposition, yadda) Work in M.

Lemma 4. *Let $\lambda \in Reg$, and define, for each $p \in P$,*

$$p^{\leq\lambda} := p \restriction \{(\kappa, \alpha, \beta) \mid \kappa \leq \lambda\}$$

$$p^{>\lambda} := p \restriction \{(\kappa, \alpha, \beta) \mid \kappa > \lambda\}$$

and define $P^{\leq\lambda}$, $P^{>\lambda}$ as expected.

Then $p = p^{\leq\lambda} \sqcup p^{>\lambda}$ and $P = P^{\leq\lambda} \times P^{>\lambda}$.

Remark 5. $P^{\leq\lambda} \times P^{>\lambda}$ is just a product forcing, and hence product forcing lemmas apply: G is $P^{\leq\lambda} \times P^{>\lambda}$-generic over M iff $G = H \times K$ for H an $P^{\leq\lambda}$-generic H over M and K a $P^{>\lambda}$-generic over $M[H]$.

This decomposition is useful:

Lemma 6. *$P^{>\lambda}$ is λ-closed, i.e. every chain of length $\leq \lambda$ has a lower bound.*

Proof. Let $\langle {}_\alpha q \mid \alpha < \lambda \rangle$ be a chain and let $q = \bigcup_{\alpha<\lambda} {}_\alpha q$. Then $|s(q)| \leq \lambda$ and hence $|s(q) \cap \gamma| < \gamma$ for all regular $\gamma > \lambda$, and $|s(q) \cap \gamma| = \emptyset$ for all $\gamma < \lambda$ regular. $\square$

Lemma 7. *$P^{\leq\lambda}$ has the λ^+-chain condition, i.e. if $W \subseteq P^{>\lambda}$ is an antichain then $|W| \leq \lambda$.*

Proof. Theorem 15.17 in Jech, for the curious. Alternatively, each P_κ is λ^{++}-Knaster, and Knaster is preserved under products. $\square$

Theorem 8. *Let $H \times K$ be an M-generic filter on $R \times Q$ where R is λ-closed and Q satisfies λ^+-cc. Then G adds no new $f : \lambda \to M$, i.e. if $f : \lambda \to M$ is in $M[G \times H]$ then $f \in M[H]$. In particular, $\mathcal{P}^{M[G\times H]}(\lambda) = \mathcal{P}^{M[H]}(\lambda)$.*

The proof is in Jech, Lemma 15.19. It's involved and technical.

Now we have preservation of cardinals: if κ is a regular cardinal in M but not in $M[G]$, then there is some $f : \lambda \to \kappa$ in $M[G]$ that is cofinal with $M \models \lambda < \kappa$; letting $G^{>\lambda} = G \cap P^{>\lambda}$ and $G^{\leq\lambda} = G \cap P^{\leq\lambda}$, we have by the above theorem that κ is not regular in $M[G^{\leq\lambda}]$. But this is impossible; $P^{\leq\lambda}$ satisfies the λ^+-chain condition and hence the κ-chain condition.

Now it just remains to accurately count 2^λ in $M[G]$. We already have that $M[G] \models 2^\lambda \geq F(\lambda)$ for each $\lambda \in dom(F)$. By the above theorem, $(2^\lambda)^{M[G]} = (2^\lambda)^{M[G^{\leq\lambda}]}$. By cardinal arithmetic, $|P^{\leq\lambda}| = F(\lambda)$ and there are at most $F(\lambda)$-many nice names (and this is where we use the assumptions on $F(\lambda)$) hence $(2^\lambda)^{M[G^{\leq\lambda}]} = F(\lambda)$.

Remark 9. Easton forcing preserves that $2^\kappa = \kappa^+$ for singular κ (this is an exercise in Jech).

Class Forcing

The above happened on a subset of Reg, but we would like to obtain $2^\kappa = F(\kappa)$ on *all* of Reg. We'll sketch how.

(decomposition into layers by rank) We may define P as before, and obtain a proper class poset; however, when we decompose $P = P^{\leq\lambda} \sqcup P^{>\lambda}$, we still have that $P^{\leq\lambda}$ is a set; so we define the P-names M^P as $\bigcup_{\lambda\in Reg} M^{P^{\leq\lambda}}$ where there is a clear embedding $P^{\leq\lambda} \hookrightarrow P^{\leq\mu}$ when $\lambda < \mu$.

(forcing as by induction on rank) We define $\Vdash$ by induction on formula complexity. Generics are defined similarly, except that we will have *class* generics, that meet dense *classes* over M.

We still get generics G, except they will be class-sized; and we define $M[G] = \bigcup_{\lambda\in Reg} M[G \cap P^{\leq\lambda}]$.

We still obtain the Forcing Theorems (i.e. $p \Vdash \phi$ iff for every generic $G \ni p$, $M[G] \models \phi$, and $M[G] \models \phi$ iff there is a $p \in G$ such that $p \Vdash \phi$).

(ZFC is interesting. Usual proofs go through except for a few, of which do one.) We show that $M[G] \models ZFC$ much in the same way that we do for set-size forcing, except that the proofs of Power Set and Replacement are nontrivial.

For power set, we still have that $P^{\leq\lambda}$ is λ^+-cc and $P^{>\lambda}$ is λ-closed, and Theorem 8 still applies. So power set just reduces to power set for $M[G \cap P^{\leq\lambda}]$.

Replacement is more involved, but again reduces to $M[G \cap P^{\leq\lambda}]$ with an application of Theorem 8.

The same argument as above properly counts 2^κ in $M[G]$.