# LHLS Notes

Noah Schoem

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I am talking today about Reverse Mathematics.

Post-talk note: The talk did not cover the full breadth of what these notes contain: the talk did not carefully cover the proof of Lemma 4, or some of the proofs that some of the various systems are stronger than others.

## 1 Introduction

Reverse Mathematics is a late twentieth century branch of logic concerned with characterizing theorems of classical (and even modern) mathematics in terms of the set-theoretic tools required to prove them. Simpson formulates the aim of Reverse Mathematics as asking, "Which set existence axioms are needed to prove the theorems of ordinary, non-set-theoretic mathematics?"

A word of warning: there is a fair amount of computability theory in this talk, because a lot of the theory of Reverse Mathematics has a natural computability-theoretic formulation (and this gives a very natural way of thinking about the theorems of Reverse Mathematics!). I will do my best to introduce, without going off on tangents, the necessary computability theory as it arises.

The content herein comes from [4], unless otherwise noted.

Without further ado:

## 2 Preliminaries

**Definition 1.** The *language of second order arithmetic* $\mathbf{L}_2$ is a second-order language that allows for quantification over elements and sets. $\mathbf{L}_2$ consists of the usual $+$, $\cdot$, $<$, countably many variables for elements (generally denoted by lowercase letters), sets (denoted by uppercase letters), and the familiar fashion of building terms, formulae, and sentences.

**Definition 2.** A *model* of $\mathbf{L}_2$ is a structure

$$M = (|M|, S_M, +^M, \cdot^M, 0^M, 1^M, <^M)$$

where $|M|$ is the set of numbers (or elements), $S_M$ is the collection of sets, and the rest of the symbols are the usual ones.

An $\omega$-*model* is a model whose domain of elements is $\omega$; the definition of *non-$\omega$ model* is analogous.

We have a hierarchy of sentences of $\mathbf{L}_2$ similar to the Levy hierarchy:

**Definition 3.**

1. We say that a formula is quantifier-free if it has no quantifiers.
2. We say that a formula is $\Pi^0_n$ if it is logically equivalent to a sentence of the form $\forall x_1 \ldots \forall x_k \phi(x_1, \ldots, x_k)$ where $\phi(x_1, \ldots, x_k)$ is $\Sigma^0_{n-1}$, and that a formula is $\Sigma^0_n$ if it is logically equivalent to a sentence of the form $\exists x_1 \ldots \exists x_k \phi(x_1, \ldots, x_k)$ where $\phi(x_1, \ldots, x_k)$ is $\Pi^0_{n-1}$. ($\Sigma^0_0$ and $\Pi^0_0$ have the obvious meanings.)
3. We say that a formula is arithmetical if it has no set quantifiers.
4. We say that a formula is $\Pi^1_n$ if it is logically equivalent to a sentence of the form $\forall X_1 \ldots \forall X_k \phi(X_1, \ldots, X_k)$ where $\phi(X_1, \ldots, X_k)$ is $\Sigma^0_{n-1}$, and that a formula is $\Sigma^0_n$ if it is logically equivalent to a sentence of the form $\exists X_1 \ldots \exists X_k \phi(X_1, \ldots, X_k)$ where $\phi(X_1, \ldots, X_k)$ is $\Pi^0_{n-1}$. ($\Sigma^1_0$ and $\Pi^1_0$ have the obvious meanings.)

5. A formula is $\Delta^i_n$ if it is logically equivalent to a $\Sigma^i_n$ formula and to a $\Pi^i_n$ formula.

**Remark 1.** Adding on dummy quantifiers shows that $\Sigma^i_n \subseteq \Pi^i_{n+1}$, $\Pi^i_n \subseteq \Sigma^i_{n+1}$, and $\Pi^i_n \cup \Sigma^i_n \subseteq \Pi^i_{n+1} \cap \Sigma^i_{n+1}$.

**Definition 4.** The *intended model* of $\mathbf{L}_2$ is the familiar $(\omega, \mathcal{P}(\omega), +, \cdot, 0, 1, <)$.

**Definition 5.** The *axioms of second order arithmetic* are the usual rules of how $+, \cdot, <$ work:

- $\forall n(n+1 \neq 0)$
- $\forall m \forall n(m+1 = n+1 \to m = n)$
- $\forall m(m+0 = m)$
- $\forall m \forall n(m+(n+1) = (m+n)+1)$
- $\forall m(m \cdot 0 = 0)$
- $\forall m \forall n(m \cdot (n+1) = (m \cdot n) + m)$
- $\forall m(m \not< 0)$
- $\forall m \forall n(m < n+1 \leftrightarrow (m < n \vee m = n))$

along with induction:

- $\forall X(0 \in X \wedge \forall n(n \in X \to n+1 = x)) \to \forall n(n \in X)$

and the comprehension scheme:

- $\exists X \forall n(n \in X \leftrightarrow \phi(n))$

over all formulae $\phi(n)$ having no free occurrences of $X$.

We call $\mathbf{Z}_2$ the closure of these axioms under logical consequence.

$\mathbf{Z}_2$ is a natural operating environment for (classical) mathematics; it allows us to build, say $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, and gives us analysis, algebra, combinatorics, number theory, etc.

**Remark 2.** $\phi(n)$ may have parameters in addition, say $\phi(n) \equiv \phi(n, a, Y, Z)$. Then we have the comprehension axiom $\forall Y \forall Z \forall a \exists X \forall n(n \in X \leftrightarrow \phi(n, a, Y, Z))$.

Now that we have established our language, we want to look at some weaker subtheories.

**Definition 6.** A *subsystem* of $\mathbf{Z}_2$ is a formal system whose axioms are theorems of $\mathbf{Z}_2$.

There are five canonical subsystems that capture a remarkably large amount of mathematics: $RCA_0$, $WKL_0$, $ACA_0$, $ATR_0$, and $\Pi^1_1 - CA_0$.

# 3 $RCA_0$

**Definition 7.** $RCA_0$ is the system containing the familiar rules of $+, \cdot, <$, the first-order induction scheme restricted to $\Sigma^0_1$ formulae (i.e. over sets definable by $\Sigma^0_1$ formulae), and the comprehension axiom restricted to $\Delta^0_1$ induction.

$RCA_0$ can be thought of as computable mathematics, not only because computable subsets of $\omega$ are precisely those that are $\Delta^0_1$ (which is a straightforward exercise if you're familiar with how Turing machines work), but also in the following sense:

**Theorem 3.** *An $\omega$-model is a model of $RCA_0$ if and only if its subsets are closed under Turing reducibility (that is, given set $A$ and $B$, $A \leq_T B$ if any relativized Turing machine that can compute $A$ can also compute $B$) and computable joins (where the computable join $A \oplus B$ is the smallest set that computes both $A$ and $B$).*

$RCA_0$ allows for the construction of a fair amount of classical mathematics: $RCA_0$ allows for a construction of real numbers, a decent amount of the theory of separable metric spaces, etc. Here are some things that can be proven in $RCA_0$:

- The Baire Category Theorem
- The Intermediate Value Theorem
- The Soundness Theorem
- The existence of algebraic closures of countable fields
- Existence of a unique real closure of a countable ordered field
- The Uniform Boundedness Principle

*Proof.* We read through the classical proofs of these theorems and observe that they only use computable tools, or we construct proofs of these using only computable tools. □

In the case of algebraic closures of countable fields, for instance, we take roots of polynomials over that field instead of using Zorn's Lemma.

$RCA_0$ has a minimum $\omega$-model $REC$, whose subsets are precisely the computable subsets of $\omega$. We will use this fact to argue that $RCA_0$ is different from our next subsystem:

## 4 $WKL_0$

**Definition 8.** $WKL_0$ is $RCA_0$ with the addition of Weak König's Lemma, which states that every infinite binary tree has an infinite path.

**Lemma 4.** *$REC$ contains an infinite binary tree with no path in $REC$.*

*Proof.* This construction comes from [1]. Take the halting problem, that is, the function $H$ (along with an enumeration of all the Turing machines $\Phi_n$) such that

$$H(n) = \begin{cases} 1 & \text{if } \Phi_n(n)\downarrow \\ 0 & \text{if } \Phi_n(n)\uparrow \end{cases}$$

Then define

$$H_k(n) = \begin{cases} H(n) & \text{if } \Phi_n(n) \text{ halts in } k \text{ steps} \\ -1 & \text{otherwise} \end{cases}$$

Observe that $H$ is computably enumerable but not computable, and $H_k$ is computable. Then we define the Kleene Tree $K$ to be a set of finite strings of 0's and 1's as follows:

$$K = \{a \in 2^* : \forall 1 \leq k \leq a(H_k(n) \neq -1 \to \text{ the } n\text{th digit of } a \text{ is } H_k(n))\}$$

Then order $K$ by saying that $a \leq a'$ if $a'$ extends $a$. Now $K$ is infinite and computable, but any path in $K$ would compute $H$, so no path in $K$ can be computable. □

We then get the following result:

**Corollary.** *$WKL_0$ can prove classical theorems that $RCA_0$ cannot.*

**Theorem 5.** *$WKL_0$ is equivalent to the following results, with these equivalences being provable in $RCA_0$:*

1. *Heine-Borel, that is, that every open cover of $[0, 1]$ admits a finite subcover*
2. *Every continuous real-valued function on $[0, 1]$ is bounded*
3. *Every Every continuous real-valued function on $[0, 1]$ is uniformly continuous*

4. *Every continuous real-valued function on* $[0, 1]$ *is Riemann integrable*

5. *The Extremal Value Theorem*

6. *Lindenbaum's Theorem that every consistent set of sentences can be enlarged to a maximal consistent set of sentences*

7. *The Compactness Theorem of First-order logic*

8. *Brouwer Fixed Point*

9. *Separable Hahn-Banach*

10. *The uniqueness of the algebraic closure of a countable field of characteristic* $0$

11. *Every countable commutative ring has a prime ideal*

So as you can see, restricting yourself only to computable mathematics gets rid of a lot of important results. We'll prove the equivalence of Lindenbaum's Theorem, first by proving Lindenbaum's Theorem in $WKL_0$ and then doing a reversal:

*Proof.* This proof is from [2]. First, we prove that $WKL_0 \implies$ Lindenbaum. Suppose that $\Gamma$ is a consistent set of sentences; then there is a computable enumeration $C$ of all of the theorems of $\Gamma$. (Just start systematically writing down proofs.) There is also a computable enumeration $\phi_n$ of all of the sentences of our language. Let $\theta^0 = \neg\theta$, and let $\theta^1 = \theta$. For a (finite) binary string $\sigma$, let $\phi_\sigma = \bigwedge_{i \leq |\sigma|} \phi_i^{\sigma(i)}$ (where $|\sigma|$ is the length of $\sigma$). Let $T$ be the set of all binary strings $\sigma$ such that there is no initial segment $\tau$ of $\sigma$ such that $\neg\theta_\tau$ is one of the first $|\sigma|$-many theorems that $C$ produces, and partially order $T$ by $\sigma \leq \tau$ if $\tau$ extends $\sigma$.

$T$ is computable from $\Gamma$ since checking whether $\sigma \in \Gamma$ is a matter of checking whether a finite set of sentences is within the first such-and-such many outputs of $C$. $T$ is infinite since $\Gamma$ is consistent. Therefore, by Weak König's Lemma, $T$ has an infinite path $\alpha$, and then $\{\theta_i^{\alpha(i)} : i \in \mathbb{N}\}$ (which is just a recursive comprehension) is a maximal consistent theory containing $\Gamma$.

Thus $WKL_0 \implies$ Lindenbaum's Theorem.

For the converse, suppose $RCA_0$+Lindenbaum's Theorem, and let $T$ be an infinite binary tree (that is, a set of finite binary strings with the usual partial order). Fix $\mathcal{L}$ to have unary relation symbols $\{R_n : n \in \mathbb{N}\}$ and a constant symbol $c$, and let $\Gamma$ be the set of all sentences of the form $\bigvee_{i \leq \sigma} (R_i(c))^{1-\sigma(i)}$ over all $\sigma \notin T$. After checking that $\Gamma$ is consistent, apply Lindenbaum's Theorem to obtain a $\Sigma \supseteq \Gamma$ that is maximal consistent. Then define $\alpha$ by $\alpha(i) = j$ iff $R_i(c)^j \in \Sigma$. Then $\alpha$ is an infinite path of $T$. $\square$

In particular, the equivalence of compactness to $WKL_0$ means that there are models of $RCA_0$ in which consistent first-order theories need not be satisfiable.

But there are subsystems that are still stronger than $WKL_0$:

## 5 $ACA_0$

**Definition 9.** $ACA_0$ is the system axiomatized by the familiar rules of $+$, $\cdot$, and $<$; induction; and the comprehension axiom over arithmetical formulae.

**Theorem 6.** *$ACA_0$ is equivalet to the following theorems, with this equivalence provable over $RCA_0$:*

- *König's Lemma that every infinite, finitely branching tree has an infinite path*
- *The Bolzano-Weierstrass Theorem that every bounded sequence in* $\mathbb{R}$ *contains a convergent subsequence*
- *Bolzano-Weierstrass for compact metric spaces*
- *Every (countable) vector space over* $\mathbb{Q}$ *(or any countable field) has a basis*
- *Infinitary Ramsey's Theorem for colorings of* $[\mathbb{N}]^n$, $n \geq 3$

Since König's Lemma trivially implies Weak König's Lemma, every theorem of $ACA_0$ is a theorem of $WKL_0$. The fact that $ACA_0$ is strictly stronger than $WKL_0$ is less clear than the analogous statement for $WKL_0$ and $RCA_0$, but can be shown from the following characterization of $\omega$-models of $ACA_0$:

**Theorem 7.** *An $\omega$-model $M$ is a model of $ACA_0$ if and only if the collection of subsets of $M$ is closed under computable join, Turing reducibility, and the Turing jump (where the Turing Jump of $A$, denoted $TJ(A)$ or $A'$, is the Turing degree (or equicomputability equivalence class of sets) that computes the halting problem of Turing machines relativized to $A$).*

Then results about low sets from computability theory yield a model of $WKL_0$ consisting entirely of low sets, and thus a model of $WKL_0$ that does not satisfy $ACA_0$.

**Remark 8.** The above characterization of models of $ACA_0$ gives a minimum $\omega$-model $ARITH$ with collection of subsets $S = \{A \in \mathcal{P}(\omega) : \exists n(A \leq_T \emptyset^{(n)})\}$, that is, all of the sets computable from finite Turing jumps of the computable sets.

We will use the above to prove that our next system is stronger than $ACA_0$:

# 6 $ATR_0$

**Definition 10.** The system $ATR_0$ is $ACA_0$ with the addition of arithmetical transfinite recursion, that is, we are allowed to do transfinite recursion along any (countable) well order and any arithmetical formula.

Observe that within $ARITH$, $\emptyset^{(n)}$, the $n$th Turing jump of $\emptyset$, is in $ARITH$; additionally, the Turing Jump is an arithmetical operator, where $A = TJ(B)$ if and only if there is an $n$ such that $\Phi_n^B$, the $n$th Turing machine relativized to $A$, computes $A$. However, $\emptyset^{(\omega)} = \bigcup_{n\in\omega} \emptyset^{(n)}$ is not in $ARITH$. Thus $ARITH$ is not a model of $ATR_0$, and $ATR_0$ is strictly stronger than $ACA_0$.

$ATR_0$ is, in a sense, the minimum theory in which ordinal arithmetic makes sense, in the following way:

**Theorem 9.** *$ATR_0$ is equivalent to the following, with the equivalence provable in $RCA_0$:*

- *Any two (countable) ordinals are comparable*
- *Strong comparability of ordinals (i.e. $\alpha \leq_s \beta$ iff there is an $f : \alpha \to \beta$ that is an order-preserving isomorphism between $\alpha$ and an initial segment of $\beta$, i.e. the familiar ordering) is equivalent to weak comparability or ordinals (i.e. $\alpha \leq_w \beta$ iff there is an order-preserving injection $f : \alpha \to \beta$)*
- *The Perfect Set theorem: Any two disjoint analytic sets can be separated by a Borel set*

The fact that $ATR_0 \implies$ any two ordinals are comprable, and that $\alpha \leq_w \beta \implies \alpha \leq_s \beta$ are fairly intuitive. The reversals are long and technical.

**Remark 10.** There are some equivalences that are provable over $ACA_0$ instead of $RCA_0$. These tend to be theorems that require $ACA_0$ for the relevant definitions, such as ordinal exponentiation.

$ATR_0$ is strong enough to capture a great amount of classical mathematical content, but there is a yet stronger system that we should note:

# 7 $\Pi_1^1 - CA_0$

**Definition 11.** $\Pi_1^1 - CA_0$ is the subsystem axiomatized by the usual rules of $+$, $\cdot$, and $<$; full induction; and comprehension over $\Pi_1^1$ formulae.

**Remark 11.** There are analogous systems $\Pi_k^1 - CA_0$ with the analogous axiomatizations.

**Remark 12.** We don't talk about $\Sigma_n^1 - CA_0$ in its own right because $\Sigma_n^1 - CA_0$, it turns out, is equivalent to $\Pi_n^1 - CA_0$.

There is no minimum, or minimal, $\omega$-model of $\Pi^1_1 - CA_0$, but there are a smaller class of models of $\Pi^1_1 - CA_0$ about which there are nice model-theoretic results:

**Definition 12.** A $\beta$-model $M$ is an $\omega$-model with power set $S$ such that for every $\Pi^1_1$ or $\Sigma^1_1$ sentence $\sigma$ with parameters from $S$, $M$ models $\sigma$ if and only if the intended model $(\omega, \mathcal{P}(\omega), +, \cdot, 0, 1, <)$ models $\sigma$.

**Theorem 13.** *An $\beta$-model $M$ is a model of $\Pi^1_1 - CA_0$ if and only if the power set in $M$ is closed under computable join, hyperreducibility (where $A \leq_H B$ iff $A \leq_T B^{(n)}$ for some $n$), and hyperjumps.*

This theorem gives a minimum $\beta$-model of $\Pi^1_1 - CA_0$, with power set $S$ equal to all of the subsets of $\omega$ that are hyperreducible to a finite hyperjump of $\emptyset$.

However, $ATR_0$ has no minimum $\beta$-model, since the intersection of all $\beta$-models of $ATR_0$ is a model $HYP$, which has power set $S$ equal to all of the hyperarithmetical sets; but $HYP$ does not contain the $\omega$th hyperjump of $\emptyset$, and is thus not a model of $ATR_0$. Therefore, $ATR_0$ is strictly weaker than $\Pi^1_1 - CA_0$.

**Theorem 14.** *The following are equivalent to $\Pi^1_1 - CA_0$, with this equivalence provable over $RCA_0$:*

- *Cantor-Bendixson, that every closed subset of $\mathbb{R}$ (or any complete separable metric space) is the union of a countable set and a perfect set*

So a lot of the research of Reverse Mathematics is characterizing other theorems that don't fit into the above five. For example, well-foundedness of countable ordinals is $\Sigma^1_1 - AC_0$, which is $ACA_0$ plus existence of choice functions for $\Sigma^1_1$ formulae, or Julien's Indecomposability Theorem (see [3]).

# References


[1] Andrej Bauer, *König's lemma and kleene tree*, 2006.

[2] Denis Hirschfeldt, *Slicing the truth : on the computable and reverse mathematics of combinatorial principles*, World Scientific, Hackensack, NJ, 2015.

[3] Itay Neeman, *The strength of jullien's indecomposability theorem*, 2008.

[4] Stephen Simpson, *Subsystems of second order arithmetic*, Cambridge University Press, Cambridge New York, 2009.