PROJECT DESCRIPTION

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The goal of my research is to better understand both the structure of local cohomology modules and the information they reveal about algebraic objects. In this project description, I will provide some background on the $\mathcal{D}$-module approach to studying local cohomology. I will then describe my past work on local cohomology, including relevant results from my thesis, and propose related research projects in three areas:

- local cohomology multiplicities (Lyubeznik numbers) for projective varieties;
- Hodge-de Rham spectral sequences for complete local rings;
- finiteness of the set of associated primes for local cohomology modules.

I will explain why Wenliang Zhang, my sponsoring scientist, is an ideal choice of postdoctoral mentor and why my proposed host institution, the University of Illinois at Chicago, is a natural home for the proposed research. I will conclude by outlining the broader impacts of my academic activities and describing the effect that an NSF MSPRF would have on my career development.

1. Background: local cohomology and $\mathcal{D}$-modules

If R is a commutative Noetherian ring, $I \subset R$ is an ideal, and M is an R-module, there is a family $\{H^i_I(M)\}$ of R-modules called the local cohomology modules of M supported at I. Local cohomology was introduced by Grothendieck in a Harvard seminar of 1961 [15] and has been a fruitful object of study ever since. Local cohomology modules carry important algebraic and geometric information. For example, if R is a local ring with maximal ideal $\mathfrak{m}$, the Cohen-Macaulay and Gorenstein conditions on R, pervasive in the commutative algebra literature, correspond to conditions on the local cohomology modules $H^i_{\mathfrak{m}}(R)$. If R is a polynomial ring $k[x_1, \dots, x_n]$ in finitely many variables with coefficients in an algebraically closed field k, and X is the set of common zeros in k^n of a collection of polynomials in these variables, the vanishing or non-vanishing of local cohomology modules $H^i_I(R)$ supported at the ideal these polynomials generate encodes information about the *minimal* number of equations needed to describe X.

Local cohomology as a $\mathcal{D}$-module. Part of the difficulty in understanding local cohomology modules is their tendency to be huge. Even for ideals I of *Noetherian* rings R and *finitely generated* R-modules M, the local cohomology modules $H^i_I(M)$ are almost never finitely generated [4, Ch. 9]. If R contains a field of characteristic zero, however, the size of local cohomology can be partially tamed by exploiting a formal algebraic version of the differential calculus: $\mathcal{D}$-modules. For simplicity, assume R is either a polynomial ring $k[x_1, \dots, x_n]$ or a formal power series ring $k[[x_1, \dots, x_n]]$ over a field k of characteristic zero. The ring $\mathcal{D} = \mathrm{Diff}(R, k)$ of k-linear differential operators is free as an R-module, generated by monomials $\partial_1^{a_1} \cdots \partial_n^{a_n}$ in the partial differentiation operators $\partial_i = \frac{\partial}{\partial x_i}$ [12, Thm. 16.12.1]. This ring contains R as a subring but is non-commutative, due to the Leibniz rule (we have $\partial_i x_i - x_i \partial_i = 1$ as operators on R). Lyubeznik in [19] observed that local cohomology modules $H^i_I(R)$ can be regarded as left modules over this larger ring $\mathcal{D}$, and as such, they satisfy strong finiteness properties. They are *holonomic* (in some sense, "as small as possible"), implying in particular that they are cyclic and of finite length as left $\mathcal{D}$-modules. Given any left $\mathcal{D}$-module M, we can construct its *de Rham complex*

$$M \otimes \Omega_R^\bullet = (0 \to M \to \oplus M\, dx_i \to \oplus M\, dx_i \wedge dx_j \to \cdots \to M\, dx_1 \wedge \cdots dx_n \to 0)$$

by the usual exterior derivative formula, and we write $H^i_{dR}(M)$ for the cohomology objects of this complex. If M is a holonomic $\mathcal{D}$-module, every $H^i_{dR}(M)$ is a finite-dimensional vector space over k [1, 9]. In particular, we can take M to be a local cohomology module. The associated de Rham complexes (and their finite-dimensional cohomology spaces) are key ingredients in my study of Hartshorne's de Rham homology and cohomology theories for complete local rings described in section 3.

2. Lyubeznik numbers for projective varieties

In [19], Lyubeznik introduced a family of invariants, denoted $\lambda_{i,j}(A)$, for complete local rings A with coefficient fields (of any characteristic). These invariants, now known as the *Lyubeznik numbers*, have been shown to have surprising relations with étale cohomology [2] and singular cohomology of complex varieties [10]; see [22] for a survey of results concerning these numbers. If X is a projective variety over a field k, we can fix an embedding ι of X into $\mathbb{P}^n_k$ for some n, take the affine cone $C(\iota(X))$ on the image of X, and compute the Lyubeznik numbers of the local ring at the vertex of this cone. Lyubeznik asked in [20, p, 133] whether the numbers so obtained are intrinsic to X, depending neither on n nor on the embedding ι. Zhang gave an affirmative answer in the case where k has characteristic $p > 0$ [30]. I gave an affirmative answer in characteristic zero for X *nonsingular* [26], using tools from [16] and [23]:

Theorem 1. [26, Thm. 1.2] *Let X be a nonsingular projective variety over a field k of characteristic zero, and choose an embedding $\iota : X \hookrightarrow \mathbb{P}^n_k$ for some n. The Lyubeznik numbers of the local ring at the vertex of the affine cone $C(\iota(X))$ depend only on X, and can be expressed in terms of the dimensions $\beta_i = \dim_k H^i_{dR}(X)$, where the de Rham cohomology $H^i_{dR}(X)$ is that defined in* [16].

Research proposal: Lyubeznik numbers for arbitrary projective varieties. Many of the results in my thesis [29] arose from an attempt to prove Theorem 1 in general, that is, without the nonsingular hypothesis on X. I believe that the techniques of [16] can be put to worthwhile use here. The key result used in [16] to prove that the de Rham *homology* theory constructed in that paper is well-defined is the following [16, Lemma II.3.1]: if $f : X \to Y$ is a smooth morphism of schemes of relative dimension n over a characteristic-zero field k, and $Z \subset X$ is a closed subscheme such that $f|_Z$ is a closed immersion, then there is a quasi-isomorphism

$$f_* \underline{\Gamma}_Z(E^\bullet(\Omega^\bullet_{X/k}))[2n] \xrightarrow{\sim} \underline{\Gamma}_Z(E^\bullet(\Omega^\bullet_{Y/k}))$$

of complexes of sheaves on Z, where $\underline{\Gamma}_Z$ is the *sheafified* "sections supported at Z" functor, $E^\bullet$ is a canonical injective resolution built from Cousin complexes, and $\Omega^\bullet$ denotes the de Rham complex of a scheme over k. The proof requires the trace map formalism of [14], which is especially well-behaved in the case where the smooth morphism f is a "projective space" $\mathbb{P}^n_Y \to Y$ [14, §III.4]. Analogous quasi-isomorphisms constructed in this special case (where the de Rham complex $\Omega^\bullet_X$ is replaced with the canonical sheaf ω_X) may form part of a promising line of attack on the general case of Theorem 1, which I propose to pursue.

3. Matlis duality for $\mathcal{D}$-modules and de Rham homology and cohomology

Matlis duality. If R is a local ring with maximal ideal $\mathfrak{m}$, the injective hull $E = E(R/\mathfrak{m})$ of the residue field $k = R/\mathfrak{m}$ as an R-module sets up a duality theory for R-modules. In particular, if R is complete, the exact contravariant functor $D(M) = \mathrm{Hom}_R(M, E)$ (for an R-module M) defines an anti-equivalence between the categories of finitely generated and Artinian R-modules [21]. According to this definition, it only makes sense to take the Matlis dual of an R-linear map between R-modules. However, inspired by [13, Exp. IV, Ex. 5.2], I have given an alternate description of the Matlis dual for a local ring $(R, \mathfrak{m})$ with coefficient field k in terms of continuous maps to the coefficient field. I have formulated a condition on k-linear maps between R-modules, called Σ-*continuity*, that guarantees that such maps admit Matlis duals. The utility of this condition lies in the fact that in the case of a complete local ring R with coefficient field k, elements of the ring $\mathcal{D} = \mathrm{Diff}(R, k)$ of k-linear differential operators on R always act on left $\mathcal{D}$-modules M via Σ-continuous maps. Therefore the Matlis duals $D(M)$ become *right* $\mathcal{D}$-modules. Even more can be said in the case of a formal power series ring, where there is a "transposition" operation converting right $\mathcal{D}$-modules to left $\mathcal{D}$-modules. In this case, I was able to prove the following:

Theorem 2. [28] *Let k be a field of characteristic zero, let $R = k[[x_1, \ldots x_n]]$ be a formal power series ring over k, and let $\mathcal{D} = \mathrm{Diff}(R, k)$ be the ring of k-linear differential operators on R. If M is a left $\mathcal{D}$-module, the Matlis dual $D(M)$ of M with respect to R can also be given a natural structure of left $\mathcal{D}$-module. If M is a* holonomic *left $\mathcal{D}$-module, then for every i, we have an isomorphism of k-spaces*

$$(H^i_{dR}(M))^* \simeq H^{n-i}_{dR}(D(M))$$

where the asterisk denotes k-linear dual.

The proof of Theorem 2 depends heavily on results of van den Essen [5, 6, 7, 8, 9] concerning the kernels and cokernels of differential operators. I have written a self-contained expository account [27] of these results.

De Rham homology and cohomology for complete local rings. Let A be a complete local ring with a coefficient field k of characteristic zero, and choose a surjection $\pi : R \to A$ where $R = k[[x_1, \ldots, x_n]]$ for some n (such a surjection exists by Cohen's structure theorem). In [16], Hartshorne defined the *de Rham homology* of $Y = \mathrm{Spec}(A)$ as the local hypercohomology of the complex $\Omega^\bullet_X$ of *continuous* differential forms on $X = \mathrm{Spec}(R)$: by definition, $H^{dR}_i(Y) = \mathbb{H}^{2n-i}_Y(X, \Omega^\bullet_X)$, where we regard Y as a closed subscheme of X via the surjection π. Hartshorne also defined the *local de Rham cohomology* of Y to be $H^i_{P,dR}(Y) = \mathbb{H}^i_P(\widehat{X}, \widehat{\Omega}^\bullet_X)$, where P is the closed point of Y and $\widehat{X}$ the formal completion of X along Y. In each case, the hypercohomology is computed in the category of sheaves of k-spaces. By reducing to the global case (finite-type schemes over k) and using his analogous global homology and cohomology theories, Hartshorne proved [16, Thm. III.1.1, Thm. III.2.1] that for each i, $H^{dR}_i(Y)$ and $H^i_{P,dR}(Y)$ are independent of X and the chosen embedding $Y \hookrightarrow X$, are finite-dimensional k-spaces, and are k-dual to each other.

There are well-known spectral sequences abutting to the hypercohomology of a complex [11, 11.4.3]. In our case, these take the form of the *Hodge-de Rham spectral sequences*, the spectral sequence for homology beginning $E_1^{n-p,n-q} = H^{n-q}_Y(X, \Omega^{n-p}_X)$ and abutting to $H^{dR}_{p+q}(Y)$, and the spectral sequence for cohomology beginning $\tilde{E}_1^{p,q} = H^q_P(\widehat{X}, \widehat{\Omega}^p_X)$ and abutting to $H^{p+q}_{P,dR}(Y)$. *A priori*, these spectral sequences depend on the choices of X and the embedding $Y \hookrightarrow X$. I have proved that, beginning with the E_2-terms, the spectral sequences are intrinsic to Y and consist of finite-dimensional k-spaces (immediately recovering the same results for the abutments):

Theorem 3. [28] *Let A be a complete local ring with coefficient field k of characteristic zero, let Y be its spectrum, and define the de Rham homology and cohomology of Y using a surjection $R = k[[x_1, \ldots, x_n]] \to A$ as above.*

(a) *Beginning with their E_2-terms, the isomorphism classes of the homology spectral sequence $E_r^{n-p,n-q}$ and cohomology spectral sequence $\tilde{E}_r^{p,q}$ are independent of R and of the surjection $R \to A$.*

(b) *For all $r \geq 2$ and all p and q, $E_r^{n-p,n-q}$ and $\tilde{E}_r^{p,q}$ are finite-dimensional k-spaces.*

(c) *For all p and q, $E_2^{n-p,n-q}$ is the k-dual space of $\tilde{E}_2^{p,q}$.*

In particular, the numbers $\rho_{p,q} = \dim_k(E_2^{n-p,n-q}) = \dim_k(\tilde{E}_2^{p,q})$ are finite and are invariants of A, analogous to the Lyubeznik numbers. The proof of part (c) of this theorem requires the theory of Matlis duality for differential operators described above.

Research proposal: duality and degeneration for Hodge-de Rham spectral sequences. The methods of my thesis do not immediately imply that the E_2-*differentials* of the two spectral sequences are k-dual to each other, and therefore the statement of part (c) for the E_r-terms ($r > 2$) remains open. I conjecture that the entire spectral sequences should be k-dual to each other, beginning at E_2, and I propose to extend the methods of my thesis in an effort to prove this conjecture. Another important open question concerns degeneration of the spectral sequences. The objects in the E_1-terms are local cohomology modules over R (or inverse limits of such), and therefore are seldom finitely generated as R-modules. Theorem 3 implies that a great deal of collapsing must occur, as the E_2-terms consist of finite-dimensional k-spaces. Sufficient conditions on the ring A such that the spectral sequences degenerate at their E_2-terms are not known, and I propose to investigate this question.

4. Associated primes of local cohomology modules

In [17], Huneke asked whether the local cohomology modules $H_I^i(R)$, where $I \subset R$ is an ideal of any commutative Noetherian ring, always have only finitely many associated primes, despite their propensity to be non-finitely generated. Singh [25, §4] constructed an example showing that the answer is negative in general, but there are many positive results in the case where R is *regular*. For example, the finiteness of associated primes of local cohomology is known

- if R is regular and contains a field of characteristic $p > 0$ [18, Cor. 2.3];
- if R is regular, local, and contains a field of characteristic zero [19, Thm. 3.4];
- if R is regular and finitely generated over a field of characteristic zero [19, Remark 3.7];
- and if R is a smooth $\mathbb{Z}$-algebra or an unramified regular local ring [3, Thm. 1.2, Cor. 4.2].

Lyubeznik has conjectured [19, Remark 3.7] that the finiteness should hold for any regular ring R.

Research proposal: finiteness of associated primes for local cohomology over regular rings. The proof in [18] localizes in a way that the proof in [19] does not, which is why the case of a regular ring (not necessarily local) containing a field of characteristic zero remains open. I am particularly interested in working on this case. It is well-known that if R is a regular ring of dimension d and I is any ideal, then the "top" local cohomology module $H_I^d(R)$ has finitely many associated primes. Recent work of Puthenpurakal [24, Thm. 1.3] shows that if R is moreover *excellent* (and contains a field of characteristic zero) then $H_I^{d-1}(R)$ also has finitely many associated primes. I propose to search for more results in this direction, beginning with the case of $H_I^{d-2}(R)$ and the weaker (but still open) question whether there are finitely many *maximal* associated primes.

5. Sponsoring scientist and host institution

In [30], Wenliang Zhang, my sponsoring scientist, made the most significant advance to date on the question of Lyubeznik numbers for projective varieties. As an expert in this subject, Zhang is particularly well-suited to supervise the first research project proposed above. As one of the organizers of the 2015 AMS Research Communities in commutative algebra, Zhang directly oversaw my own group, which focused on the associated primes of local cohomology modules. The questions he asked us led directly to the last research project proposed above.

My thesis research as well as the specific topics proposed here have all dealt with commutative algebra and algebraic geometry in characteristic zero. In the future, I would also like to diversify my research by considering a project in positive characteristic. In addition to Zhang, who has done significant work on local cohomology in positive characteristic, the University of Illinois at Chicago is home to Kevin Tucker, a specialist in positive-characteristic commutative algebra and algebraic geometry. I believe that the UIC department would be an excellent base of operations for the research proposed here as well as fruitful future collaborations.

6. Broader impacts and career development

I have worked extensively with the University of Minnesota Talented Youth Mathematics Program (UMTYMP) as a teaching assistant and workshop leader, as well as in an intervention program for younger students denied admission to the program on their first attempt. UMTYMP, directed at middle and high school students, covers the high school mathematics curriculum rapidly, followed by the equivalent of a lower-division honors calculus sequence and a transition course to advanced mathematics. It provides an invaluable experience for gifted students, either giving them a massive head start in case they choose to pursue mathematics further, or giving them the necessary background early in case their passion leads them to other STEM fields (as it does for many of them). Away from the UMTYMP context, I have a serious interest in and experience with the transition or "bridge" course. This course is a place where many students' feelings about mathematics as an abstract subject begin to stabilize for good or ill, and so a deft pedagogical hand is critical.

My desires to work with secondary students interested in mathematics and to teach and– ideally– design transition courses to advanced mathematics persist, and I intend to seek out opportunities at UIC to do both. The NSF Postdoctoral Research Fellowship would provide me not only with these opportunities, but would install me in an excellent place to pursue my research interests, beginning with the specific projects described in this proposal. This fellowship would position me well for my intended career as a research mathematician.