1. Let X be a *Pareto*-distributed random variable; i.e. the pdf of X for $k, \theta > 0$ is
$$f(x) = \begin{cases} \frac{k\theta^k}{x^{k+1}} & x \geq \theta \\ 0 & x < \theta \end{cases}.$$

(a) Verify that the total area under the graph equals 1.

$$\int_\theta^\infty \frac{k\theta^k}{x^{k+1}} dx = \frac{k\theta^k}{-kx^k}|_\theta^\infty = 0 + \frac{k\theta^k}{k\theta^k} = 1$$

(b) Find an expression for the cdf $P(X \leq x)$.

$$\int_\theta^x \frac{k\theta^k}{t^{k+1}} dt = \frac{k\theta^k}{-kt^k}|_\theta^x = 1 - \frac{\theta^k}{x^k}$$

(c) Calculate the population mean and variance, $E(X)$ and $V(X)$. What condition(s) do you need on k in order for the mean and variance to be finite?

$$E(X) = \int_\theta^\infty \frac{k\theta^k}{x^k} dx = \frac{k\theta^k}{(-k+1)x^{k-1}}|_\theta^\infty = \frac{k\theta}{k-1},\ k > 1$$
$$E(X^2) = \frac{k\theta^k}{(2-k)x^{k-2}}|_\theta^\infty = \frac{k\theta^2}{k-2},\ k > 2.$$

2. Let Z be a standard normal random variable. Find the value of c that makes the following probability statements correct.

(a) $P(0 \leq Z \leq c) = .4838$

$c = 2.14$

(b) $P(c \leq Z) = .121$

$c = 1.17$

(c) $P(-c \leq Z \leq c) = .668$

$c = 0.97$

(d) $P(c \leq |Z|) = 0.016$

$c = 2.41$

3. Let X be a normal random variable with mean 80 and standard deviation 10. Compute the following probabilities by standardizing.

(a) $P(X \leq 100)$

$P(Z \leq \frac{100-80}{10} = 2) = .9772$

(b) $P(65 \leq Z \leq 90)$

$P(Z \leq 1) - P(Z \leq -1.5) = 0.7745$

(c) $P(|X - 80| \leq 10)$

$P(70 \leq X \leq 90) = P(Z \leq 1) - P(Z \leq -1) = 0.6826$

4. Consider the following two situations in which you want to compute binomial probabilities: (a) $X \sim$ Binomial $n = 1000$, $q = 0.01$,
(b) $X \sim$ Binomial $n = 100$, $q = 0.20$. Suppose you have access to Poisson and standard normal probability tables, but you do not have access to a binomial table or any other means to efficiently calculate binomial probabilities.

   Question 1. For situations (a) and (b), would you use the normal approximation or the Poisson approximation and why?

   The Poisson approximation applies to the situation in which $nq \to \lambda$, so n should be large and q small. This is (a). The normal approximation generally works well for large n and moderate q. In fact, you want $nq - 2\sqrt{nq(1-q)} > 0$ so that $\mu - 2\sigma > 0$ and the normal places almost all of its probability on the positive reals. So, normal should work in (b).

   Question 2. For (a) calculate $P(X \leq 5)$, $P(X \leq 10)$, $P(X \leq 15)$ using both approximations and the binomial table. Which approximation did better?

   Using the normal approximation, the mean is $\mu = nq = 1000 * 0.01 = 10$ and the standard deviation is $\sigma = \sqrt{nq(1-q)} = \sqrt{1000 * 0.01 * 0.99} =$ 3.146427. We get about 0.076,0.56, and 0.96 for the three probabilities.

   Using the Poisson, we have $\lambda = nq = 10$. The Poisson probabilities are

0.067, 0.58, 0.95.

Using the binomial, the probabilities are 0.066, 0.58, and 0.95.

The Poisson approximation is better than the normal.

Question 3. For (b) calculate $P(X \leq 10)$, $P(X \leq 20)$, $P(X \leq 30)$ using both approximations and the binomial table. Which approximation did better?

For the normal, the mean is $\mu = nq = 100 * 0.2 = 20$ while standard deviation is $\sigma = \sqrt{nq(1-q)} = 4$. The normal probabilities are 0.006, 0.5, and 0.99.

For the Poisson, $\lambda = 20$ and the probabilities are 0.01, 0.56, and 0.9865. The binomial probabilities are 0.0057, 0.56 and 0.99. Both approximations work well.

Table entry for z is the area under the standard normal curve to the left of z.

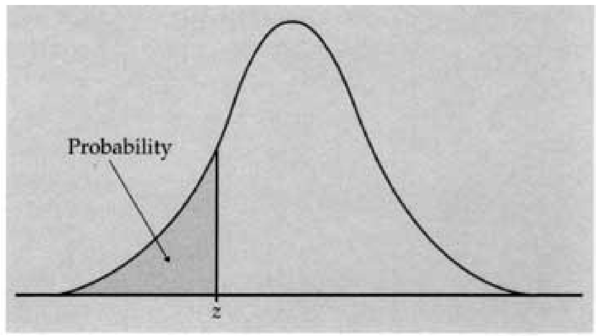

TABLE A Standard normal probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

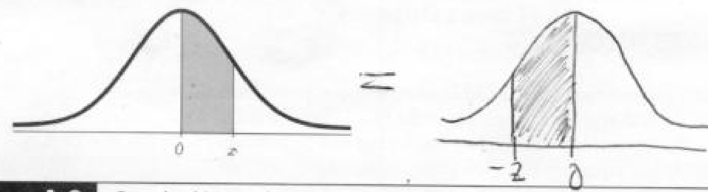

TABLE A-2 Standard Normal (z) Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495 *	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	$\boxed{.4750}$	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949 *	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.10 and higher	.4999									

NOTE: For values of z above 3.09, use 0.4999 for the area.
*Use these common values that result from interpolation:

z score	Area
1.645	0.4500
2.575	0.4950

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