1. Professor Stan der Deviation can take one of two routes on his way home from work. On the first route, there are four railroad crossings. The porbability that her will be stopped by a train at any particular crossing is 0.1, and trains operate independently at the four crossings. The other route is longer but there are only two crossings, independent of one another, with the same stoppage probability for each as on the first route. On a particular day, Professor Stan has a meeting scheduled at home for a certain time. Whichever route he takes, he calculates that he will be late if he is stopped by trains at least at half the crossings he encounters.

   a) Which route should he take to minimize the probability of being late?

   b) If he tosses a fair coin to decide on a route and he is late, what is the probability that he took the four-crossing route?

   Solutions:

   a) The probability of being stopped by two or more trains along the first route is $\binom{4}{2}(0.1^2)(0.9^2) + \binom{4}{3}(0.1^3)(0.9) + \binom{4}{4}(0.1^4) = 0.0523$. The probability of being stopped by 1 or more trains along the second route is $\binom{2}{1}(0.1)(0.9) + \binom{2}{2}(0.1^2) = 0.19$. So, he should take the first route.

   b) $P(\text{he takes the first route}|\text{ he is late}) = \frac{0.0523}{0.0523+0.19} = 0.215848$. Note that the law of total probability is being used to calculate the denominator probability.

2. For two events A and B, we say that "A *attracts* B" if $P(B|A) > P(B)$. If "A *attracts* B" prove "B *attracts* A".

   Solution:

From the definition of conditional probability, $P(B|A) = \frac{P(A\cap B)}{P(A)}$ and vice versa $P(A|B) = \frac{P(A\cap B)}{P(B)}$. By *Bayes' Formula*,

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}.$$

Dividing both sides by $P(A)$, we have

$$\frac{P(A|B)}{P(A)} = \frac{P(B|A)}{P(B)}.$$

Hence, if $P(B|A) > P(B)$ then $P(A|B) > P(A)$.

3. Consider a woman whose brother is afflicted with hemophilia, which implies that the woman's mother has the hemophilia gene on one of her two X chromosomes (almost surely not both because that is generally fatal). Thus, there is a $50-50$ chance that the woman's mother has passed on the bad gene to her. The woman has two sons, each of whom independently inherit one of her two chromosomes. If the woman herself has a bad gene, there is a $50-50$ chance she will pass it on to a son. Suppose that neither her sons is afflicted with hemophilia. What then is the probability that the woman is indeed a carrier of the hemophilia gene? How does this probability change given she has a third son who is also not afflicted?

   Solution:

   We want to calculate $P($ woman has bad gene $|$ neither of her two sons have bad gene $)$. Using the formula for conditional probability, we first calculate the probability of the intersection: $P($ woman has bad gene AND neither of her two sons have it $)$. The probability that the woman has the bad gene is 0.5. The probability that a son inherits the bad gene is 0.5, and sons inherit independently, so that $P($ woman has bad gene AND neither of her two sons have it $) = 0.5 * 0.5 * 0.5 = 0.125$. For the denominator, we must calculate $P($neither of the two sons have the bad gene$)$. We can calculate this using the *Law of Total Probability* as $P($neither of the two sons have the bad gene$) = P($neither of the two sons have the bad gene$|$ the woman has the bad gene $) * P($ the woman has the bad gene $) + P($neither of the two sons have the bad gene$|$ the woman does not have the bad gene $) * P($ the woman does not have the bad gene $)$. This is equal to $0.5 * 0.5 * 0.5 + 1 * 1 * 0.5 = 0.625$, since the gene can only be inherited if the woman has it. So, the conditional probability we want is $P($ woman has bad gene — neither of her two sons have bad gene $) = \frac{0.125}{0.625}$. The second question can be computed similarly.

4. *Gambler's Ruin.* Allan and Beth currently have \$2 and \$3, respectively. A fair coin is tossed. If the result of the toss is H, Allan wins \$1 from Beth, whereas if the coin toss results in T, then Beth wins \$1 from Allan. The process is then repeated, with a coin toss followed by the exchange of \$1, until one of the two players goes broke (hence *gambler's ruin*). We wish to determine

$$a_2 = P(Allan\, is\, the\, winner | he\, starts\, with\, \$2).$$

To do so, let's also consider $a_i = P(\text{Allan is the winner} \mid \text{he starts with}$ \$i and Beth starts with \$(5-i)) for $i = 0, 1, 3, 4, 5$.

a) What are the values of a_0 and a_5?

b) Wse the law of total probability to obtain an equation relating a_2 to a_1 and a_3. *Hint:* condition on the result of the first coin toss, realizing that if it is a H, then from that point Allan starts with \$3.

c) Using the logic developed in b), develop a system of equations relating a_i to a_{i-1} and a_{i+1}. Then solve these equations. *Hint:* write each equation so that $a_i - a_{i-1}$ is on the left hand side. Then use the result of the first equation to express each other $a_i - a_{i-1}$ as a function of a_1, and add together all four of these expressions for $i = 2, 3, 4, 5$.

d) Generalize the result to the situation in which Allan's initial fortune is \$$a$ and Beth's is \$$b$.

Solutions:

a) a_0 means Allan has no money and Beth has all the money, so Beth wins and $a_0 = 0$. Likewise, a_5 means Allan has all the money so $a_5 = 1$.

b) See that $a_2 = 0.5a_1 + 0.5a_3$.

c) In general, $a_i = 0.5(a_{i-1} + a_{i+1})$. Following the hint, $a_i - a_{i-1} = 0.5(a_{i+1} - a_{i-1})$ and see that $a_1 = 0.5a_2$. Next, $a_2 - a_1 = 0.5(a_3 - a_1) = 0.5(a_3 - 0.5a_2)$. So, $a_2 = \frac{2}{3}a_3$. Next, $a_3 - a_2 = 0.5(a_4 - a_2)$ so that $\frac{1}{3}a_3 = 0.5(a_4 - \frac{2}{3}a_3)$ and $a_3 = \frac{3}{4}a_4$. Finally, $a_4 - a_3 = 0.5(a_5 - a_3)$. Since we know $a_5 = 1$, we can get $a_4 = 0.8$ and from there backsolve to get $a_3 = .6$, $a_2 = .4$, and $a_1 = 0.2$.

d) Part c) reveals a pattern. If Allan starts with \$$a$ and Beth stars with

\$$b$, then $P($ Allan wins | they start with \$$a$ and \$$b$, respectively $) = \frac{a}{a+b}$.

(It only works this way if the probability of winning each round is 1/2.)