Rules. You may use only your formula sheet, calculator, and pen or pencil. No phones, notes, or books are allowed. You will have 50 minutes to complete the exam.

1. One method used to distinguish between granitic (G) and balsaltic (B) rock is to examine a portion of the infrared spectrum of the sun's energy reflected from the rock surface. Let W_1, W_2, and W_3 denote measured spectrum intensities at three different wavelengths. Typically, for granite, $W_1 < W_2 < W_3$ whereas for basalt $W_3 < W_1 < W_2$. When remote measurements using aircraft are taken, various orders of W_1, W_2, and W_3 may be observed. Based on previous research, the distributions of spectrum intensities given rock type are the following:

	Granite	Basalt
$W_1 < W_2 < W_3$	60%	10%
$W_1 < W_3 < W_2$	25%	20%
$W_3 < W_1 < W_2$	15%	70%

Suppose for a randomly selected rock in a given region,
$P(\text{ granite }) = 0.25$ and $P(\text{ basalt }) = 0.75$. (*Turn page.*)

16 pts. (*a*) Show that $P($ granite $|W_1 < W_2 < W_3) > P($ basalt $|W_1 < W_2 < W_3)$. If measurements yielded $W_1 < W_2 < W_3$, would you classify the rock as granite or basalt? Why?

$$P(\text{granite} \mid W_1 < W_2 < W_3) = \frac{P(\text{granite} \overset{\text{"and"}}{\cap} W_1 < W_2 < W_3)}{P(W_1 < W_2 < W_3)}$$ conditional probability

$$= \frac{P(W_1 < W_2 < W_3 \mid \text{granite})\, P(\text{granite})}{P(W_1 < W_2 < W_3)}$$ Bayes Formula

$$= \frac{P(W_1 < W_2 < W_3 \mid \text{granite})\, P(\text{granite})}{P(W_1<W_2<W_3 \mid \text{granite})P(\text{granite}) + P(W_1<W_2<W_3 \mid \text{basalt})P(\text{basalt})}$$ Law of Total Probability

$$= \frac{0.6(0.25)}{0.6(0.25) + 0.1(0.75)} = \frac{2}{3}$$ using given information

Similarly,

$$P(\text{basalt} \mid W_1 < W_2 < W_3) = \frac{0.1(0.75)}{0.6(0.25) + 0.1(0.75)} = \frac{1}{3}$$

Given that measurements yield $W_i < W_2 < W_3$, classify rock as granite because it is 2 times as likely as basalt.

* If you do w/o calculating denominator, that's ok.

5 pts for proper use of Bayes Formula

5 pts for Law of Total Probability   1 pt explanation

5 pts for calculation using given numbers

6 pts

(*b*) If measurements yielded $W_1 < W_3 < W_2$, how would you classify the rock? Answer the same question for the observation $W_3 < W_1 < W_2$ and explain your answers.

Same approach as (a)

$$P(\text{granite} \mid W_1 < W_3 < W_2) = \frac{0.25(0.25)}{0.25(0.25) + 0.2(0.75)} \approx 0.294$$

$$P(\text{basalt} \mid W_1 < W_3 < W_2) = \frac{0.2(0.75)}{0.25(0.25) + 0.2(0.75)} \approx 0.706$$

$$P(\text{granite} \mid W_3 < W_1 < W_2) = \frac{0.15(0.25)}{0.15(0.25) + 0.7(0.75)} \approx 0.0667$$

$$P(\text{basalt} \mid W_3 < W_1 < W_2) = \frac{0.7(0.75)}{0.15(0.25) + 0.7(0.75)} \approx 0.9333$$

For $W_1 < W_3 < W_2$ classify as basalt because it is more likely than granite. Same for $W_3 < W_1 < W_2$.

4 pts for calculations

2 pts for explanation of classification.

EC 4 pts.
Extra Credit, Won't count against you.

(c) Using the classification rules developed in (a) and (b) calculate the probability of an erroneous classification. That is,

P(classify as granite | rock is actually basalt) $+ P$(classify as basalt | rock is actually granite).

P(classify as granite | rock is basalt)

$= P(W_1 < W_2 < W_3 \mid \text{basalt}) = 10\%$

P(classify as basalt | rock is granite)

$= P(W_3 < W_1 < W_2 \mid \text{granite}) + P(W_1 < W_3 < W_2 \mid \text{granite})$

$= 25\% + 15\%$

So, $10\% + 25\% + 15\% = 50\%$.

If you did this, then, you got full points. Of course, I made a typo. The probability of erroneous classification is not the conditional I wrote above. It is actually correctly stated as

P(classify as granite $\overset{\text{"and"}}{\cap}$ rock is basalt) +
P(classify as basalt $\overset{\text{"and"}}{\cap}$ rock is granite).

This is calculated $10\%(75\%) + 25\%(25\%) + 15\%(25\%)$

$= 17.5\%$

This is also an accepted answer.

10 pts.

2. Consider a random variable X with a mean μ and a standard deviation σ (X is not necessarily normal). Let $g(X)$ be a specified function of X. The first order Taylor approximation of $g(X)$ in a neighborhood of μ is

$$g(X) \approx g(\mu) + g'(\mu)(X - \mu).$$

The right hand side is a linear function of X.

(a) Using rules of expectation and variance for a linear function $aX + b$, find approximations to $E(g(X))$ and $V(g(X))$ using the Taylor expansion.

If you remembered your formulas... $E(aX+b) = aE(X)+b = a\mu+b$.

$V(aX+b) = a^2V(X) = a^2\sigma^2$. If not...

$E(aX+b) = \int (aX+b)f(x)\,dx = a\int xf(x)\,dx + b\int f(x)\,dx = aE(X)+b$ ✓ 3pts

$E[(aX+b)^2] = E(a^2X^2 + 2abX + b^2) = a^2E(X^2) + 2abE(X) + b^2$

$= a^2E(X^2) + 2ab\mu + b^2$.

$E((aX+b)^2) - E(aX+b)^2 = V(X) = a^2E(X^2) + 2abE(X) + b^2$

$- a^2E(X)^2 - 2abE(X) - b^2 = a^2(E(X^2) - E(X)^2)$

$= a^2V(X)$ ✓ 3 pts

Then, $a = g'(\mu)$ $b = g(\mu) - \mu g'(\mu)$ So,

$E(g(X)) \approx g'(\mu)\mu + g(\mu) - \mu g'(\mu) = g(\mu)$. 2 pts

$V(g(X)) \approx (g'(\mu))^2\sigma^2$. 2 pts

6 pts for formulas and 4 for calculation.

4 pts.

(*b*) If the voltage ν across a medium is fixed, but the current I is a random variable, then the resistance $R = \nu/I$ is also a random variable. If $\mu_I = 20$ and $\sigma_I = 0.5$, use your work in part (*a*) to find approximations to μ_R and σ_R.

$R = g(I) = \frac{V}{I}$ 1pt $\qquad g'(I) = \frac{-V}{I^2}$ 1pt

$E(R) \approx g(\mu) = \frac{V}{20}$ 1pt

$V(R) \approx (g'(\mu))^2 \sigma^2 = \left(\frac{-V}{20^2}\right)^2 (0.5)^2 = \frac{0.25V^2}{400^2} = \frac{V^2}{1600 \cdot 400}$ 1pt

4 pts

3. Suppose that 10% of steel rods produced at a manufacturing plant are nonconforming and must be reforged. Let X denote the number among the next 200 rods sampled that must be reforged.

(*a*) What is the distribution of X? Be sure to include the values of any parameters belonging to this distribution. What is the mean and variance of this distribution?

$X \sim \text{Binomial}(n = 200,\ q = 10\%)$ 2pts

$E(X) = nq = 20$ 1pt

$V(X) = nq(1-q) = 20(.90) = 18.$ 1pt

6 pts

(*b*) Consider using the Poisson and normal distributions to furnish approximations to probabilities involving X. What are the means and variances of these approximating distributions? Which would you rather use and why?

$X \sim$ Poisson

$\lambda = nq = 20$

$E(X) = \lambda = 20$ 2 pts

$V(X) = \lambda = 20$

$X \sim Normal(\mu = nq, \sigma^2 = nq(1-q))$ 2pts

$= 20 \qquad = 18$

Explanations: Is q small and n large? Maybe, Poisson might work. The mean of Poisson matches mean of Binomial, but variances are slightly different. The normal matches the mean and variance, so it could be better than Poisson.

$\mu - 2\sigma = 20 - 2\sqrt{18} \approx 11.5$ so normal won't have problem with putting a lot of probability on negative values. Conclude normal is probably more accurate than Poisson in this case. 2 pts

5 pts (c) Calculate $P(X \leq 30)$ using the normal approximation. Use the z-table provided.

$$P\left(Z \leq \frac{30 - \mu + 0.5}{\sigma}\right)$$

$$= P\left(Z \leq \frac{30 - 20 + 0.5}{\sqrt{18}}\right) \approx P(Z \leq 2.47) \quad 4\text{pts}$$

$$= 0.9932 \quad 1\text{pt}$$

736 *Appendix A Statistical Tables and Proofs*

Table **A.3** (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998