Math320 : Practice problems
Spring 2014

Problem 1. Define the following notions:

(1) row echelon form/reduced row echelon form
(2) vector space
(3) subspace
(4) linearly dependent/independent set
(5) basis
(6) dimension
(7) row space/column space/nullspace
(8) linear transformation

Problem 2. Give an example of

(1) a set in $\mathbb{R}^3$ that is linearly independent but not a basis of $\mathbb{R}^3$
(2) a set in P_3 that is a spanning set for P_3 but not a basis of P_3
(3) a basis in $\mathbb{R}^{2\times 2}$
(4) a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$
(5) a map from $\mathbb{R}^2$ to $\mathbb{R}^3$ that is not a linear transformation

Problem 3. Prove or disprove

(1) $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,|\, x + y = 3 \right\}$ is a subspace of $\mathbb{R}^2$.

(2) $T = \{p(x) \in P_3 \,|\, p'(3) = 0\}$ is a subspace of P_3.

Problem 4. Let $A = \begin{pmatrix} 1 & -2 & 1 & 0 & 3 \\ 0 & 0 & 2 & 2 & -1 \\ 1 & -2 & -1 & -2 & 0 \\ -1 & 2 & -2 & -1 & 2 \end{pmatrix}$.

(1) Find R, a matrix in reduced row echelon form that is row equivalent to A, and matrix P such that $R = PA$.
(2) Find a basis for the row space of A. What is the dimension of the row space of A?
(3) Find a basis for the column space of A. What is the dimension of the column space of A? Find the relations x_1, x_2, x_3, x_4 have to satisfy for $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ to be in the column space of A.

(4) Find a basis for the nullspace of A. What is the dimension of the nullspace of A?

Problem 5. (15pts) Let $\mathbf{u_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $\mathbf{u_2} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ and $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{v_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

a) Find the transition matrix S corresponding to the change of basis from $\{\mathbf{u_1}, \mathbf{u_2}\}$ to $\{\mathbf{v_1}, \mathbf{v_2}\}$.

b) If $\mathbf{w} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ find the coordinate representation of $\mathbf{w}$ with respect to $\{\mathbf{v_1}, \mathbf{v_2}\}$.

Problem 6. Determine if $\{x^2 - x + 1,\ x + 3,\ x^2 - 5\}$ is

(1) a spanning set for P_3
(2) linearly independent
(3) a basis of P_3

Problem 7. Let V be a vector space, $S \subset V$, $u \in S$ and $v \in V \setminus S$. Prove the following statements

(1) If $W_1 < V$ and $W_2 < V$ then $W_1 \cap W_2 < V$.
(2) If S is linearly independent then $S - \{u\}$ is linearly independent.
(3) If S spans V then $S \cup \{v\}$ is linearly dependent.

Answers
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Problem 2.

(1) $\{(1,2,3)\}$

(2) $\{1, x, x^2, x+1\}$

(3) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

(4) $L : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $L((x,y,z)) = (x+y, x-y)$

(5) $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f((x,y)) = (x+1, y^2, \cos x)$

Problem 3.

(1) S is not a subspace of $\mathbb{R}^2$ since $(0,0) \notin S$.

(2) T is a subspace of P_3, since it is a nonempty set closed under linear combinations. Indeed, the zero polynomial belongs to T. Also if $p_1, p_2 \in T$ and $a, b \in \mathbb{R}$ then $(ap_1 + bp_2)'(3) = ap_1'(3) + bp_2'(3) = 0 + 0 = 0$, that is $ap_1 + bp_2 \in T$.

Problem 4.

(1) $R = \begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 1/8 & 3/8 & 7/8 & 0 \\ 1/8 & 3/8 & -1/8 & 0 \\ 1/4 & -1/4 & -1/4 & 0 \\ -1/8 & 13/8 & 9/8 & 1 \end{pmatrix}$ (P is not unique.)

(2) $\{(1,-2,0,-1,0), (0,0,1,1,0), (0,0,0,0,1)\}$, $dim(RowSpace) = 3$

(3) $\{(1,0,1,-1), (1,2,-1,-2), (3,-1,0,2)\}$, $dim(ColumnSpace) = 3$; $-x_1 + 13x_2 + 9x_3 + 8x_4 = 0$

(4) $\{(2,1,0,0,0), (1,0,-1,1,0)\}$, $dim(Nullspace) = 2$

Problem 5. a) $S = \begin{pmatrix} 3/2 & 5/2 \\ -1/2 & -1/2 \end{pmatrix}$

b) $[w]_{\{v_1, v_2\}} = (4, -1)$

Problem 6. $\{x^2 - x + 1,\ x + 3,\ x^2 - 5\}$ is a basis of P_3. It is a linearly independent set, and it is a spanning set for P_3.