

Math 516: Exam 2

Problem 1. (25 points) Prove that in a UFD a nonzero element is prime if and only if it is irreducible.

Problem 2. (20 points) Let F be a field and $f(x)$ be a polynomial in $F[x]$. Prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible.

Problem 3. (30 points) Let $\varphi : R \rightarrow S$ be a ring homomorphism. (R and S are not assumed to be commutative.)

a) Show that if J is an ideal of S then $\varphi^{-1}(J)$ is an ideal of R .

b) Show that if φ is surjective and I is an ideal of R then $\varphi(I)$ is an ideal of S . Give an example where this fails if φ is not surjective.

c) Assume that R and S are nonzero rings with identities 1_R and 1_S , respectively. Prove that if $\varphi(1_R) \neq 1_S$ then $\varphi(1_R)$ is zero or a zero divisor in S .

Problem 4. (25 points) Let $R = \mathbb{Z}[x]/(5, (x-1)^3(x^2+1)^2)$. Describe the ring structure of R , list all the ideals of R . Use the Chinese remainder theorem to present R as a direct product of local rings, list all the ideals of R using this representation. How many maximal ideals does R have?