Math215: HW3 solutions

Proposition 1. $3|(4^n + 5)$ *for all* $n \in \mathbb{Z}^+$.

Proof. We use the induction principle. When $n = 1$ the statement is true, since $3|9$.

Assume that $3|(4^k + 5)$ for $k \geq 1$ and show that $3|(4^{k+1} + 5)$. Let q be an integer such that $4^k + 5 = 3q$. Then $4^k = 3q - 5$ and

$$4^{k+1} + 5 = 4 \cdot 4^k + 5 = 4(3q - 5) + 5 = 12q - 15 = 3(4q - 5).$$

Since $4q - 5$ is an integer, we get$3|(4^{k+1} + 5)$.

Thus $3|(4^n + 5)$ for all $n \in \mathbb{Z}^+$. $\square$

Proposition 2. $(1 + x)^n \geq 1 + nx$ *for all nonnegative integers* n *and real numbers* $x > -1$.

Proof. We use the induction principle. When $n = 0$ the statement is true, since $1 \geq 1$.

Assume that $(1+x)^k \geq 1+kx$ for $k \geq 0$ and show that $(1+x)^{k+1} \geq 1+(k+1)x$. Using the fact that $(1 + x) > 0$, inductive hypothesis, order axioms and inequality $kx^2 \geq 0$ we get

$$(1 + x)^{k+1} = (1 + x)(1 + x)^k \geq (1 + x)(1 + kx) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x$$

That is $(1 + x)^{k+1} \geq 1 + (k + 1)x$, as required.

Thus $(1 + x)^n \geq 1 + nx$ for all nonnegative integers n. $\square$

Proposition 3.

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for all integers $n \geq 2$.

Proof. For $n = 2$ the statement is true, since $3/4 = 3/4$.

Assume that $\prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) = \frac{k+1}{2k}$ and show that $\prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) = \frac{(k+1)+1}{2(k+1)}$.

$$\prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) = \left(1 - \frac{1}{(k+1)^2}\right) \prod_{i=2}^{k} \left(1 - \frac{1}{i^2}\right) = \frac{(k+1)^2 - 1}{(k+1)^2} \cdot \frac{k+1}{2k} = \frac{k^2 + 2k}{2k(k+1)} = \frac{k+2}{2(k+1)}$$

This completes the inductive step. Thus the statement is true for all integers $n \geq 2$. $\square$