Math 220: Exam 1
Fall 2007

Problem 1. (40pts) State the method you will use and then find the general solution to the equation (you may leave your answer in implicit form)

$$a) \quad y' = \frac{xe^{x+y}}{y}$$

$$b) \quad y' = \frac{y}{x} + \frac{x}{x^2+1}$$

$$c) \quad (2xy + \cos y)dy + y^2 dx = 0$$

Problem 2. (20pts) Consider the initial value problem $\frac{dy}{dx} = \frac{x}{y} + \cos(2\pi x)$ with $y(2) = -1$.

a)Use Euler's method with step size $h = 0.5$ to approximate $y(x)$ at the point $x = 2.5$.

b)Use the midpoint method with step size $h = 0.5$ to approximate $y(x)$ at the point $x = 2.5$.

Problem 3.(15pts) Solve the initial value problem (your answer should NOT contain imaginary number i)

$$y'' - 4y' + 5y = 0 \quad \text{with} \quad y(0) = 3, \quad y'(0) = 1.$$

Problem 4. (25pts) Consider a large tank holding 100 L of brine solution, initially containing 5 kg of salt. At time $t = 0$, more brine solution begins to flow into the tank at the rate of 2 L/min. The solution inside the tank is well-stirred and is flowing out of the tank at the rate of 2 L/min. The concentration of salt in the solution entering the tank is $te^{-t/50}$ kg/L, i.e. varies in time. Set up and solve the problem for $A(t)$, the amount of salt in the tank at time t.

Extra Credit. (10pts) Find the most general function $N(x, y)$ so that the equation is exact

$$(ye^{xy} - 4x^3y + 2)dx + N(x, y)dy = 0.$$

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Problem 1.

a) Separable:
$$ye^{-y}dy = xe^x dx$$
$$-e^{-y} - ye^{-y} = xe^x - e^x + C$$

b) Linear:
$$y' - \frac{1}{x}y = \frac{x}{x^2+1}, \quad P(x) = -\frac{1}{x}$$
$$\int -\frac{dx}{x} = -\ln|x| + C, \quad \mu(x) = \frac{1}{x}$$
$$\frac{y'}{x} - \frac{y}{x^2} = \frac{1}{x^2+1}, \quad \left(\frac{y}{x}\right)' = \frac{1}{x^2+1}$$
$$\frac{y}{x} = \arctan(x) + C, \quad y = x\arctan(x) + Cx$$

c) Since $\dfrac{\partial}{\partial x}(2xy\cos y) = 2y$ and $\dfrac{\partial}{\partial y}(y^2) = 2y$, the equation is exact.
$$F(x,y) = \int y^2 dx = xy^2 + g(y), \quad \frac{\partial F}{\partial y} = 2xy + g'(y)$$
$$2xy + g'(y) = 2xy + \cos x, \quad g'(y) = \cos y, \quad g(y) = \sin y + C$$
$$xy^2 + \sin y = D$$

Problem 2.
$$f(x,y) = \frac{x}{y} + \cos(2\pi x), \quad x_0 = 2,\ y_0 = -1, \quad h = .5$$

a) $y_1 = y_0 + hf(x_0,y_0) = -1 + .5(2/(-1) + \cos(4\pi)) = -1.5, \quad y(2.5) \approx -1.5$

b) $x_{mid} = x_0 + \dfrac{h}{2} = 2.25,\ y_{mid} = y_0 + \dfrac{h}{2}f(x_0,y_0) = -1 + .25(2/(-1) + \cos(4\pi)) = -1.25$

$y_1 = y_0 + hf(x_{mid},y_{mid}) = -1 + .5(2.25/(-1.25) + \cos(2.5\pi)) = -1.9, \quad y(2.5) \approx -1.9$

Problem 3.

Characteristic equation is: $r^2 - 4r + 5 = 0,\ r = 2 \pm i$

$y(t) = C_1e^{2t}\cos t + C_2e^{2t}\sin t, \quad y'(t) = 2C_1e^{2t}\cos t - C_1e^{2t}\sin t + 2C_2\sin t + C_2e^{2t}\cos t$

Substitute $t = 0,\ y = 3:\ 3 = C_1$

Substitute $t = 0,\ y' = 1:\ 1 = 2C_1 + C_2, \quad C_2 = -5$

$$y(t) = 3e^{2t}\cos t - 5e^{2t}\sin t$$

Problem 4.

$A(t)$=amount of salt in the tank at time t, $\quad A(0) = 5$

Input rate: (concentration)$\times$(rate of flow) $= 2te^{-t/50}$

Output rate: (concentration)$\times$(rate of flow) $= 2\dfrac{A(t)}{100}$

$$\frac{\partial A}{\partial t} = 2te^{-t/50} - \frac{A}{50}$$

Linear: $$A' + \frac{1}{50}A = 2te^{-t/50}, \quad P(t) = \frac{1}{50}$$

$$\int \frac{dt}{50} = \frac{t}{50} + C, \quad \mu(x) = e^{t/50}$$

$$e^{t/50}A' - \frac{e^{t/50}}{50}A = 2t, \quad (e^{t/50}A)' = 2t$$

$$e^{t/50}A = t^2 + C, \quad A = t^2e^{-t/50} + Ce^{-t/50}$$

Substitute $t = 0,\ A = 5:\quad 5 = C$

$$A(t) = t^2e^{-t/50} + 5e^{-t/50}$$

Problem 5.

Assume that this equation is exact then

$$F(x,y) = \int (ye^{xy} - 4x^3y + 2)dx = e^{xy} - x^4y + 2x + g(y)$$

$$N(x,y) = \frac{\partial F}{\partial y} = xe^{xy} - x^4 + g'(y)$$

or

$$N(x,y) = \frac{\partial F}{\partial y} = xe^{xy} - x^4 + f(y)$$

where $f(y)$ is an arbitrary function of y.