Boundary Expression for Chern Classes of the Hodge Bundle on Spaces of Cyclic Covers

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Abstract

We compute an explicit formula for the first Chern class of the Hodge Bundle over the space of admissible μ_3-covers of n-pointed rational stable curves as a linear combination of boundary strata. We then apply this formula to give a recursive formula for calculating certain families of Hodge integrals containing λ_1. We also consider covers with a μ_2-action for which we compute λ_2 as a linear combination of codimension two boundary strata.

1 Introduction

This paper studies the intersection theory of moduli spaces of cyclic admissible covers. The first Chern class of the Hodge bundle over spaces of degree-two admissible covers was described as a linear combination of boundary divisors first in an unpublished undergraduate honors thesis [12], and two years later in [2, Proposition 10.20]. In this paper we extend the computation to the case of cyclic degree-three covers. For a positive integer $T \geq 3$, consider the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,T}$ of the moduli space of T-pointed curves of genus 0. Let $\omega = (123), \overline{\omega} = (132) \in \mu_3$. The moduli space $Adm_{g,(n\omega+m\overline{\omega})}$ parameterizes admissible genus g covers of $(n+m)$-pointed genus 0 curves which ramify over the marked points of the covered curve, such that over the first n branch points the monodromy of the cover is ω and over the last m branch points it is $\overline{\omega}$. We denote by λ_i the i-th Chern class of the Hodge Bundle over $Adm_{g,(n\omega+m\overline{\omega})}$ and by D_i^j the sum of all irreducible codimension one boundary strata parameterizing nodal covers with i branch points of monodromy ω and j branch points with monodromy $\overline{\omega}$ on one component. The branch morphism $br : Adm_{g,(n\omega+m\overline{\omega})} \to \overline{\mathcal{M}}_{0,T}$ realizes $br : Adm_{g,(n\omega+m\overline{\omega})}$ as a $B\mu_3$-gerbe over $\overline{\mathcal{M}}_{0,T}$; intuitively, this means that br is a bijection on the closed points of the moduli spaces, but has degree one third due to the fact that every admissible cover has a cyclic automorphism group of order three. It follows in particular that the intersection rings (with rational coefficients) of the two spaces agree. From [7, Theorem 1], we know that the Chow ring $A^*(\overline{\mathcal{M}}_{0,T})$ is generated by its boundary strata. The class λ_1, a codimension one tautological class of $Adm_{g,(n\omega+m\overline{\omega})}$, can therefore be expressed as the pullback via the branch morphism of a linear combination of boundary divisors of $\overline{\mathcal{M}}_{0,T}$. The main result in this paper gives an explicit formula for computing the coefficients of this linear combination.

Theorem 1.1. *The class λ_1 may be expressed as $\sum \alpha_i^j br^*(D_i^j)$, where the sum runs over all symmetrized boundary divisors D_i^j of $\overline{\mathcal{M}}_{0,T}$ as in Definition 3.2. The coefficients are:*

$$\alpha_i^j = \begin{cases} \frac{2(i+j)(T-i-j)}{9(T-1)} & i-j \equiv 0 \ mod\ 3 \\ \\ \frac{2(i+j-1)(T-i-j-1)}{9(T-1)} & i-j \equiv \pm 1 \ mod\ 3 \end{cases}$$

The strategy of proof is as follows. For any irreducible boundary stratum of dimension one γ in $\overline{\mathcal{M}}_{0,T}$, we compute $br_*(\lambda_1)\cdot\gamma$ and $\sum \alpha_i^j D_i^j \cdot \gamma$ using the formula described in Theorem 1.1 and verify these two expressions always agree. Since dimension one boundary strata form a set of generators for $A_1(\overline{\mathcal{M}}_{0,T})$ and the intersection pairing is non-degenerate, the result follows.

As an application of Theorem 1.1 we give the following recursive method for calculating certain families of Hodge integrals containing λ_1.

Theorem 1.2. *For any $n+m \geq 3$, the Hodge integral $\int_{Adm_{g,(n\omega+m\overline{\omega})}} \lambda_1^{n+m-3}$ is given by the recursive* *formula:*

$$\int_{Adm_{g,(n\omega+m\overline{\omega})}} \lambda_1^{n+m-3} = 3\sum_{i=0}^{n}\sum_{j=0}^{m} \frac{2(i+j-1)(T-i-j-1)}{9(T-1)} \binom{n+m-3}{i+j-2}\binom{n}{i}\binom{m}{j} \int_{Adm_{g_1,((i+1)\omega+j\overline{\omega})}} \lambda_1^{i+j-2} \int_{Adm_{g_2,((n-i)\omega+(m-j+1)\overline{\omega})}} \lambda_1^{n+m-i-j-2}$$

This theorem follows naturally from Theorem 1.1 as well as how the Hodge bundle, and consequently its Chern classes, split when restricted to boundary strata.

In the case of degree two admissible covers we also study the second Chern class of the Hodge bundle, λ_2. Using Mumford's relations and the expression for λ_1 in [2, Proposition 10.20] we prove the following theorem.

Theorem 1.3. *Let Δ_{i_1,i_2,i_3} denote the codimension 2 symmetrized stratum in Adm_g with i_1 branch points on the left component, i_2 branch points on the middle component, and i_3 branch points on the right component. Then $\lambda_2 = \sum \alpha_{i_1,i_2,i_3}\Delta_{i_1,i_2,i_3}$, where*

$$\alpha_{i_1,i_2,i_3} = \begin{cases} \frac{i_1 i_2 i_3(2i_1i_2+2i_1i_3+2i_2i_3-i_1-2i_2-i_3)}{32(i_1+i_2+i_3-1)(i_1+i_2-1)(i_2+i_3-1)} & i_1,i_2,i_3 \equiv 0 \ mod\ 2 \\ \frac{(i_1-1)(i_2)(i_3-1)((i_2+i_3-1)(i_1+i_2)+(i_1+i_2-1)(i_2+i_3))}{32(i_1+i_2+i_3-1)(i_1+i_2)(i_2+i_3)} & i_1,i_3 \equiv 1, \quad i_2 \equiv 0 \ mod\ 2 \\ \frac{(i_1-1)(i_2+i_3-1)(i_2+1)(i_3)(i_1+i_2-1)+(i_3)(i_1+i_2)(i_2-1)(i_1-1)(i_2+i_3)}{32(i_1+i_2+i_3-1)(i_2+i_3)(i_1+i_2-1)} & i_1,i_2 \equiv 1, \quad i_3 \equiv 0 \ mod\ 2 \end{cases}$$

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2 The Moduli Spaces of Rational, Pointed, Stable Curves

In this section we recall some of the notions we need about moduli spaces of rational, pointed, stable curves; a friendly introduction to this material is [9, Chapter 1]. For $T \geq 3$, denote by $\mathcal{M}_{0,T}$ the moduli space of T marked points on $\mathbb{P}^1(\mathbb{C})$, up to the action of $\mathbb{P}$GL(2,$\mathbb{C}$). That is, each point on $\mathcal{M}_{0,T}$ corresponds to an isomorphism class of T distinct marked points on $\mathbb{P}^1(\mathbb{C})$. The theory of Möbius transformations tells us that there exists a unique automorphism on $\mathbb{P}^1(\mathbb{C})$ that sends any triple of points to any other triple of points: given a T-tuple p_1, ..., p_T, there exists a unique Möbius transformation $\Phi : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ such that $\Phi(p_1) = 0, \Phi(p_2) = 1, \Phi(p_3) = \infty$ (these numbers are chosen by convention and could be chosen to be any other three points in $\mathbb{P}^1(\mathbb{C})$) and the other marked points are determined uniquely as the images by Φ of $p_4, ..., p_T$. Since any $\mathbb{P}^1(\mathbb{C})$ marked by some T-tuple is isomorphic in a unique way to a $\mathbb{P}^1(\mathbb{C})$ marked with a T-tuple whose first three coordinates are $0, 1, \infty$ by some Möbius Transformation, we may pick the latter configuration of points to be the representative for the isomorphism class. Having chosen these distinguished representatives, we may parametrize isomorphism classes by the positions of the remaining $T - 3$ points. Therefore, $dim_{\mathbb{C}}(\mathcal{M}_{0,T}) = T - 3$. For example, $\mathcal{M}_{0,3}$ is a single point since all copies of $\mathbb{P}^1(\mathbb{C})$ marked with three points are isomorphic to $\mathbb{P}^1(\mathbb{C})$ marked with $0, 1, \infty$. While $\mathcal{M}_{0,T}$ is not a compact topological space, it admits a compactification known as the Deligne-Mumford compactification, denoted $\overline{\mathcal{M}}_{0,T}$, which parameterizes stable curves.

Definition 2.1. *A **stable curve** is a tree of projective lines, with the following properties:*

- *Components of the tree are copies of* $\mathbb{P}^1(\mathbb{C})$ *connected at nodes.*
- *Each component has at least three special points (marked points or nodes).*

The complement of $\mathcal{M}_{0,T}$ in $\overline{\mathcal{M}}_{0,T}$, called the **boundary** of $\overline{\mathcal{M}}_{0,T}$, is the set of points parameterizing marked stable nodal curves with at least one node. The boundary is stratified by topological type, with each stratum being indexed by a dual graph, which we now define.

Definition 2.2. *Given a nodal stable curve C, the **dual graph** is a connected graph with the following properties:*

- *Each vertex corresponds to a copy of* $\mathbb{P}^1(\mathbb{C})$.

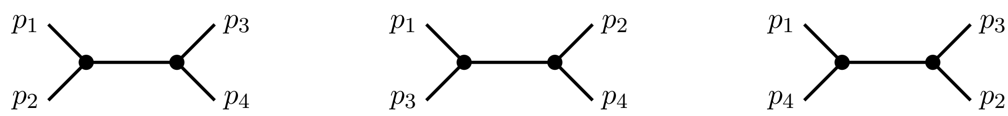


Figure 1: Dual graphs representing the three boundary points in $\overline{\mathcal{M}}_{0,4}$.

- *Half-edges connected to a vertex correspond to marked points on the component corresponding to that vertex.*
- *Edges between vertices correspond to nodes connecting the components corresponding to the vertices.*

A given dual graph represents how each component is connected to the others and which marked points are on which components. Therefore, dual graphs represent strata (loci) of points rather than a single boundary point.

We use this combinatorial representation for calculations in this paper. As an example, consider the three boundary points of $\overline{\mathcal{M}}_{0,4}$, each of which parameterizes a nodal curve consisting of two copies of $\mathbb{P}^1(\mathbb{C})$ each containing two marked points. They are represented by the dual graphs in Figure 1.

Definition 2.3. *Consider $\overline{\mathcal{M}}_{0,n+m}$ and the action of $S_n \times S_m$ where the element in S_n permutes the first n points and the element in S_m permutes the last m points. Then, a (n,m)-**symmetrized stratum** is the orbit of a stratum via this action.*

For example, if we let $n = 4, m = 0$, then the three boundary points in Figure 1 all belong to the same orbit of the S_4 action. There is a single symmetrized stratum which is the union of the three boundary points. We denote this symmetrized stratum as an unlabeled copy of the dual graphs of the boundary points in the symmetrized stratum (see Figure 2).

If we consider the case $n = 2, m = 2$, then we have a copy of S_2 permuting p_1 and p_2, and

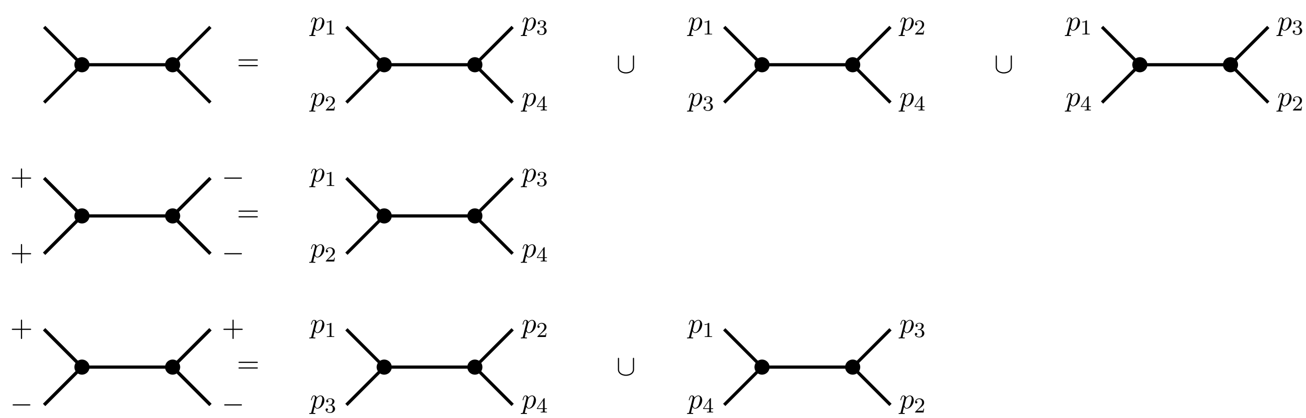


Figure 2: The first line represents the unique $(4,0)$ symmetrized boundary stratum in $\overline{\mathcal{M}}_{0,4}$. The second and third lines show the two $(2,2)$ symmetrized boundary strata in $\overline{\mathcal{M}}_{0,4}$.

another copy of S_2 permuting p_3 and p_4. The boundary of $\overline{\mathcal{M}}_{0,4}$ now consists of two orbits. The corresponding symmetrized strata are illustrated in Figure 2. To represent an (n, m)-symmetrized stratum we use the symbol $+$ for any mark permuted by S_n and $-$ for any mark permuted by S_m.

The codimension 1 boundary strata of $\overline{\mathcal{M}}_{0,T}$ are called **boundary divisors**. These are especially important for us as $br_*(\lambda_1)$ is expressible as a linear combination of boundary divisors of $\overline{\mathcal{M}}_{0,T}$. A nice combinatorial representation of the codimension of boundary strata is the number of edges in the dual graph of the stratum. For example, a boundary divisor has one edge connecting two vertices, each with some number of half-edges connected to it.

2.1 The Chow Ring $A^*(\overline{\mathcal{M}}_{0,T})$

The Chow ring of $\overline{\mathcal{M}}_{0,T}$ is a ring whose elements are equivalence classes of subvarieties of $\overline{\mathcal{M}}_{0,T}$, graded by codimension ([4]). $A^i(\overline{\mathcal{M}}_{0,T})$ is then the section of the ring corresponding to codimension i subvarieties of $\overline{\mathcal{M}}_{0,T}$. From Keel [7, Theorem 1], we know that $A^*(\overline{\mathcal{M}}_{0,T})$ is generated by the boundary strata of $\overline{\mathcal{M}}_{0,T}$. The binary operations on $A^*(\overline{\mathcal{M}}_{0,T})$ are formal addition and a multiplication, which corresponds to intersection of strata. Naively, we may expect this multiplication to simply be set theoretic intersection, however, as we will see, this is not always the case. To see why, note that since $A^*(\overline{\mathcal{M}}_{0,T})$ is a graded ring, the multiplication must be a map

$$\cdot : A^i(\overline{\mathcal{M}}_{0,T}) \times A^j(\overline{\mathcal{M}}_{0,T}) \to A^{i+j}(\overline{\mathcal{M}}_{0,T})$$

That is, $codim(X \cdot Y) = codim(X) + codim(Y)$. For an easy example of where set theoretic intersections fail to satisfy this codimension requirement, consider a self-intersection. Obviously, $X \cap X = X$, so the codimensions can only follow the above condition if $X = \overline{\mathcal{M}}_{0,T}$. Therefore, we must consider two types of intersections: transverse and non-transverse, where multiplication on the Chow Ring is set theoretic intersection if and only if the intersection is transverse. If the intersection is non-transverse, then we must algebraically deform one of the subvarieties so that the intersection respects codimension. We discuss this in Section 2.4.

2.2 Chern Classes

Let E be a complex rank r vector bundle on a space X. The i-th Chern class, $c_i(E)$ is a class in $A^i(X)$. We omit a formal definition or an extensive discussion, which may be found, for example, in [10, §14], [4, Chapter 3]. Here we recall the properties which we use throughout this paper:

functoriality: if $f : Y \to X$ is a flat morphism, $c_i(f^*(E)) = f^*(c_i(E))$.

normalization: $c_0(E) = 1$.

rank vanishing: $c_i(E) = 0$ for $i > r$.

Whitney formula: If E is a vector bundle of rank r and F a vector bundle of rank s, then

$$\sum_{i=0}^{r+s} c_i(E \oplus F) = \left(\sum_{i=0}^{r} c_i(E)\right)\left(\sum_{i=0}^{s} c_i(F)\right) \tag{1}$$

In particular, we will often use the degree one part of this relation, $c_1(E \oplus F) = c_1(E) + c_1(F)$.

2.3 ψ-Classes

In order to fully understand the intersection theory of $\overline{\mathcal{M}}_{0,T}$ we must first introduce the idea of ψ-classes (see [8] for a friendly introduction). Consider the universal family $\pi : \overline{\mathcal{M}}_{0,T+1} \to \overline{\mathcal{M}}_{0,T}$, and the T tautological sections $s_i : \overline{\mathcal{M}}_{0,T} \to \overline{\mathcal{M}}_{0,T+1}$. There is a sheaf on the universal family whose stalks are differential 1-forms on the curves parametrized; it is called the *relative dualizing sheaf*, and denoted ω_π. This gives a means to define the i-th ψ-class.

Definition 2.4. *The i-th ψ-class, denoted ψ_i, is the first Chern class of the restriction of ω_π to the i-th section of $\overline{\mathcal{M}}_{0,T}$, that is*

$$\psi_i = c_1(s_i^*(\omega_\pi))$$

There is a combinatorial description of ψ classes on $\overline{\mathcal{M}}_{0,T}$ as a linear combination of boundary divisors. It states that ψ_i on $\overline{\mathcal{M}}_{0,T}$ is equal to the sum of all boundary divisors where the i-th point is fixed on one twig and two other marked points are fixed on the other ([8, §1.5.2]). For our work, the two important computations are that ψ_i on $\overline{\mathcal{M}}_{0,3}$ is zero (there is no way to distribute three points to obtain a stable boundary divisor) and ψ_i on $\overline{\mathcal{M}}_{0,4}$ is the class of a point (there is only one way to distribute the last point so the dual graph is stable).

We introduce some classes related to ψ classes but supported on boundary strata. Consider first a boundary divisor S of $\overline{\mathcal{M}}_{0,T}$, where T_1 denotes the set of marked points on one component, and T_2 the set of marked points on the other component. Then, $S \cong \overline{\mathcal{M}}_{0,T_1\cup\{\bullet\}} \times \overline{\mathcal{M}}_{0,T_2\cup\{\star\}}$ with $\bullet, \star$ the two marked points which, when glued together, form the node of the curves parametrized by S. Then, there exist projections

$$\begin{array}{ccccc} & & S \cong \overline{\mathcal{M}}_{0,T_1\cup\{\bullet\}} \times \overline{\mathcal{M}}_{0,T_2\cup\{\star\}} & \xhookrightarrow{i} & \overline{\mathcal{M}}_{0,T} \\ & \swarrow^{\rho_1} & & \searrow^{\rho_2} & \\ \overline{\mathcal{M}}_{0,T_1\cup\{\bullet\}} & & & & \overline{\mathcal{M}}_{0,T_2\cup\{\star\}} \end{array} \qquad (2)$$

With a small abuse of notation we then define ψ-classes at the half-edges of the dual graph corresponding to the marks $\bullet$ and $\star$ as follows:

$$\psi_\bullet := i_*(\rho_1^*(\psi_\bullet)),$$

$$\psi_\star := i_*(\rho_2^*(\psi_\star)).$$

In [5] it is shown that the first Chern class of the normal bundle to the divisor S is given by

$$c_1(N_S) = -\psi_\bullet - \psi_\star. \tag{3}$$

For a higher codimension stratum S_Γ corresponding to a dual graph Γ, an edge e identifies two special marked points $\bullet, \star$ in the isomorphism between S_Γ and an appropriate product of moduli spaces. The class $S_\Gamma(-\psi_\bullet - \psi_\star)$ is obtained by pull-push via a diagram analogous to (2).

2.4 Intersections of Strata

The intersection theory of boundary strata in moduli spaces of curves admits a combinatorial description, explained in great generality in [13]. In this section we present the algorithm for intersecting strata in the generality that is needed for the later computations.

Take two strata S_{Γ_1} and S_{Γ_2} in $\overline{\mathcal{M}}_{0,T}$ with dual graphs Γ_1 and Γ_2, respectively. In order to calculate the intersection of the two strata, we must find the minimal refinement of the graphs Γ_1 and Γ_2. A refinement is a graph Γ such that both Γ_1 and Γ_2 can be obtained by contracting edges. Color the edges which are *not* contracted to obtain Γ_1 red and the edges which are *not* contracted to obtain Γ_2 blue. The refinement is said to be minimal if all edges are colored red, blue, or both.

This method of coloring edges gives a simple test for whether an intersection is transverse or non-transverse. The intersection is transverse if no edge in the minimal refinement is colored both red and blue, and the minimal refinement Γ is dual to the product $S_{\Gamma_1} \cdot S_{\Gamma_2}$. An intersection is non-transverse if there exist edges in Γ which are colored both red and blue; we call any such edge a common edge for the two graphs.

Let the i-th bicolored edge of Γ break into half-edges $\bullet_i, \star_i$. Then, it follows from (3) that

$$S_{\Gamma_1} \cdot S_{\Gamma_2} = S_\Gamma \left(\prod_i -\psi_{\bullet_i} - \psi_{\star_i} \right),$$

where the product ranges over the common edges for Γ_1 and Γ_2 in Γ.

3 The Space of Cyclic Admissible Covers

In this Section we introduce moduli spaces of admissible cyclic covers of rational curves; while the term admissible covers was introduced in [6], we adopt the more modern perspective ([1]) that one should call admissible μ_d-covers the stack of twisted stable maps to $B\mu_d$. A comprehensive treatment of this theory may be found in [2].

Definition 3.1. *A degree d* ***cyclic admissible cover*** *is a curve C along with a map $\pi : C \to X$ of degree d which satisfies the following:*

- *C has an action by the cyclic group μ_d of order d , and π is the quotient map.*
- *π is étale everywhere except at a finite set of points, called the ramification locus. That is, except at branch points (points in the image of the ramification locus), the fiber consists of d points. The fiber of a branch point will always consist of fewer than d points.*
- *C and X are nodal curves, and the image of a node in C by π is a node in X.*
- *Over a node, locally in analytic coordinates, X, C, π are described as follows, for some positive integer r not larger than d:*
  - $C : c_1 c_2 = 0.$

- $X : x_1x_2 = 0$.
- $x_1 = c_1^r, x_2 = c_2^r$.
- *Each branch point has a mondromy representation in the cyclic group of degree d .*
- *the monodromies of the two inverse images of a node in its normalization are inverses of each other.*

We consider the moduli space of admissible covers of a marked rational curve, which are allowed to ramify only over the marked points and the nodes. For this paper, we are focusing on the case of degree two and degree three covers.

In the degree two case, the group μ_2 acts on C, meaning that the inverse image of each branch point is a point of full ramification (that is, each branch point has exactly one inverse image). We say the branch points have monodromy $-1 \in \mu_2$. When a marked point in the base curve X is not a branch point, we say it has monodromy $1 \in \mu_2$. We denote the moduli space of genus g admissible covers of degree two of a marked rational curve, with n-branch points and e marked non-branch points by $Adm_{g,(n(-1)+e(1))}$. Since $n = 2g + 2$ by the Riemann-Hurwitz formula, we may omit some part of the notation when there is no risk of confusion. There exists a natural function $br : Adm_{g,(n(-1)+e(1))} \to \overline{\mathcal{M}}_{0,n+e}$ that forgets the cover and only remembers the marked base. The function br is called the branch morphism, and it is a bijection on closed points. Since $Adm_{g,(n(-1)+e(1))}$ is a $B\mu_2$ gerbe over $\overline{\mathcal{M}}_{0,n+e}$, the branch morphism has degree $\frac{1}{2}$.

In the degree three case there is a μ_3 action on the admissible covers. Note that $\mu_3 \cong \langle(123)\rangle \leq S_3$. This is why each branch point has monodromy (123), or (132), which we denote as ω and $\overline{\omega}$, respectively. Marked points that are not branch points are again said to have monodromy $1 \in \mu_3$. We denote by $Adm_{g,(n\omega+m\overline{\omega}+e(1))}$ the moduli space parameterizing admissible covers with n branch points of monodromy ω, m branch points of monodromy $\overline{\omega}$ and e marked points above which the cover is unramified. This moduli space has dimension $n + m + e - 3$ and parametrizes curves of genus $n+m-2$. The branch morphism is defined analogously, and in the case of μ_3 admissible covers it has degree $\frac{1}{3}$.

Remark 3.1. Let Δ be a boundary divisor in $Adm_{g,(n\omega+m\overline{\omega}+e(1))}$. Then

$$\Delta \cong 3Adm_{g,(n_1\omega+m_1\overline{\omega}+e_1(1))} \times Adm_{g,(n_2\omega+m_2\overline{\omega}+e_2(1))} \tag{4}$$

for some $n_1, n_2, m_1, m_2, e_1, e_2$. It is clear that the closed points of Δ are in bijection with the cartesian product in (4). The multiplicity of 3, called the gluing factor, arises from the fact that the boundary divisor is a fiber product over the inertia stack of $B\mu_3$, as explained in [3, Section 1.6].

In our computations, we will be comparing strata on spaces of admissible covers and on $\overline{\mathcal{M}}_{0,T}$ via the branch morphism. There is a natural action of $S_n \times S_m$ where the first factor permutes the points of monodromy ω and the second factor the points of monodromy $\overline{\omega}$.

Definition 3.2. *We denote by D_i^j the symmetrized boundary divisor in $\overline{\mathcal{M}}_{0,T}$ whose pullback via br is a sum of irreducible divisors parameterizing admissible covers with i ω-points and j $\overline{\omega}$-points on one component.*

3.1 The Hodge Bundle

The Hodge bundle $\mathbb{E}^g$ is a complex rank g vector bundle over $Adm_{g(n\omega+m\overline{\omega}+e(1))}$, where the fiber of a moduli point $[f : C \to \mathbb{P}^1(\mathbb{C})]$ is the vector space of holomorphic one-forms on C. We denote $\lambda_i := c_i(\mathbb{E}^g)$. Along with the properties of Chern classes given earlier, the Chern classes of the Hodge bundle also satisfy Mumford's relation [11], which states that

$$(1+\lambda_1+\lambda_2+\lambda_g)(1-\lambda_1+\lambda_2-\ldots\pm\lambda_g) = 1. \tag{5}$$

Another important concept which is important for our result is the projection formula ([4]). Given the branch morphism $br : Adm_{g,(n\omega+m\overline{\omega}+e(1))} \to \overline{\mathcal{M}}_{0,T}$, the projection formula states that for any curve C in $\overline{\mathcal{M}}_{0,T}$,

$$\int_C br_*(\lambda_1) \;=\; \int_{br^*(C)} \lambda_1.$$

Let Δ_i^j be a boundary divisor in $Adm_{g,(n\omega+m\overline{\omega})}$. Then, Δ_i^j is a sum of irreducible components each isomorphic to a (gerbe over a) product of two moduli spaces of admissible covers of genus g_1 and g_2. Then, from [11] we have the behavior of the Hodge bundle restricted to Δ_i^j. If the node is a branch point:

$$\mathbb{E}^g|_{\Delta_i^j} \cong \mathbb{E}^{g_1} \oplus \mathbb{E}^{g_2} \tag{6}$$

If the node is not a branch point:

$$\mathbb{E}^g|_{\Delta_i^j} \cong \mathbb{E}^{g_1} \oplus \mathbb{E}^{g_2} \oplus \mathcal{O}^{\oplus 2} \tag{7}$$

where $\mathcal{O}$ denotes the trivial line bundle.

In either case, the following relation between Chern classes of the Hodge bundle and Chern classes of the Hodge bundle restricted to a divisor can be derived. Let λ_1^L, λ_1^R denote λ_1 pulled back from the left and right factors in the product of moduli spaces isomorphic to the divisor Δ. Then, from the relations above together with the Whitney decomposition formula (1), we have:

$$1 + \lambda_1 + \lambda_2 + \cdots + \lambda_g = (1 + \lambda_1^L + \lambda_2^L + \cdots + \lambda_{g_1}^L)(1 + \lambda_1^R + \lambda_2^R + \cdots \lambda_{g_2}^R) \tag{8}$$

This is used to compute λ_1 of the Hodge bundle restricted to a boundary divisor. Namely,

$$\lambda_1 = \lambda_1^R + \lambda_1^L. \tag{9}$$

This result generalizes to the Hodge bundle restricted to boundary strata of higher codimension. In this case, λ_1 is equal to the sum of λ_1 pulled back from each component of the boundary stratum.

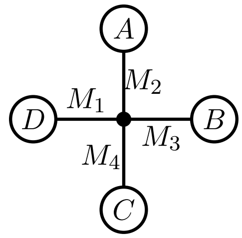


Figure 3: Graph dual to a one-dimensional stratum in $Adm_{g(n,(\omega+m\overline{\omega}))}$; A, B, C, D represent either trivalent trees or legs.

4 The Class λ_1 on Spaces of μ_3 Covers

In this Section we compute an expression for the class λ_1 on moduli spaces of μ_3-admissible covers as a linear combination of boundary divisors. Our main technique of proof consists of intersecting divisors with boundary curves. We now introduce the relevant concepts and notation.

Let $\mathcal{C}$ be a boundary curve in $Adm_{g,(n\omega+m\overline{\omega})}$. Then $\mathcal{C}$ is a gerbe over the product

$$Adm_{g_1,(n_1\omega+m_1\overline{\omega}+e_1(1))} \times ... \times Adm_{g_k,(n_k\omega+m_k\overline{\omega}+e_k(1))},$$

where $k = dim_{\mathbb{C}}(Adm_{g,(n\omega+m\overline{\omega})}) - 1 = n+m-4$. Since $\mathcal{C}$ is a curve, there is a single factor, denoted $\mathcal{C}_4$, for which $n_i + m_i + e_i = 4$; for all other moduli spaces in this product, this sum equals three. The component $\mathcal{C}_4$ is represented by a four-valent vertex in the dual graph identifying $\mathcal{C}$, as shown in Figure 3. Let $\rho : \mathcal{C} \to \mathcal{C}_4$ be the projection function, which is a bijection on closed points.

Denote by M_i the monodromies of the four marked points of elements of $\mathcal{C}_4$. To simplify the exposition in the proof of the main theorem, we refer to these monodromies as decorating the edges of the dual graph corresponding to the node connecting the marked point on the four-valent component to either A, B, C, or D; for example, in Figure 3 we would say the monodromy M_2 is the monodromy at the edge connecting A to the four-valent component.

Remark 4.1. We adopt a mild abuse of notation and talk about monodromies decorating edges of a dual graph for more general graphs: in this case the notion is not well-defined, as each edge of a dual graph corresponds to two inverse images in the normalization of a node, which come with inverse monodromies. So it would be more appropriate to talk about monodromies at the half-edges of a graph. However, we will only be interested in whether the two monodromies are equal to the identity or not, hence we can get away with associating such notion with the edge.

A stratum in $\overline{\mathcal{M}}_{0,T}$ dual to a trivalent tree has the highest possible codimension and so is of dimension zero. Making a single node 4-valent decreases the codimension by one and thus creates a curve. Given a curve in $\overline{\mathcal{M}}_{0,T}$, we pull it back via the branch morphism to obtain a curve in the space of admissible covers. The monodromy of each node on the four-valent component is determined by the congruence of the number of ω points minus the number of $\overline{\omega}$ points on the trivalent tree or single leg attached to the four-valent component at that node. That is, let i_A, i_B, i_C, i_D denote the number of ω points on A, B, C, D, respectively and j_A, j_B, j_C, j_D the number of $\overline{\omega}$ points on A, B, C, D, respectively. Then, the edge on the four-valent vertex connected to A will have monodromy (1) if $i_A - j_A \equiv 0 \ mod \ 3$, ω if $i_A - j_A \equiv -1 \ mod \ 3$ and $\overline{\omega}$ if $i_A - j_A \equiv 1 \ mod \ 3$, and likewise for B, C, D.

We group boundary curves in spaces of admissible covers based on the monodromies at the edges of the four-valent vertex of the dual graph. We will see later that the intersection of a boundary curve with λ_1 depends only on such monodromies.

Lemma 4.1. *Every 1 dimensional boundary stratum in $\overline{\mathcal{M}}_{0,T}$ pulls back via the branch morphism to one of the five types in Figure 4.*

Proof. Since the product of the monodromies at all branch points must equal 1, the sum $i_A - j_A + i_B - j_B + i_C - j_C + i_D - j_D$ must be congruent to $0 \ mod \ 3$. It is a simple problem of modular arithmetic to verify that there are only five possible ways to add four (unordered) integers modulo 3 to be congruent to 0, which correspond to the five families depicted in Figure 4. □

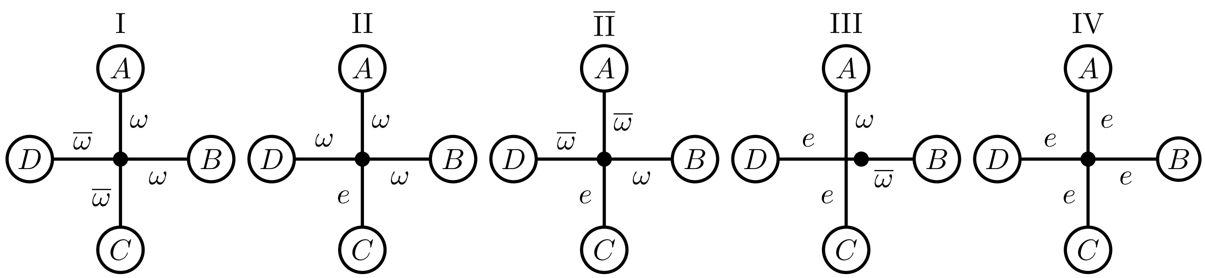


Figure 4: Pullbacks of boundary strata in $\overline{\mathcal{M}}_{0,T}$ via the branch morphism

The next Lemma studies how certain powers of the class λ_1 vanish on moduli spaces of admissible covers with a positive number of marked points with monodromy (1).

Lemma 4.2. $\lambda_1^{n+m-2+k} = 0$ *over* $Adm_{g,(n\omega+m\overline{\omega}+e(1))}$ *for any* $e, k \in \mathbb{Z}_{\geq 0}$

Proof. Consider the forgetful morphism $\phi : Adm_{g,(n\omega+m\overline{\omega}+e(1))} \to Adm_{g,(n\omega+m\overline{\omega})}$, which forgets each unramified marked point. Then,

$$[\lambda_1]_{Adm_{g,(n\omega+m\overline{\omega}+e(1))}} = \phi^*([\lambda_1]_{Adm_{g,(n\omega+m\overline{\omega})}})$$

$$\text{Thus, } ([\lambda_1]_{Adm_{g,(n\omega+m\overline{\omega}+e(1))}})^{n+m-2} = \phi^*([\lambda_1]_{Adm_{g,(n\omega+m\overline{\omega})}})^{n+m-2})$$

Since $dim_{\mathbb{C}}(Adm_{g,(n\omega+m\overline{\omega})}) = n+m-3$ and the codimension of $([\lambda_1]_{Adm_{g,(n\omega+m\overline{\omega})}})^{n+m-2+k} = n+m-2+k > n+m-3$ for any $k \in \mathbb{Z}_{\geq 0}$,

$$([\lambda_1]_{Adm_{g,(n\omega+m\overline{\omega})}})^{n+m-2+k} = 0$$

$$\text{And so, } ([\lambda_1]_{Adm_{g,(n\omega+m\overline{\omega}+e(1))}})^{n+m-2+k} = \phi^*(0) = 0$$

□

Next, we investigate how λ_1 restricts to boundary strata of moduli spaces of admissible covers.

Lemma 4.3. *Let* Δ *be a boundary stratum in* $Adm_{g,(n\omega+m\overline{\omega})}$. *The stratum* Δ *is a gerbe over a product* $Adm_{g_1,(n_1\omega+m_1\overline{\omega}+e_1(1))}\times\ldots\times Adm_{g_k,(n_k\omega+m_k\overline{\omega}+e_k(1))}$. *Denoting by* λ_1 *the class on* $Adm_{g,(n\omega+m\overline{\omega})}$, *by* $[\lambda_1]_{Adm_{g_i,(n_i\omega+m_i\overline{\omega}+e_i(1))}}$ *the class on the i-th factor, and by* ρ_i *the projection function to the i-th factor, we have that*

$$\lambda_1 \cdot \Delta = \sum \rho_i^*([\lambda_1]_{Adm_{g_i,(n_i\omega+m_i\overline{\omega}+e_i(1))}}).$$

We say λ_1 *splits additively over the components of* Δ.

Proof. This follows from (6), (7) and (1). □

As a corollary we obtain the vanishing of λ_1 when restricted to all boundary curves except for those of type I.

Lemma 4.4. *Let* γ *be a boundary curve in* $\overline{\mathcal{M}}_{0,T}$*. Then,* $br_*(\lambda_1)\cdot\gamma = 0$ *for curves which pull back to Families II,* $\overline{II}$*, III and IV.*

Proof. The boundary curve $br^*(\gamma)$ is a gerbe over a product of admissible cover spaces. All but one of the factors have exactly three marked points, and are therefore zero-dimensional. By dimension reasons, $\lambda_1 = 0$ on such factors.

Denote the four-valent component as $\mathcal{C}_4$. If $br^*(\gamma)$ is a curve of types II, $\overline{II}$, III or IV, $\mathcal{C}_4$ has at least one marked point with monodromy (1). Then Lemma 4.2 implies that $\lambda_1 = 0$ on $\mathcal{C}_4$.

Finally, by Lemma 4.3,

$$\lambda_1 \cdot br^*(\gamma) = \sum_i \rho_i^*([\lambda_1]_{Adm_{g_i,(n_i\omega+m_i\overline{\omega}+e_i(1))}}) = \sum_i \rho_i^*(0) = 0.$$

□

Next we compute the intersection of λ_1 with boundary curves of type I.

Lemma 4.5. *Let* γ *be a boundary curve in* $\overline{\mathcal{M}}_{0,T}$ *such that* $br^*(\gamma)$ *is of type* I*. Then,*

$$br_*(\lambda_1)\cdot\gamma = \frac{2}{9}.$$

Proof. Consider again the boundary curve $br^*(\gamma)$ as a (gerbe over a) product of moduli spaces of admissible covers. As before the class λ_1 is zero on the three-pointed moduli spaces, and therefore with notation as in the previous lemma,

$$\lambda_1 \cdot br^*(\gamma) = \sum_i \rho_i^*([\lambda_1]_{Adm_{g_i,(n_i\omega+m_i\overline{\omega}+e_i(1))}}) = \rho^*([\lambda_1]_{\mathcal{C}4}).$$

To evaluate that intersection number we must now analyze the multiplicities coming from the stacky structure of $br^*(\gamma)$. All three-pointed factors are isomorphic to $B\mu_3$, since the unique admissible cover parameterized has group of automorphisms equal to μ_3. For each edge, we must also multiply by the gluing factor of 3, as explained in Remark 3.1. Let V be the set of tri-valent vertices in the graph dual to γ and E be the set of edges in the dual graph. We have:

$$\begin{aligned}\lambda_1 \cdot br^*(\gamma) &= \left(\int_{\mathcal{C}_4} \lambda_1\right) \cdot \prod_E 3 \cdot \prod_{V \neq \mathcal{C}_4} \frac{1}{3} \\ &= \int_{\mathcal{C}_4} \lambda_1\end{aligned}$$

since $|E| = |V|$ for a one-dimensional stratum.

We must therefore compute $\int_{Adm_{2,(2\omega+2\overline{\omega})}} \lambda_1$. This evaluation may be extracted from the computation of equivariant Gromov-Witten invariants of the orbifold $\left[\mathbb{C}^3 / \mathbb{Z}_3\right]$ contained in [3]: the table of invariants in Section 6 gives $\langle \omega^2, \overline{\omega}^2 \rangle = -\frac{1}{3}$. Section 1.2 describes the relationship between the orbifold invariants and Hodge integrals on spaces of admissible covers. In particular, one obtains:

$$\langle \omega^2, \overline{\omega}^2 \rangle = -\frac{3}{2} \int_{Adm_{2,(2\omega+2\overline{\omega})}} \lambda_1.$$

Solving this linear equation completes the proof of this lemma. □

Let us restate the main technical result of this article.

Theorem 1.1. *The class λ_1 on $Adm_{g,(n\omega+m\overline{\omega})}$ can be expressed as $\sum \alpha_i^j br^*(D_i^j)$; the sum is over* *all symmetrized boundary divisors* D_i^j *of* $\overline{\mathcal{M}}_{0,T}$ *and the coefficients* α_i^j *are defined as follows:*

$$\alpha_i^j = \begin{cases} \frac{2(i+j)(T-i-j)}{9(T-1)} & i-j \equiv 0 \; mod\; 3 \\ \frac{2(i+j-1)(T-i-j-1)}{9(T-1)} & i-j \equiv \pm 1 \; mod \; 3 \end{cases}$$

Proof. Note that the coefficient in front of each symmetrized boundary divisor depends only on the sum of the numbers of ω and $\overline{\omega}$ points on one component. Therefore, we can denote α_i^j as α_{i+j}, and if we let $t_1 = i+j$ denote the number of marked points on one component and $t_2 = T-i-j$ the number of marked points on the other component, we can denote

$$\alpha_{t_1} = \begin{cases} \frac{2t_1t_2}{9(T-1)} & \text{monodromy at node: } (1) \\ \frac{2(t_1-1)(t_2-1)}{9(T-1)} & \text{monodromy at node: } \omega, \overline{\omega} \end{cases}$$

We use this notation in the proof for simplicity.

Since br is a bijection on closed points but has degree $\frac{1}{3}$, the statement of this theorem is equivalent to saying that $br_*\lambda_1 = \frac{1}{3}\sum \alpha_i^j D_i^j$. This together with the fact that $\overline{\mathcal{M}}_{0,T}$ is a smooth projective variety and its Chow group of curve classes is generated by boundary curves, imply that to prove this theorem, it is sufficient to show that for every one-dimensional boundary stratum $\gamma \subseteq \overline{\mathcal{M}}_{0,T}$,

$$\gamma \cdot br_*\lambda_1 = \frac{1}{3}\gamma \cdot \sum \alpha_i^j D_i^j \tag{10}$$

The dual graph of a one-dimensional boundary stratum contains a single four-valent vertex with each edge connected to either a trivalent tree or to a single leg (that is, it is a half-edge representing a marked point). We denote this as in Figure 5.

Let γ be a curve in $\overline{\mathcal{M}}_{0,T}$ and let t_a, t_b, t_c, t_d be the total number of marked points in A, B, C, D, respectively, where A, B, C, D are all either trivalent trees or single half-edges (if $t_A, t_B, t_C, t_D = 1$, respectively). For this proof, we calculate

$$\sum \alpha_{i+j}\gamma \cdot D_i^j \tag{11}$$

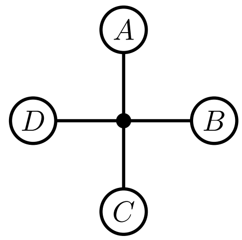
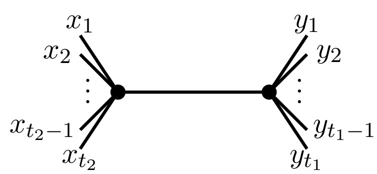


Figure 5: Arbitrary irreducible boundary curve in $\overline{\mathcal{M}}_{0,T}$.



where x_i are the marked points in $A \cup B$ and y_i are the marked points in $C \cup D$

Figure 6: The graph dual to the divisor $D_{A\cup B}$

and show that (10) holds. By Lemma 4.3, since λ_1 vanishes on components with three special points, the intersection number of λ_1 with $br^*(\gamma)$ is determined uniquely by the evaluation of λ_1 on the four-valent component, so we only need to prove the cases where γ pulls back to each of the five types described in Lemma 4.1. It is easy to see that the only non-zero intersections between γ and the sum of symmetrized boundary divisors are the non-transverse intersections with D_A, D_B, D_C, and D_D and the transverse intersections with $D_{A\cup B}, D_{A\cup C}$, and $D_{A\cup D}$, where D_X represents the boundary divisor with all the points in X on one component and all other points on the other component. For example, $D_{A\cup B}$ would be represented by the dual graph in Figure 6. The intersections with the divisors $D_{A\cup B}, D_{A\cup C}$, and $D_{A\cup C}$ are supported on the zero-dimensional boundary stratum which adds a single edge separating the points on $A\cup B, A\cup C$, or $A\cup D$ from all other points, respectively. For example, the intersection between γ and the divisor $D_{A\cup B}$ is supported by the dual graph in Figure 7. Therefore, $\gamma \cdot D_{A\cup B}$ is the class of a point and thus contributes 1 to the sum in the right hand side of (10). The intersections with D_A, D_B, D_C, and

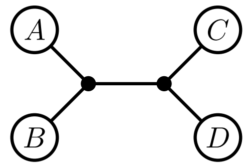


Figure 7: The transverse intersection $\gamma \cdot D_{A\cup B}$.

D_D are non-transverse intersections. These each have a common node with γ, connecting the four-valent component to a trivalent component. Therefore, intersecting these two strata reduces to computing

$$-\psi_{n_1} - \psi_{n_2}$$

where n_1, n_2 are the two marks labeling the inverse images of the node dual to the edge in the normalization of the curve. Since the common node is between an $\overline{\mathcal{M}}_{0,3}$ and an $\overline{\mathcal{M}}_{0,4}$, the intersection is computed by

$$-[\psi_{n_1}]_{\overline{\mathcal{M}}_{0,3}} - [\psi_{n_1}]_{\overline{\mathcal{M}}_{0,4}} = 0 - 1 = -1.$$

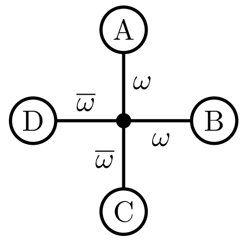
This means that each intersection of the curve γ with a divisor D_A, D_B, D_C, D_D contributes -1 to the sum on the right hand side of (10). Therefore, verifying (10) reduces to showing that

$$\begin{aligned} \gamma \cdot br_*(\lambda_1) &= \gamma \cdot (\alpha_A D_A + \alpha_B D_B + \alpha_C D_C + \alpha_D D_D + \alpha_{A\cup B} D_{A\cup B} + \alpha_{A\cup C} D_{A\cup C} + \alpha_{A\cup D} D_{A\cup D}) \\ &= -\alpha_A - \alpha_B - \alpha_C - \alpha_D + \alpha_{A\cup B} + \alpha_{A\cup C} + \alpha_{A\cup D} \end{aligned} \tag{12}$$

Note: if any of A, B, C, D are single half-edges rather than trivalent trees, the divisor D_A, D_B, D_C, or D_D is not a valid boundary divisor given our compactification of $\mathcal{M}_{0,T}$ as its dual graph would not be stable. However, this problem resolves itself combinatorially as the coefficient in front of these divisors becomes zero since $1 - 0 \equiv 1 \ mod \ 3$ so $\alpha_X = (t_X - 1)(\ldots) = (1-1)(\ldots) = 0$ for whichever set X is a single half-edge.

We now verify (10) for each of the cases described in Lemma 4.1.

Type I



First, we determine which of the two cases each coefficient $\alpha_A, \alpha_B, \alpha_C, \alpha_D, \alpha_{A\cup B}, \alpha_{A\cup C}$, and $\alpha_{A\cup D}$ fall into. We do this by analyzing the monodromy at the edge of each of the above divisors, see Remark 4.1. For example, the monodromy at the node of the four valent vertex connected to A is of type ω. It follows that a general cover in the divisor D_A is fully ramified over the unique node of the base, which is equivalent to $i_A - j_A \neq 0 \ mod\ 3$. We collect this information for all relevant divisors in the following table:

Divisor	D_A	D_B	D_C	D_D	$D_{A\cup B}$	$D_{A\cup C}$	$D_{A\cup D}$
Monodromy at node	$\neq 1$	$\neq 1$	$\neq 1$	$\neq 1$	$\neq 1$	1	1

From Lemma 4.5 we have

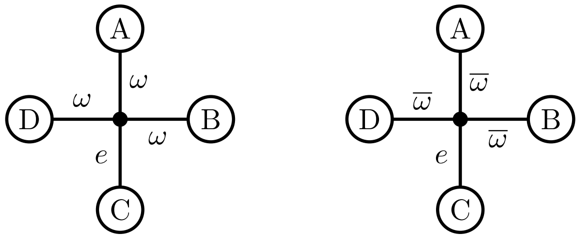
$$br_*\lambda_1 \cdot C = \frac{2}{9},$$

so we are left to show that

$$\frac{2}{9} = \frac{2}{27(T-1)}(-(t_A-1)(t_B+t_C+t_D-1) - (t_B-1)(t_A+t_C+t_D-1)$$
$$-(t_C-1)(t_A+t_B+t_D-1) - (t_D-1)(t_A+t_B+t_C-1) + (t_A+t_B-1)(t_C+t_D-1)$$
$$+(t_A+t_C)(t_B+t_D) + (t_A+t_D)(t_B+t_C))$$

This equality is easily verifiable. Therefore, all curves which pull back to type I intersect correctly with $br_*\lambda_1$.

For both of these types, the ramification data at the node of the general covers of the divisors that intersect the curve C are collected in the following table:

Types II, $\overline{\text{II}}$ 

Divisor	D_A	D_B	D_C	D_D	$D_{A\cup B}$	$D_{A\cup C}$	$D_{A\cup D}$
Monodromy at node	$\neq 1$	$\neq 1$	1	$\neq 1$	$\neq 1$	$\neq 1$	$\neq 1$

From Lemma 4.4, we have

$$br_*\lambda_1 \cdot C = 0.$$

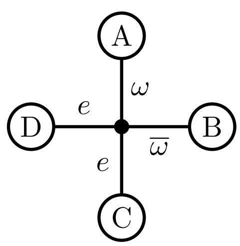
Hence, verifying (10) for curves of this type reduces to showing that

$$0 = \frac{2}{27(T-1)}(-(t_A-1)(t_B+t_C+t_D-1)-(t_B-1)(t_A+t_C+t_D-1)-t_C(t_A+t_B+t_D)$$

$$-(t_D-1)(t_A+t_B+t_C-1)+(t_A+t_B-1)(t_C+t_D-1)+(t_A+t_C-1)(t_B+t_D-1))$$

$$+(t_A+t_D-1)(t_B+t_C-1)$$

Again, it is easy to verify this equality.

Type III 

A curve of type III intersects seven boundary divisors parameterizing covers with the following monodormies at the node:

Divisor	D_A	D_B	D_C	D_D	$D_{A\cup B}$	$D_{A\cup C}$	$D_{A\cup D}$
Monodromy at node	$\neq 1$	$\neq 1$	1	1	1	$\neq 1$	$\neq 1$

By Lemma 4.4, we have

$$br_*(\lambda_1) \cdot \gamma = 0.$$

Therefore the compatibility condition imposed by (10) is:

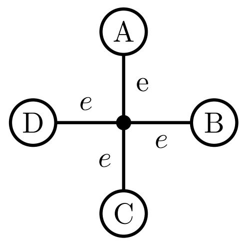
$$0 = \frac{2}{27(T-1)}(-(t_A - 1)(t_B + t_C + t_D - 1) - (t_B - 1)(t_A + t_C + t_D - 1) - t_C(t_A + t_B + t_D)$$

$$- t_D(t_A + t_B + t_C) + (t_A + t_B)(t_C + t_D) + (t_A + t_C - 1)(t_B + t_D - 1)$$

$$+ (t_A + t_D - 1)(t_B + t_C - 1))$$

Again, this equality is easily verifiable.

Type IV



In this case every node has monodromy 1, and therefore every coefficient is in the case of equivalence to 0 mod 3. From Lemma 4.4, we know that

$$br_*\lambda_1 \cdot C = 0,$$

so the equality which needs to be verified in this case is

$$0 = \frac{2}{27(T-1)}(-t_A(t_B + t_C + t_D) - t_B(t_A + t_C + t_D) - t_C(t_A + t_B + t_D)$$

$$- t_D(t_A + t_B + t_C) + (t_A + t_B)(t_C + t_D) + (t_A + t_C)(t_B + t_D)$$

$$+ (t_A + t_D)(t_B + t_C)) \tag{13}$$

Again, it is a simple matter to check that this equality holds.

We have shown that for any one-dimensional boundary stratum in $\overline{\mathcal{M}}_{0,T}$ (10) holds, thus proving Theorem 1.1. □

5 Hodge Integrals

In this Section we use Theorem 1.1 to derive a recursive structure among certain Hodge integrals on spaces of cyclic degree three covers.

Theorem 5.1. *The family of Hodge integrals* $\int \lambda_1^{n+m-3}$ *can be computed using the recursive formula:*

$$\int_{Adm_{g,(n\omega+m\overline{\omega})}} \lambda_1^{n+m-3} = 3\sum_{i=0}^{n}\sum_{j=0}^{m} \frac{2(i+j-1)(T-i-j-1)}{9(T-1)} \binom{n+m-3}{i+j-2}\binom{n}{i}\binom{m}{j} \int_{Adm_{g,((i+1)\omega+j\overline{\omega})}} \lambda_1^{i+j-2} \int_{Adm_{g,((n-i)\omega+(m-j+1)\overline{\omega})}} \lambda_1^{n+m-i-j-2}.$$

Proof. From Theorem 1.1, we can replace one λ_1 with its boundary expression, so the Hodge integral becomes

$$\int_{Adm_{g,(n\omega+m\overline{\omega})}} \lambda_1^{n+m-3} = 3\sum_{i=0}^{n}\sum_{j=0}^{m} \alpha_i^j \int_{Adm_{g,(n\omega+m\overline{\omega})}} \lambda_1^{n+m-4} br^*(D_i^j). \tag{14}$$

Now for a given D_i^j we must evaluate $\int_{Adm_{g,(n\omega+m\overline{\omega})}} \lambda_1^{n+m-4} br^*(D_i^j)$. When we restrict λ_1 to the pull back of D_i^j we use the Whitney formula and notation as in (9): ${\lambda_1}_{|br^*(D_i^j)} = \lambda_1^L + \lambda_1^R$. We raise this to the $n+m-4$ power using the binomial expansion theorem and have

$${\lambda_1}^{n+m-4}_{|_{br^*(D_i^j)}} = \sum_{k=0}^{n+m-4} \binom{n+m-4}{k} (\lambda_1^L)^k (\lambda_1^R)^{n+m-4-k}.$$

First, note that if $i-j$ is divisible by 3 the above expression is zero. This follows from the splitting formula and the fact that $\lambda_1^{n+m-4}|_{br^*(D_i^j)} = 0$, which was shown in Lemma 4.2. Furthermore, by dimension count, we can see that the only non-zero term in this sum is when $k = i+j-2$. If we account for the fact that there are $\binom{n}{i}\binom{m}{j}$ irreducible boundary divisors in a given D_i^j, then we can see that

$$\int_{Adm_{g,(n\omega+m\overline{\omega})}} \lambda_1^{n+m-4} br^*(D_i^j) = \binom{n+m-3}{i+j-2}\binom{n}{i}\binom{m}{j} \int_{Adm_{g,((i+1)\omega+j\overline{\omega})}} \lambda_1^{i+j-2} \int_{Adm_{g,((n-i)\omega+(m-j+1)\overline{\omega})}} \lambda_1^{n+m-i-j-2}.$$

Placing this and the expression for α_i^j when $i-j$ is not divisible by 3 (as terms where we would need the other coefficient vanish anyways) into (14) completes the proof. $\square$

Remark 5.1. Observe that many of the terms in the summation on the right hand side of the statement of Theorem 5.1 vanish. In order for given values of i, j to define stable admissible cover spaces in the terms in Theorem 5.1, they must satisfy the following properties:

1. $i - j \equiv 2 \; mod \; 3$

2. $i + j \geq 2$

That is, any terms in which i, j do not satisfy the above conditions must vanish. It is also easy to note that from the combinatorial factor, the term when $i = n-1, j = m$ must also always vanish.

The formula from Theorem 5.1 allows us to compute

$$\int_{Adm_{g,(n\omega+m\overline{\omega})}} \lambda_1^{n+m-3}$$

for some values of n, m in the following table.

n	m	$\int \lambda_1^{n+m-3}$
3	0	$\frac{1}{3}$
2	2	$\frac{2}{9}$
4	1	$\frac{4}{27}$
6	0	$\frac{8}{27}$
3	3	$\frac{128}{135}$
5	2	$\frac{3392}{729}$
4	4	$\frac{446923}{5103}$

Note that we do not include symmetric cases (where the values of n and m are switched) as the Hodge integrals are equal due to the symmetry of boundary divisors.

6 The Class λ_2 on Spaces of Hyperelliptic Covers.

In this Section we study the second Chern class of the Hodge bundle over spaces of degree two cyclic admissible covers. We first set up notation.

We denote the space of admissible covers with m branch points and k identity points $Adm_{(m(-1)+k(1))}$, where we omit the g from the notation as it can be recovered via Riemann-Hurwitz formula.

Our notation of boundary divisors in the space of admissible covers will vary as well in places. We let Δ_i denote the boundary divisor with i branch points on the left component. Some proofs necessitate that we denote a divisor by its number of branch points on the left component and by the space it resides in. In these cases we let Δ_i^n denote that the divisor is in the space of admissible covers with n marked points of monodromy -1. If an identity point is also present, the divisor is denoted $\Delta_{i,1}$

Lemma 6.1. *[2, Proposition 10.20] Let λ_1 be the first Chern class of the Hodge bundle over Adm_g. Furthermore, let Δ_i denote the codimenesion 1 symmetrized stratum parameterizing nodal covers with i branch points on the left twig. Let $n = 2g + 2$ denote the total number of branch points. Then, λ_1 can be expressed as $\sum \alpha_i \Delta_i$ over all symmetrized boundary divisors Δ_i of Adm_g*

$$\text{where } \alpha_i = \begin{cases} \frac{i(n-i)}{8(n-1)} & i \equiv 0 \; mod \; 2 \\ \frac{(i-1)(n-i-1)}{8(n-1)} & i \equiv 1 \; mod \; 2 \end{cases} \tag{15}$$

Our main goal in this Section is to extend these results to the second Chern class. The key tool is that λ_2 is related to λ_1 as follows.

Lemma 6.2. *The equality:*

$$\lambda_2 = \frac{1}{2}\lambda_1^2 = \frac{1}{2}\sum \alpha_i \lambda_1 \Delta_i$$

holds.

Proof. The first equality follows from Mumford's relations [11], the second from Lemma 6.1 □

Similarly to the interchangeable notation Δ_i and Δ_i^n, also α_i will be denoted α_i^n to emphasize the n marked points. Following notation from Section (3.1), (9) we have:

Lemma 6.3.

$$\alpha_i^n \lambda_1 \Delta_i^n = \alpha_i^n([1]^L[\lambda_1]^R \oplus [1]^R[\lambda_1^L])$$

$$= \begin{cases} \alpha_i^n([1]|_{Adm_{(m(-1)+1(1))}}[\lambda_1]|_{Adm_{((n-m)(-1)+1(1))}} \oplus [\lambda_1]|_{Adm_{(m(-1)+1(1))}}[1]|_{Adm_{((n-m)(-1)+1(1))}}) & i \equiv 0 \; mod \; 2. \\ \alpha_i^n([1]|_{Adm_{((m+1)(-1))}}[\lambda_1]|_{Adm_{((n-m+1)(-1))}} \oplus [\lambda_1]|_{Adm_{((m+1)(-1))}}[1]|_{Adm_{((n-m+1)(-1))}}) & i \equiv 1 \; mod \; 2. \end{cases}$$

The next Lemma sets up notation and clarifies what the coefficients of the boundary expression for λ_1 are for moduli spaces of covers with one marked point of trivial monodromy.

Lemma 6.4.

$$[\lambda_1]|_{Adm_{((n-m)(-1)+1(1))}} = \sum \alpha_j^{n-m} \Delta_{j,1}^{n-m}.$$

$$[\lambda_1]|_{Adm_{(m(-1)+1(1))}} = \sum \alpha_j^m \Delta_{j,1}^m.$$

$$[\lambda_1]|_{Adm_{((n-m+1)(-1))}} = \sum \alpha_{j+1}^{n-m+1} \Delta_{j+1}^{n-m+1}.$$

$$[\lambda_1]|_{Adm_{((m+1)(-1))}} = \sum \alpha_{j+1}^{m+1} \Delta_{j+1}^{m+1}.$$

Proof. For the case with with no identity point simply apply Lemma 6.1. For the case with an identity point note that $[\lambda_1]|_{Adm_{(m(-1)+1(1))}} = \sum \alpha_{j,1}^m \Delta_{j,1}^m$. Now consider the forgetful morphism $\phi : Adm_{((m)(-1)+1(1))} \to Adm_{(m(-1))}$. Then $\alpha_{j,1}^m \Delta_{j,1}^m = \phi^*(\alpha_j \Delta_j^m)$ so that $\alpha_{j,1} = \alpha_j$. □

We can now state the main result in this Section.

Theorem 6.1. *Let Δ_{i_1,i_2,i_3} denote the codimension 2 symmetrized stratum in Adm_g with i_1 branch points on the left component, i_2 branch points on the middle component, and i_3 branch points on* *the right component. Then* $\lambda_2 = 2\sum \alpha_{i_1,i_2,i_3}\Delta_{i_1,i_2,i_3}$ *where*

$$\alpha_{i_1,i_2,i_3} = \begin{cases} \frac{i_1 i_2 i_3(2i_1i_2+2i_1i_3+2i_2i_3-i_1-2i_2-i_3)}{32(i_1+i_2+i_3-1)(i_1+i_2-1)(i_2+i_3-1)} & i_1,i_2,i_3 \equiv 0 \; mod\; 2 \\ \frac{(i_1-1)(i_2)(i_3-1)((i_2+i_3-1)(i_1+i_2)+(i_1+i_2-1)(i_2+i_3))}{32(i_1+i_2+i_3-1)(i_1+i_2)(i_2+i_3)} & i_1,i_3 \equiv 1, \quad i_2 \equiv 0 \; mod\; 2 \\ \frac{(i_1-1)(i_2+i_3-1)(i_2+1)(i_3)(i_1+i_2-1)+(i_3)(i_1+i_2)(i_2-1)(i_1-1)(i_2+i_3)}{32(i_1+i_2+i_3-1)(i_2+i_3)(i_1+i_2-1)} & i_1,i_2 \equiv 1, \quad i_3 \equiv 0 \; mod\; 2 \end{cases}$$

The strategy of the proof for Theorem 6.1 is to fix a stratum Δ_{i_1,i_2,i_3} in the boundary expression for λ_2 and track when it appears in the expression of Lemma 6.3. This allows us write the coefficient α_{i_1,i_2,i_3} in terms of the coefficients α_i, the latter of which we have an explicit formula for.

Proof. **Case 1:** i_1, i_2, i_3 even

Let $i_1 + i_2 + i_3 = n$. Since i_1, i_2, i_3 are even, all nodes of Δ_{i_1,i_2,i_3} are unramified. Thus the stratum Δ_{i_1,i_2,i_3} only appears in the case i even of Lemma 6.3. Using this, we have that

$$\sum \alpha_{i_1,i_2,i_3}\Delta_{i_1,i_2,i_3} = \sum \alpha_i^n([1]_{Adm_{(i(-1)+1(1))}} \times \sum \alpha_j^{n-i}\Delta_{j,1}^{n-i} \oplus [1]_{Adm_{((n-i)(-1)+1(1))}} \times \sum \alpha_j^i \Delta_{j,1}^i). \tag{16}$$

Now we fix a given $\Delta_{i_1i_2i_3}$ and compute its coefficient in (16). The expression $[1]_{Adm_{(i(-1)+1(1))}} \times \Delta_{j,1}^{n-i}$ gives the divisor $\Delta_{i_1i_2i_3}$ when $i = i_1$ and $j = i_2$, or $i = i_3$ and $j = i_2$. The expression $[1]_{Adm_{((n-i)(-1)+1(1))}} \times \Delta_{j,1}^i$ gives the divisor $\Delta_{i_1i_2i_3}$ when $i = i_2 + i_3$ and $j = i_2$, or when $i = i_1 + i_2$ and $j = i_2$. Combining the above lemmas we have the following for the coefficient of a fixed $\Delta_{i_1i_2i_3}$:

$$\alpha_{i_1}^n\alpha_{i_2}^{i_2+i_3} \; + \; \alpha_{i_3}^n\alpha_{i_2}^{i_1+i_2} \; + \; \alpha_{i_2+i_3}^n\alpha_{i_2}^{i_2+i_3} \; + \; \alpha_{i_1+i_2}^n\alpha_{i_2}^{i_1+i_2} = \alpha_{i_1i_2i_3}.$$

Using the expressions $\alpha_{i_1}^n = \alpha_{i_2+i_3}^n$ and $\alpha_{i_3}^n = \alpha_{i_1+i_2}^n$ we have the expression

$$\alpha_{i_1}^n\alpha_{i_2}^{i_2+i_3} \; + \; \alpha_{i_3}^n\alpha_{i_2}^{i_1+i_2} = \frac{1}{2}\alpha_{i_1i_2i_3}. \tag{17}$$

which holds when i_1, i_2, i_3 are even. Plugging in the formula for each α from Lemma 6.1 into (17) we get the coefficient $2(\frac{(i_1+i_2)i_3}{8(i_1+i_2+i_3-1)}\frac{i_1i_2}{8(i_1+i_2-1)} + \frac{i_1(i_2+i_3)}{8(i_1+i_2+i_3-1)}\frac{i_2i_3}{8(i_2+i_3-1)})$ for the divisor Δ_{i_1,i_2,i_3}. Upon simplification we arrive at the expression:

$$\frac{i_1i_2i_3(2i_1i_2 + 2i_1i_3 + 2i_2i_3 - i_1 - 2i_2 - i_3)}{32(i_1 + i_2 + i_3 - 1)(i_1 + i_2 - 1)(i_2 + i_3 - 1)}.$$

Case 2: i_1, i_3 odd, i_2 even

Since i_1, i_3 are odd, all nodes of Δ_{i_1,i_2,i_3} are ramified. Thus, the divisor Δ_{i_1,i_2,i_3} only appears in the case i odd of Lemma 6.3. We have

$$\sum \alpha_{i_1,i_2,i_3}\Delta_{i_1,i_2,i_3} = \sum \alpha_i^n([1]_{Adm_{((i+1)(-1))}} \times \sum \alpha_{j+1}^{n-i+1}\Delta_{j+1}^{n-i+1} \oplus [1]_{Adm_{((n-i+1)(-1))}} \times \sum \alpha_{j+1}^{i+1}\Delta_{j+1}^{i+1}) \tag{18}$$

Again we fix a given $\Delta_{i_1i_2i_3}$ and compute its coefficient in (18). The expression $[1]_{Adm_{((i+1)(-1))}} \times \Delta_{j+1}^{n-i+1}$ gives the divisor Δ_{i_1,i_2,i_3} when $i = i_1$ and $j = i_2$ or when $i = i_3$ and $j = i_2$. The expression $[1]_{Adm_{(n-i+1|0)}} \times \Delta_{j+1}^{i+1}$ gives the divisor Δ_{i_1,i_2,i_3} when $i = i_2 + i_3$ and $j = i_2 + 1$ or when $i = i_1 + i_2$ and $j = i_2$. Combining these gives the following coefficient for a fixed Δ_{i_1,i_2,i_3} :

$$\alpha_{i_1}^n\alpha_{i_2+1}^{i_2+i_3+1} + \alpha_{i_3}^n\alpha_{i_2+1}^{i_1+i_2+1} + \alpha_{i_2+i_3}^n\alpha_{i_2+1}^{i_2+i_3+1} + \alpha_{i_1+i_2}^n\alpha_{i_2+1}^{i_1+i_2+1} = \alpha_{i_1i_2i_3}.$$

which can be regrouped into

$$\alpha_{i_1}^n\alpha_{i_2+1}^{i_2+i_3+1} + \alpha_{i_3}^n\alpha_{i_2+1}^{i_1+i_2+1} = \frac{1}{2}\alpha_{i_1i_2i_3}.$$

Here, i_1 is odd, $i_2 + 1$ is odd and i_3 is odd. Using this we plug in the expressions from Lemma 6.1.

$$\alpha_{i_1,i_2,i_3} = 2\left(\frac{(i_1 - 1)(n - i_1 - 1)}{8(n - 1)}\frac{(i_2)(i_3 - 1)}{8(i_2 + i_3)} + \frac{(i_3 - 1)(n - i_3 - 1)}{8(n - 1)}\frac{(i_2)(i_1 - 1)}{8(i_1 + i_2)}\right)$$

or

$$\alpha_{i_1,i_2,i_3} = \frac{(i_1 - 1)(i_2)(i_3 - 1)((i_2 + i_3 - 1)(i_1 + i_2) + (i_1 + i_2 - 1)(i_2 + i_3))}{32(i_1 + i_2 + i_3 - 1)(i_1 + i_2)(i_2 + i_3)}.$$

Case 3: i_1 odd i_2 odd i_3 even

In this case the left node is ramified and the other is unramified. Thus the divisor Δ_{i_1,i_2,i_3} appears in Lemma 6.3 as follows.

$$\sum \alpha_{i_1,i_2,i_3}\Delta_{i_1,i_2,i_3} = \sum \alpha_i^n([1]_{Adm_{((i+1)(-1))}} \times \sum \alpha_{j+1}^{((n-i+1)(-1))}\Delta_{j+1}^{n-i+1} \oplus [1]_{Adm_{((n-i+1)(-1))}} \times \sum \alpha_{j+1}^{i+1}\Delta_{j+1}^{i+1})$$
$$+ \sum \alpha_t^n([1]_{Adm_{((n-t)(-1)+1(1))}} \times \sum \alpha_j^t \Delta_{j,1}^t \oplus [1]_{Adm_{((n-t)(-1)+1(1))}} \times \sum \alpha_j^t \Delta_{j,1}^t)). \quad (19)$$

Now we fix the divisor Δ_{i_1,i_2,i_3} and see when it appears in (19). The divisor is given when $i = i_1$ and $j = i_2$, $n - i = i_1$ and $j = i_2$ or when $t = i_1 + i_2$ and $j = i_2$, $n - t = i_3$ and $j = i_2$. This gives the following equation for the fixed coefficient α_{i_1,i_2,i_3}:

$$\alpha_{i_1,i_2,i_3} = 2(\alpha_{i_1}^n \alpha_{i_2+1}^{i_2+i_3+1} + \alpha_{i_3}^n \alpha_{i_2}^{i_1+i_2}).$$

Here, i_1 is odd, $i_2 + 1$ is even, i_3 is even and i_2 is odd. Using this and the formulas from Lemma 6.1 we have

$$\alpha_{i_1,i_2,i_3} = 2\left(\frac{(i_1-1)(n-i_1-1)}{8(n-1)}\frac{(i_2+1)(i_3)}{8(i_2+i_3)} + \frac{(i_3)(n-i_3)}{8(n-1)}\frac{(i_2-1)(i_1-1)}{8(i_1+i_2-1)}\right)$$

or

$$\alpha_{i_1,i_2,i_3} = \frac{(i_1-1)(i_2+i_3-1)(i_2+1)(i_3)(i_1+i_2-1) + (i_3)(i_1+i_2)(i_2-1)(i_1-1)(i_2+i_3)}{32(i_1+i_2+i_3-1)(i_2+i_3)(i_1+i_2-1)}.$$

□

References

[1] Dan Abramovich, Alessio Corti, and Angelo Vistoli. Twisted bundles and admissible covers. volume 31, pages 3547–3618. 2003. Special issue in honor of Steven L. Kleiman.

[2] José Bertin and Matthieu Romagny. Champs de Hurwitz. *Mém. Soc. Math. Fr. (N.S.)*, (125-126):219, 2011.

[3] Charles Cadman and Renzo Cavalieri. Gerby localization, Z_3-Hodge integrals and the GW theory of $[\mathbb{C}^3/Z_3]$. *Amer. J. Math.*, 131(4):1009–1046, 2009.

[4] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[5] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[6] Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982. With an appendix by William Fulton.

[7] Sean Keel. Intersection theory of moduli space of stable n-pointed curves of genus zero. *Trans. Amer. Math. Soc.*, 330(2):545–574, 1992.

[8] Joachim Kock. Notes on psi classes. Notes. http://mat.uab.es/∼kock/GW/notes/psi-notes.pdf, 2001.

[9] Joachim Kock and Israel Vainsencher. *An invitation to quantum cohomology*, volume 249 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2007. Kontsevich's formula for rational plane curves.

[10] John W. Milnor and James D. Stasheff. *Characteristic classes*. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974. Annals of Mathematics Studies, No. 76.

[11] David Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 271–328. Birkhäuser Boston, Boston, MA, 1983.

[12] Peter Troyan. Hyperelliptic hodge integrals., 2007. Undergraduate Honors Thesis.

[13] Stephanie Yang. Intersection numbers on $\overline{M}_{g,n}$. *J. Softw. Algebra Geom.*, 2:1–5, 2010.