2 Integrals

2.1 Antiderivatives

Calculus I is largely spent trying to answer the question "given a function $f(x)$, what is the derivative $f'(x)$?" Now, we may want to ask the opposite question "given a function $f(x)$, can we find a function $F(x)$ such that $f(x)$ is the derivative of $F(x)$, i.e. $F'(x) = f(x)$?" Since this is in some sense the opposite of taking derivatives, we will call it *anti*differentiation. That is, given a function $f(x)$ we define an ***antiderivative of*** f to be a function $F(x)$ such that $F'(x) = f(x)$. Note that another way to write this is $\frac{d}{dx}(F(x)) = f(x)$ or simply $\frac{dF}{dx} = f$.

For example, since the derivative of $\sin(x)$ (with respect to the variable x) is $\cos(x)$, we say that $\sin(x)$ is an antiderivative of $\cos(x)$ (with respect to x).

The first key observation is that antiderivatives are not unique, that is there are different functions with the same derivatives. For example, the derivative of both $4x^2+3x-5$ and $4x^2+3x+2$ is $8x + 3$, so we say $4x^2 + 3x - 5$ and $4x^2 + 3x + 2$ are both antiderivatives of $8x + 3$. In fact, if $F(x)$ is any antiderivative of $f(x)$ and c is any number, then $F(x) + c$ is also an antiderivative of $f(x)$ since the derivative of a constant is 0. Luckily, this is as bad as it gets – two antiderivatives of some function $f(x)$ can only differ by a constant. That is, if $F(x)$ and $G(x)$ are two antiderivatives of $f(x)$, then $G(x) = F(x) + c$ for some constant c.

For example, every antiderivative of $8x + 3$ is of the form $4x^2 + 3x + c$ for some number c (the examples we gave above were for $c = -5$ and $c = 2$). We therefore call $4x^2 + 3x + c$ the ***general antiderivative*** of $8x + 3$ and we use the notation $\int(8x + 3)dx = 4x^2 + 3x + c$ for this general antiderivative. This is also called the ***indefinite integral*** of $8x+3$ (indefinite means there are no number bounds on the $\int$ symbol, in contrast with *definite* integrals, which we will talk about in the next section).

Note that the dx is part of the notation along with the $\int$ symbol, and it tells us that we are taking antiderivatives "with respect" to the variable x.

You can think of the symbols $\int$ and dx as being like parentheses, where every time you write an $\int$ it must be closed with a dx.

Also, note that everything we have done so far has used the x as our input variable, but we could just as well use t or y or any other letter you wish, just make sure that the dx changes to a dt or dy etc. to match the variable used in the function.

Some more examples to help get used to the notation:

$$\int \sin(x)dx = -\cos(x) + c \qquad \int 3y^2 dy = y^3 + c \qquad \int \frac{1}{t^2} dt = -\frac{1}{t} + c$$

To take the derivative of a polynomial, we the power rule, which tells us to drop the exponent and then subtract one from it. For example, to take the derivative of x^3 we first multiply by the exponent 3, and then subtract 1 from it to get $3x^2$. That is, we do

$$x^3 \xrightarrow{\text{multiply by exponent}} 3 \cdot x^3 \xrightarrow{\text{subtract 1}} 3x^{3-1} = 3x^2$$

This can be written algebraically as

$$\frac{d}{dx}(x^n) = n \cdot x^{n-1}$$

Now, since taking antiderivatives is the "opposite" of taking derivatives, let us think what we have to do to go from the $3x^2$ to get back to the original function x^3. Well to "undo" the derivative, we should do the opposite steps in the opposite order. That is, rather than multiplying by the exponent and then subracting 1 from it, we first add one to the exponent and then divide by that new exponent. That is, we do

$$x^3 = \frac{3x^3}{3} \xleftarrow{\text{divide by exponent}} 3x^3 = 3x^{2+1} \xleftarrow{\text{add 1}} 3x^2$$

As always, if we wanted to write the general antiderivative, we would add $+c$ at the end. This can be written algebraically as

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

Recall that derivatives are "additive" meaning if $f(x)$ and $g(x)$ are two functions, then $\frac{d}{dx}(f + g) = \frac{df}{dx} + \frac{dg}{dx}$. For example, to take the derivative of a polynomial like $3x^2 + 2x - 1$, you can just take the derivative of each term individually, giving $6x + 2$.

Because of this, antiderivatives are also additive, i.e. if $f(x)$ and $g(x)$ are functions, then $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$. For example, we can take the antiderivative of a polynomial like $5x^4 + 2x^2 - 4x$ by using the power rule term-by-term, giving us $x^5 + \frac{2}{3}x^3 - 2x^2 + c$.

Let's next consider the exponential functions e^x, since this has a very simple derivative - itself! Recall that by the chain rule, if we have something other than just x (or whichever variable we are using) in the exponent, we have to multiply our final result by its derivative. For example, the derivative with respect to x of e^{x^2} is $2xe^{x^2}$.

Similarly, if we take the derivative of e^{2x} or e^{3x} or e^{ax} for any number $a \neq 0$, we get $2e^{2x}$ and $3e^{3x}$ and ae^{ax}, respectively. Therefore, if we want to take the *anti*derivative of e^{2x} or e^{3x} or e^{ax}, we divide by whichever constant is multiplied by x rather than multiplying it, giving us $\frac{1}{2}e^{2x}$, $\frac{1}{3}e^{2x}$, and $\frac{1}{a}e^{ax}$, respectively.

Notice that since the derivative of a constant is 0, the same result holds if we have e^{2x+1} or e^{3x-4} or e^{ax+b} for any numbers a, b (where again $a \neq 0$ so we don't divide by 0) since the derivatives of $2x+1, 3x-4$, and $ax+b$ are still $2, 3$, and a, respectively. That is,

$$\int e^{ax+b}dx = \frac{e^{ax+b}}{a} + c \text{ for any number } a \neq 0 \text{ and any number } b$$

Exercise 2.1. Find the following antiderivatives (remember, your answer should have a $+c$ in it). To check your work, take the derivative of your answer and see if you get the original function back.

a. $\int (4x^3 - 5x + 6)dx =$

b. $\int 2e^{2w}dw =$

c. $\int (2t^2 + \sin(3t) - 2)dt =$

d. $\int \frac{1}{y}dy =$

(!)*e.* $\int (2xe^x + 2e^x)dx =$

(!)*f.* $\int \frac{1}{x^2+1}dx =$

2.2 Definite Integrals

We will now discuss a seemingly completely different problem – finding the area under a curve. To start, let us discuss an example to demonstrate why this is a problem we may care about.

Let's say we want to know the average temperature throughout the week. We could take the temperature every day, add all of them up, and divide the total by 7 to get the average. However, this doesn't give us the full picture of the weather throughout the week since we only took the temperature once every day. We could then take the temperature several times per day, which will give us a better approximation for the average temperature throughout the week, but it still won't necessarily be entirely accurate. Suppose then we have digital thermometer which continually records the temperature throughout the week and gives us a graph of the data at the end. If we want to find the average temperature, we can no longer just add up all of the individual data points and divide by the total number since the temperature was recorded continuously rather than at individual points. Instead, we would have to find the total area under the graph and divide that by the total amount of time which passed, which would then give us the exact average temperature throughout the week.

While finding the exact area under a curve is a difficult problem, we can approximate the area using what we call ***Riemann Sums***. The basic idea is to approximate the area using a bunch of rectangles – since it is easy to calculate the area of a rectangle, just multiply the width and height – and add up the areas of all of the rectangles to give an approximation of the total area. Then, we can use limits to see what happens as we take better and better approximations, and the limit – which we will call a ***definite integral*** – will actually tell us the exact area under the curve! A lot goes into defining Riemann sums and definite integrals, and as you can see below the definition is technical and involves a lot of complicated notation we haven't discussed yet. However, I think understanding the idea of taking approximations and using limits to take better and better approximations is an important concept which will appear again later throughout the semester, so I will quickly explain the different parts of this definition.

Definition (Formal Definition of Definite Integral)**.**

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\Delta x, \ \textit{where}\ \Delta x = \frac{b-a}{n}\ \textit{and}\ x_i = a + i\Delta x$$

For anyone who has seen Riemann sums before, this definition defines definite integrals in terms of a *left* Riemann sum, and there are actually several more possible definitions we could use which define them in terms of other types of Riemann sums. But since we are not covering this topic in depth, I won't go into more detail about this, I just wanted to add this as a note for anyone who may have seen this before and was curious which Riemann sum I am using here.

Now, time to parse the definition. The $f(x_i)\Delta x$ is just the area of a rectangle (with width Δx and height $f(x_i)$), the Σ symbol means add up the area of a bunch of different rectangles, and the $\lim\limits_{n\to\infty}$ is telling us to let the number of rectangles we are using go to infinity, i.e. we are letting the approximations get better and better. To quickly demonstrate what I mean by this, look at the figure below. On the left, we can see an approximation with 10 rectangles, and you may notice that while the area of these rectangles somewhat resembles the area under the curve we are trying to compute, it isn't very accurate – there are a lot of places where the rectangles either over- or under-approximate the area. On the right, we see an approximation with 50 rectangles and you may notice that this looks like a much better approximation. This is the whole idea of Riemann sums – as you increase the number of rectangles, the error decreases, and letting the number of rectangles go to infinity, the error goes to 0 recovering the *exact* area under the curve.

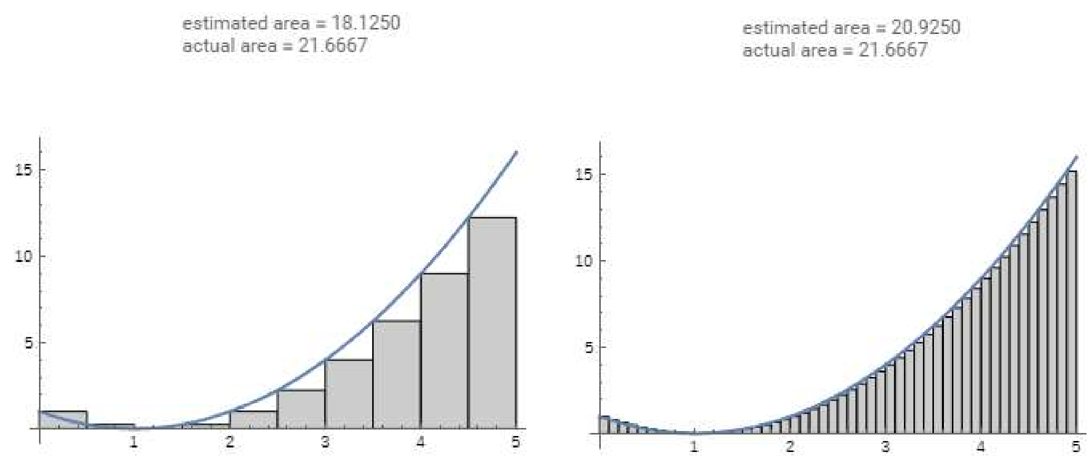

Figure 1: Images created using: Ed Pegg Jr "Riemann Sums"
http://demonstrations.wolfram.com/RiemannSums/

Now that we have talked a little bit about how we define these definite integrals, let's now spend some time actually working with them. The first thing I will start with is some notation. There is a lot of notation in calculus that can sometimes be confusing, so the key thing to remember about definite integrals is they represent the area under a curve. Anytime you see a definite integral, you should always remember it has something to do with the area under a curve.

Definition (Useful Definition of Definite Integral)**.** *The definite integral* $\int_a^b f(x)dx$ *is the total (signed) area under the curve* $f(x)$ *between the values* $x = a$ *and* $x = b$*.*

Let's look at a quick example: $\int_0^2 3xdx$. If we draw the function $f(x) = 3x$, we see this is just a straight line with slope 3. If we shade in the area underneath, we can see it is just a triangle with base 2 and height 6, so the total area is $\frac{1}{2}bh = \frac{1}{2} \cdot 2 \cdot 6 = 6$. That is, $\int_0^2 3xdx = 6$.

As of right now, these kinds of examples where the area we are trying to calculate is some nice shape like a triangle or circle are the only ones we can actually compute. This is because the definition of the definite integral, as we saw earlier, is very complicated and trying to use the definition to directly compute an integral is rather difficult. Right now, we don't yet have any tools to tell us how to actually compute these integrals by hand, and a lot of this semester will be focused on developing tools and techniques to compute integrals and antiderivatives.

Luckily for us, there is an extremely powerful computational tool called the Fundamental Theorem of Calculus, which we will discuss in the next section, which will tell us how we can actually compute these integrals. For now, however, let us quickly discuss a few important properties of definite integrals, which will help us do computations later.

While finding antiderivatives and calculating the area under a curve seem like very different problems, the fact that we use the same $\int$ symbol for both should indicate that there is some connection between the two concepts. Indeed there is, and that connection is so important we call it the *Fundamental Theorem of Calculus*, which we will discuss in the next section. This is an extremely powerful tool which is what will actually allow us to compute these integrals in a reasonable way, and it is the basis that the rest of this semester relies on.

2.3 The Fundamental Theorem of Calculus

As mentioned previously, the FTC is an extremely important tool, and it is essentially the reason all of this theory works and we can actually study integrals.

As the first two sections hopefully demonstrated, taking antiderivatives is a much more concrete and approachable problem than naively trying to compute integrals using the definition. While finding antiderivatives of more complicated functions is a difficult problem in general, and we will need to develop a lot of tools to compute these more advanced antiderivatives, we saw in the first section that we can at least already find antiderivatives of polynomials and certain exponential functions. That is where the power of the FTC comes into play – it is essentially telling us that we can use antiderivatives to compute definite integrals. First, I will state the theorem, which has two parts, and then after I will go into some detail about what the theorem actually says, why it is useful, and how we use it.

Theorem 2.2 (Fundamental Theorem of Calculus). *Let F be an antiderivative of f. Then,*

i) $\frac{d}{dx}\big(\int_0^x f(t)dt\big) = f(x)$

ii) $\int_a^b f(x)dx = F(b) - F(a)$

Let us spend some time unpacking what this theorem is telling us. First, let us try and understand what the first part is even saying. Recall that for any numbers a, b the definite integral $\int_a^b f(x)dx$ is just a number – the area under the graph of $f(x)$ between $x = a$ and $x = b$. Instead of using x as our input for the function f, let's use the letter t instead. This is simply because I want to use x somewhere else, so I need a different variable to use in the function. Notice this doesn't actually change anything, $f(t)$ is still the exact same function as $f(x)$ we are just renaming the input.

Now, if we fix $a = 0$ and let b vary, we get a function, which we can call $A(x) = \int_0^x f(t)dt$. That is, $A(x)$ is also just a number – the area under f between 0 and x (this is why I changed to using $f(t)$, so I can use x as the input to the function A). What the first part of the FTC is telling us is that if we take the derivative of $A(x)$ with respect to x, we get back $f(x)$. That is, this area function $A(x)$ is actually an antiderivative of $f(x)$!

On the other hand, the second part tells us that we can actually use antiderivatives to compute definite integrals. First find an antiderivative $F(x)$ of $f(x)$, and then you can calculate the definite integral $\int_a^b f(x)dx$ by simply plugging in a and b into $F(x)$ and taking their difference $F(b) - F(a)$. In practice, this is how we *always* solve definite integrals since, as we saw earlier, working with Riemann sums and the definition of the integral is rather difficult.

For example, say we want to compute $\displaystyle\int_1^3 (3x^2 + 1)dx$. First, we need to find an antiderivative of $x^2 + 1$, which as we saw by the power rule earlier, must be of the form $x^3 + x + c$ for any constant c. You can always choose any value of c, so most of the time we want to just let $c = 0$ for simplicity. Next, we need to plug the bounds of the integral into this antiderivative: $F(3) = 3^3 + 3 = 30$ and $F(1) = 1^3 + 1 = 2$. Finally, subtract these two numbers, remembering to always take the number which was on the top of the integral first and then subtracting off the lower bound. That is, we take $F(3) - F(1) = 30 - 2 = 28$. The FTC then tells us that this is the definite integral, i.e. that $\displaystyle\int_1^3 (3x^2 + 1)dx = 28$.

To quickly convince you that it really doesn't matter which value of c we pick, let us do the same calculation but using the general antiderivative $x^3 + x + c$. Plugging 3 into this we get $3^3 + 3 + c = 30 + c$ and plugging in 1 gives $1^3 + 1 + c = 2 + c$. Therefore, if we subtract them, the c's will cancel and we are left with $F(3) - F(1) = (30 + c) - (2 + c) = 30 + c - 2 - c = 28$, exactly the same as we got before! Thus, the value of c you pick doesn't affect the final result.

That is, the FTC is essentially telling us that antiderivatives and integrals are the same! This is why we use the $\int f(x)dx$ notation for an antiderivative of f, and we usually just call antiderivatives

indefinite integrals (indefinite meaning there are no bounds on the integral).

An important note on using the FTC: you should always plug in the top bound first, and then subtract off the bottom bound. For example, if instead we wanted to calculate $\int_3^1 (3x^2+1)dx$ with the bounds flipped, we would first take $F(1) = 2$ and then subtract off $F(3) = 30$ to get $F(1) - F(3) = 2 - 30 = -28$. This is precisely why flipping the bounds of an integral gives the same number, but with a sign change.

Exercise 2.3. Use the Fundamental Theorem of Calculus to calculate the following definite integrals (hint: use Exercise 2.1).

a. $\int_0^1 (4x^3 - 5x + 6)dx =$

b. $\int_2^1 2e^{2w} dw =$

c. $\int_{-1}^1 (2t^2 + \sin(3t) - 2)dt =$

d. $\int_{-2}^{-1} \frac{1}{y} dy =$

(!)*e.* $\int_{-3}^2 (2xe^x + 2e^x)dx =$

(!)*f.* $\int_0^1 \frac{1}{x^2+1} dx =$