4 Substitution

Let's start with an example to help motivate what we are about to do.

Let's start with a simple example such as $\int 2x dx = x^2 + c$. Our choice to use x as a variable is completely arbitrary, we could just as well use any other variable. For example, we could write this function as $2u$ and then if we want to take the integral, we also have to change our dx to du to match our new variable, giving $\int 2u du = u^2 + c$. Substituting x back in for u, this gives us $x^2 + c$ which is exactly what we got before.

Now, let's introduce a new variable u again, but this time let $u = 2x$ (why we may want to do this may not be clear right now, but for now let's just see what happens, and we will look why this is a useful thing to do later). Now, if we naively change dx to a du again, we get the integral $\int u du = \frac{u^2}{2} + c$. Now, if we plug $2x$ back in for u to get $\frac{(2x)^2}{2} = \frac{4x^2}{2} = 2x^2$ which is different from what we got before. The reason for this discrepancy is because although we are allowed to make substitutions like this, we cannot simply change the dx to a du and expect the answer not to change. That is, the symbols dx, du, etc. are more than simply notation – they actually carry important information, and simply changing the dx to a du loses this information and changes the integral we get as a result.

Luckily, there is a way we can fix this! We just need to figure out what the new du should be in terms of the original dx. The way we do this is using derivatives – recall that given a function $f(x)$ we can use the notation $\frac{df}{dx}$ to represent the derivative of f with respect to x, i.e. $\frac{df}{dx}$ is another way of writing $f'(x)$.

Now, if we go back to our example, we can think of u as being a function of x – namely the function $2x$. If we take the derivative of u with respect to x, we get 2, i.e. $\frac{du}{dx} = 2$. We can now treat this like a fraction and multiply both sides by dx to get $du = 2dx$. Why we are allowed to treat this derivative like a fraction is technical and not important, but just know that there is a way to interpret these symbols du and dx in such a way that dividing them gives us the derivative of u with respect to x.

Now, if we want to substitute $u = 2x$, we need to also use the substitution $du = 2dx$. We can divide both sides by 2 to get $dx = \frac{1}{2}du$, so doing these two substitutions gives us

$$\begin{aligned}\int 2x dx &= \int u dx \\ &= \int u \frac{1}{2} du \\ &= \frac{1}{2}\int u du\end{aligned}$$

Computing this integral gives us $\frac{1}{2}\int u du = \frac{1}{2}\cdot\frac{u^2}{2} + c = \frac{u^2}{4} + c$. Plugging $2x$ back in for u, this gives us $\frac{(2x)^2}{4} + c = \frac{4x^2}{4} + c = x^2 + c$ which is the same as we got originally!

Now, let's compute a somewhat less contrived example that shows why this technique of substitution can be so powerful. Suppose we want to compute $\int \sin(x)\cos(x)dx$. This is a function where it may be hard to just think of what the antiderivative would be, so let's use substitution to figure it out! We can start by letting $u = \sin(x)$, which then gives us $\frac{du}{dx} = \cos(x)$ and multiplying both sides by dx gives us $du = \cos(x)dx$. Doing these substitutions yields

$$\begin{aligned}\int \sin(x)\cos(x)dx &= \int u\cos(x)dx \\ &= \int u du \\ &= \frac{u^2}{2} + c \\ &= \frac{\sin^2(x)}{2} + c\end{aligned}$$

which we can verify is the correct antiderivative by taking its derivative. Indeed, by the chain rule, $\frac{d}{dx}\left(\frac{1}{2}\sin^2(x)\right) = \frac{1}{2}\cdot 2\sin(x)\cdot\cos(x) = \sin(x)\cos(x)$.

What you may notice is that in this process of doing substitution, we have to take the derivative $\frac{du}{dx} = u'(x)$ and then multiply both sides to get $du = u'(x)dx$. That is, when doing substitution we want to write part of the integrand as u in such a way that its derivative u' is multiplied by dx so we can do the appropriate substitution for du. This is why we chose $u = \sin(x)$ in the previous example, since the derivative of $\sin(x)$ is $\cos(x)$ and there is a $\cos(x)dx$ in the integrand. Notice, however, that since the order of multiplication doesn't matter, we can also write this as $\int \cos(x)\sin(x)dx$ and the derivative of $\cos(x)$ is $-\sin(x)$, so we could have also chosen to let $u = \cos(x)$. Indeed, this would have given us the same answer, we just need to be careful not to lose track of the negative sign we pick up when taking the derivative of $\cos(x)$.

Exercise 4.1. Compute $\int \sin(x)\cos(x)dx$, but this time by letting $u = \cos(x)$. Double check you get the same answer as before.

Now that we have seen how substitution works for *in*definite integrals, let us check what happens if we add bounds and make our integrals definite. Again, let's start with a simple example. Suppose we want to calculate $\int_1^2 2xdx$. We can do this by simply taking the antiderivative of $2x$ and evaluating at the bounds. That is,

$$\begin{aligned}\int_1^2 2xdx &= x^2\big|_1^2 \\ &= 2^2 - 1^2 \\ &= 3\end{aligned}$$

Now, let us use the substitution $u = 2x$ and $du = 2dx$ or $\frac{1}{2}du = dx$, which gives us $\int 2xdx = \frac{1}{2}\int udu$, as before. Let's start by leaving the bounds unchanged and seeing what happened. That is, we want to compute $\frac{1}{2}\int_1^2 udu$ and see if it gives us 3 or not. As we can see,

$$\begin{aligned}\frac{1}{2}\int_1^2 udu &= \frac{1}{2}\frac{u^2}{2}\Big|_1^2 \\ &= \frac{u^2}{4}\Big|_1^2 \\ &= \frac{2^2}{4} - \frac{1^2}{4} \\ &= 1 - \frac{1}{4} \qquad = \frac{3}{4}\end{aligned}$$

computing this integral gives us a different answer. Why? Recall that definite integrals represent areas underneath curves, and our original integral was in terms of the variable x, i.e. we were using dx, so our bounds are x values.

Thus, one option is to simply plug $2x$ back in for u after finding the antiderivative and then plugging in the original bounds. That is, once we find the antiderivative $\frac{u^2}{4}$, plugging in $u = 2x$ gives us $\frac{(2x)^2}{4} = \frac{4x^2}{4} = x^2$, which we can then evaluate at the bounds to get $2^2 - 1^2 = 3$, as before.

On the other hand, sometimes the function u will be very complicated and we may not want to substitute the x's back in. However, as we just saw, if we leave the bounds unchanged we will get a different answer. Just as we had to change the dx to a du, if we want to evaluate the final answer using u's instead of x's, we also have to change the bounds to reflect that. That is, if we think of u as the function $u(x) = 2x$, then when $x = 1$, we have $u(1) = 2(1) = 2$ and when $x = 2$ we get $u(2) = 2(2) = 4$. It may then seem reasonable that these are the new bounds we should use with our u integral, and indeed that is the case! We can simply compute

$$\begin{aligned}\frac{1}{2}\int_2^4 udu = \frac{1}{2}\frac{u^2}{2}\Big|_2^4 \\ &= \frac{u^2}{4}\Big|_2^4 \\ &= \frac{4^2}{4} - \frac{2^2}{4} \\ &= 4-1 \\ &= 3\end{aligned}$$

which is the same as when we computed $\int_1^2 2xdx$!

Next, let's compute $\displaystyle\int_0^{\frac{\pi}{2}} \sin(x)\cos(x)dx$ by using $u = \sin(x)$, which gives us $du = \cos(x)dx$. We can also compute the new bounds $u(0) = \sin(0) = 0$ and $u(\frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$ to use later. Then, just as before this gives us an antiderivative of $\frac{u^2}{2}$. To compute the definite integral, we can either evaluate $\frac{u^2}{2}$ at the new bounds $0, 1$ or substitute $\sin(x)$ back in for u and evaluate the result at the original bounds of $0, \frac{\pi}{2}$. Using the first method gives us $\frac{u^2}{2}\Big|_0^1 = \frac{1^2}{2} = \frac{0^2}{2} = \frac{1}{2}$, and the second method gives $\frac{\sin^2(x)}{2}\Big|_0^{\frac{\pi}{2}} = \frac{\sin(\frac{\pi}{2})^2}{2} - \frac{\sin(0)^2}{2} = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$, so both methods give the same answer.