Turán Numbers of Hypergraph Suspensions of Even Cycles

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Abstract

For fixed $k \geq 2$, determining the order of magnitude of the number of edges in an n-vertex bipartite graph not containing C_{2k}, the cycle of length $2k$, is a long-standing open problem. We consider an extension of this problem to triple systems. In particular, we prove that the number of triples in an n-vertex triple system which does not contain a C_6 in the link of any vertex, has order of magnitude $n^{7/3}$. Additionally, we construct new families of dense C_6-free bipartite graphs with n vertices and $n^{4/3}$ edges in order of magnitude.

1 Introduction

An r-uniform hypergraph, or simply, an r-graph H on vertex set $V(H)$ is a subset of $\binom{V(H)}{r}$. Given an r-graph H, the hypergraph Turán problem asks the following question: what is the largest size of an r-graph on n vertices that does not contain a copy of H as a subgraph? This number is known as the Turán number or the extremal number of H, and is denoted by $\mathrm{ex}_r(n, H)$. The case $r = 2$ was first introduced by Turán [19] in 1941, and several lower and upper bounds on $\mathrm{ex}_r(n, H)$ have been obtained since then for different values of r and H.

Towards analyzing the asymptotic behavior of $\mathrm{ex}_2(n, G)$ for graphs G, the seminal result of Erdős and Stone [5] states that when the chromatic number $\chi(G) \geq 3$,

$$\mathrm{ex}_2(n, G) = \left(1 - \frac{1}{\chi(G) - 1}\right)\binom{n}{2} + o(n^2).$$

This result essentially determines $\mathrm{ex}_2(n, G)$ for graphs G which are not bipartite. The analysis of $\mathrm{ex}_2(n, G)$ for bipartite graphs G turns out to be extremely difficult, and the reader is referred to [6] for a comprehensive survey of the bipartite case.

One especially well-studied class of bipartite graphs G are the even cycles C_{2k} for $k \geq 2$. For these graphs, the best known upper bound is provided by Bondy and Simonovits [2], who proved that $\mathrm{ex}_2(n, C_{2k}) \leq O(n^{1+\frac{1}{k}})$. Improvements in the constant term has been obtained in [8, 17, 4].

A major open problem for even cycles is to construct C_{2k}-free graphs on n vertices and $\Omega(n^{1+\frac{1}{k}})$ edges. There have been several bipartite constructions based on finite geometries including [18, 3, 1, 12, 21, 11] that have sequentially improved the bounds; however, they give the tight bound only for $k \in \{2, 3, 5\}$. For $k \notin \{2, 3, 5\}$, the best known lower bounds are given by the bipartite graphs $CD(k, q)$ [9, 10] for integers $k \geq 2$ and prime powers q. These graphs arise from Lie algebraic incidence structures that approximate the

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behavior of generalized polygons, and are analyzed in detail in [22]. For a recent survey on the even cycle problem, the reader is referred to [20].

In this paper, we are mainly concerned with three classes of lower bound constructions: the bipartite graphs $D(k, q)$ from [9, 10], the arc construction introduced in [13] and later generalized in [14], and Wenger's construction [21]. Our results can be divided into two sections: results about 3-graphs and results about graphs.

1.1 3-Graphs

For a graph G, the suspension $\widehat{G}$ is the graph obtained from G by adding a new vertex adjacent to all vertices of G. In [16], the author, together with Mubayi, studied the generalized Turán number $\mathrm{ex}(n, K_3, \widehat{G})$ for different bipartite graphs G. Analogously, we introduce the concept of a *hypergraph suspension*.

Let H be a 3-graph and $x \in V(H)$ be any vertex of H. The link of x in H, denoted by $L_{x,H}$, is the graph with vertex set $V(H) \setminus \{x\}$ and edges $\{uv : \{x, u, v\} \in H\}$. For a graph G, the hypergraph suspension $\widetilde{G}$ is a 3-graph defined as follows: add a new vertex x to $V(G)$, and let $\widetilde{G} = \{e \cup \{x\} : e \in E(H)\}$. By definition, $L_{x,\widetilde{G}} = G$.

Note that the numbers $\mathrm{ex}_3(n, \widetilde{G})$ and $\mathrm{ex}(n, K_3, \widehat{G})$ are closely related. In fact, given a $\widehat{G}$-free graph, we can replace all triangles in it with hyperedges to obtain a $\widetilde{G}$-free 3-graph, implying

$$\mathrm{ex}(n, K_3, \widehat{G}) \leq \mathrm{ex}_3(n, \widetilde{G}). \tag{1.1}$$

In this paper, we study $\mathrm{ex}_3(n, \widetilde{C}_{2k})$ for $k \geq 2$. When $k = 2$, observe that $\widetilde{C}_{2k}$ is the complete 3-partite 3-graph $K^{(3)}_{1,2,2}$, and its extremal number has been exactly determined to be $\Theta(n^{5/2})$ in [15]. Thus, we focus our attention on $\mathrm{ex}_3(n, \widetilde{C}_{2k})$ for $k \geq 3$.

Observe that a 3-graph H does not contain $\widetilde{C}_{2k}$ iff $L_{x,H}$ does not contain C_{2k} for every vertex $x \in V(H)$, leading us to the upper bound

$$\mathrm{ex}_3(n, \widetilde{C}_{2k}) \leq O(n \cdot n^{1+\frac{1}{k}}) = O(n^{2+\frac{1}{k}}) \tag{1.2}$$

On the other hand, a probabilistic deletion argument lets us deduce the following result:

Proposition 1.1. *For $k \geq 2$,*

$$ex_3(n, \widetilde{C}_{2k}) \geq \Omega\left(n^{2+\frac{1}{2k-1}}\right). \tag{1.3}$$

Our main result is to show a construction of $\widetilde{C}_{2k}-$free 3-graphs, which asymptotically improves the bound above for $k = 3$ and $k = 4$.

Theorem 1.2. *For every integer q that is a power of 3, there exists a 3-partite 3-graph $D_3(k, q)$ with the following properties:*

1. *$D_3(k, q)$ has $3q^k$ vertices and q^{2k+1} edges,*
2. *The link graph of every vertex of $D_3(k, q)$ is isomorphic for $k \leq 6$, and*
3. *$D_3(3, q)$ and $D_3(5, q)$ are $\widetilde{C}_6$ and $\widetilde{C}_8$-free, respectively.*

In particular, Theorem 1.2 implies that

$$\mathrm{ex}_3(n, \widetilde{C}_6) \geq \Omega(n^{7/3}) \text{ and } \mathrm{ex}_3(n, \widetilde{C}_8) \geq \Omega(n^{11/5}). \tag{1.4}$$

As a corollary of (1.2) and (1.4), we determine the exact growth rate of $\mathrm{ex}_3(n, \widetilde{C}_6)$.

Corollary 1.3. For large n, the Turán number of $\widetilde{C}_6$ grows as,

$$\mathrm{ex}_3(n, \widetilde{C}_6) = \Theta(n^{7/3}). \tag{1.5}$$

Corollary 1.3 further implies that the bound in (1.1) is not always sharp, since we demonstrated in [16] that $\mathrm{ex}(n, K_3, \widehat{C}_6) = o(n^{7/3})$.

Remark. Our proof of Theorem 1.2 heavily relies on the bipartite graphs $D(k, q)$ introduced by Lazebnik, Ustimenko and Woldar in [9], and $D_3(k, q)$ can be viewed as an extension of $D(k, q)$ to 3-graphs. $D_3(k, q)$ has the property that for every $k \geq 2$ and prime power $3 \mid q$, the link graph of any of its vertex is isomorphic to either $D(k, q)$ or another graph which we call $D'(k, q)$ (Proposition 2.5). We also make a conjecture (Conjecture 2.9) about the girth of $D'(k, q)$, which, if true, would give a bound of $\mathrm{ex}_3(n, \widetilde{C}_{2k}) \geq \Omega(n^{2+\frac{1}{2k-3}})$ for all $k \geq 3$, an asymptotic improvement on (1.3).

1.2 Graphs

We also compare two well-known constructions of C_{2k}-free graphs: the arc construction [13, 14] and Wenger's construction [21]. Let $t \geq 2$, and let q be a prime power. An arc in a projective t-space $PG(t, q)$ is a collection of points such that no $(t-1)$ of them lie in a hyperplane. The arc construction is defined as follows.

The bipartite graphs $G_{\mathtt{arc}}(k, q, \alpha)$. Let $\Sigma = PG(t, q)$, and $\Sigma_0 \subset \Sigma$ be the hyperplane consisting of points with first homogeneous coordinate 0. Note that $\Sigma_0 \cong PG(t-1, q)$. Let α be any arc in Σ_0. Then, the bipartite graph $G_{\mathtt{arc}}(k, q, \alpha)$ with parts P and L is defined as follows. Let $P = \Sigma \setminus \Sigma_0$, and L be the set of all projective lines ℓ of Σ such that $\ell \cap \Sigma_0 \in \alpha$. Vertices $p \in P$ and $\ell \in L$ are adjacent if and only if $p \in \ell$.

It was shown in [14] that $G_{\mathtt{arc}}(k, q, \alpha_0)$ is C_{2k}-free for $k = 2, 3, 5$ but contains C_8 for $k = 4$, where α_0 is the normal rational curve in Σ_0 given by

$$\alpha_0 = \{[0 : 1 : x : x^2 : \cdots : x^{t-1}] : x \in \mathbb{F}_q\} \cup \{[0 : 0 : \cdots : 0 : 1]\}.$$

In contrast, let $H(k, q)$ be the bipartite graph with parts $A = B = \mathbb{F}_q^k$ such that $(a_1, \ldots, a_k)$ is adjacent to $(b_1, \ldots, b_k)$ iff

$$a_i + b_i = a_1 b_1^{i-1} \text{ for all } 2 \leq i \leq k.$$

It was shown by Wenger in [21] that $H(k, q)$ is C_{2k}-free for $k = 2, 3, 5$.

We prove that these two constructions are in fact, isomorphic, and our proof uses the Plücker embedding [7], a tool from algebraic geometry that lets us parametrize the set of projective lines L.

Proposition 1.4. *Let α_0 be the normal rational curve in $PG(2, k)$, and $\alpha_0^- = \alpha_0 \setminus \{[0 : \cdots : 0 : 1]\}$. Then,*

$$G_{\mathtt{arc}}(k, q, \alpha_0^-) \cong H(k, q).$$

As $G_{\mathtt{arc}}(4, q, \alpha_0)$ is shown to contain C_8's in [14], Proposition 1.4 also provides a geometric explanation for why Wenger's bound is tight for $k = 3$ and $k = 5$ but not $k = 4$.

For $1 \leq s \leq r$ with $(s, r) = 1$, it is known that $\alpha = \{[1 : x : x^{2^s}] : x \in \mathbb{F}_{2^r}\}$ is an arc in the projective space $PG(2, 2^r)$. Using the proof method of Proposition 1.4 on this arc α, we are able to construct a new family of C_6-free graphs with $\Omega(n^{4/3})$ edges, given as follows.

Theorem 1.5. *Let $q = 2^r$ and $1 \leq s \leq r$ be such that $(s, r) = 1$. Let $G(2^r, s)$ denote the bipartite graph with parts $A = B = \mathbb{F}_q^3$ such that $(a_1, a_2, a_3) \in A$ is adjacent to $(b_1, b_2, b_3) \in B$ iff*

$$b_2 + a_2 = a_1 b_1 \text{ and } b_3 + a_3 = a_1 b_1^{2^s}.$$

Then, $G(2^r, s)$ is C_6-free.

Note that the graphs $G(2^r, s)$ extend Wenger's C_6-free construction in even characteristic, as $G(2^r, 1) \cong H(3, 2^r)$.

This paper is organized as follows. In Section 2, we prove Proposition 1.1, recapitulate on the graphs $D(k, q)$, extend them to the 3-graphs $D_3(k, q)$, and investigate its link graphs, finally proving Theorem 1.2. Section 3 is devoted to proving Proposition 1.4 and Theorem 1.5.

2 Lower bounds on ex$(n, \widetilde{C}_{2k})$

Our goal in this section is to extend the graphs $D(k, q)$ to a family of 3-graphs, and build up the tools required to prove Theorem 1.2. We start with a proof of Proposition 1.1. Recall that we wish to show $\mathrm{ex}_3(n, \widetilde{C}_{2k}) \geq \Omega(n^{2+\frac{1}{2k-1}})$.

Proof of Proposition 1.1. Let $H \sim G_3(n, p)$ be the Erdős-Rényi 3-graph, where each edge of the complete 3-graph on n vertices is selected with probability $p = \frac{1}{10}k^{-100}n^{-\frac{2k-2}{2k-1}}$. Then, $\mathbb{E}(|H|) = p\binom{n}{3}$. For every $\widetilde{C}_{2k}$ in H, we remove one edge from it. Let $H' \subset H$ be the new 3-graph obtained via the deletion of edges. Note that the probability that any $2k+1$ vertices forms a $\widetilde{C}_{2k}$ is $(2k+1)p^{2k}$, and therefore, the expected number of them is at most $(2k+1)n^{2k+1}p^{2k}$. Now, $E(H') = p\binom{n}{3} - (2k+1)n^{2k+1}p^{2k}$. As

$$n^{2k+1}p^{2k-1} = n^{2k+1} \cdot 10^{-(2k-1)}k^{-100(2k-1)}n^{-(2k-2)} \leq 10^{-(2k-1)}n^3k^{-100(2k-1)},$$

and

$$p\left(\binom{n}{3} - (2k+1)n^{2k+1}p^{2k}\right) \geq pn^3\left(\frac{1}{10} - \frac{2k+1}{10^{2k-1}}\right) \geq \frac{pn^3}{100},$$

This implies that $\mathbb{E}(|H'|) \geq \frac{1}{1000}k^{-100}n^{3-\frac{2k-2}{2k-1}}$. Thus, there exists a 3-graph H' with $\Omega(n^{3-\frac{2k-2}{2k-1}})$ edges with no copy of $\widetilde{C}_{2k}$. This completes our proof.

□

Since probabilistic lower bounds for 3-graphs tend to be weak, we try to strengthen this result via a look at the graphs $D(k, q)$. Here we present a summary of the properties of $D(k, q)$; for more details, the reader is referred to [9, 10, 22].

Definition 2.1 (The bipartite graphs $D(q)$)**.** For a prime power q, let A and B be two copies of the countably infinite dimensional vector space V over $\mathbb{F}_q$. Use the following coordinate representations for elements $a \in A$ and $b \in B$:

$$\begin{aligned} a &= (a_1, a_{11}, a_{12}, a_{21}, a_{22}, a'_{22}, a_{23}, \ldots, a_{ii}, a'_{ii}, a_{i,i+1}, a_{i+1,i}, \ldots), \\ b &= (b_1, b_{11}, b_{12}, b_{21}, b_{22}, b'_{22}, b_{23}, \ldots, b_{ii}, b'_{ii}, b_{i,i+1}, b_{i+1,i}, \ldots). \end{aligned} \tag{2.1}$$

Let $A \sqcup B$ be the vertex set of $D(q)$, and join $a \in A$ to $b \in B$ if the following coordinate relations hold ($i \geq 2$):

$$\begin{aligned} a_{11} + b_{11} + a_1b_1 &= 0 \\ a_{12} + b_{12} + a_1b_{11} &= 0 \\ a_{21} + b_{21} + a_{11}b_1 &= 0 \\ &\vdots \\ a_{ii} + b_{ii} + a_{i-1,i}b_1 &= 0 \\ a'_{ii} + b'_{ii} + a_1b_{i,i-1} &= 0 \\ a_{i,i+1} + b_{i,i+1} + a_1b_{ii} &= 0 \\ a_{i+1,i} + b_{i+1,i} + a'_{ii}b_1 &= 0. \end{aligned} \tag{2.2}$$

Define $D(k,q)$ to be the graph obtained by truncation of A and B to the first k coordinates in (2.1) and, the first $k-1$ relations in (2.2).

The key properties of the graphs $D(k,q)$ are summarized in the following proposition.

Proposition 2.2. *For any prime power q and $k \geq 2$, the following holds:*

1. *$D(k,q)$ is a q-regular bipartite graph of order $2q^k$;*
2. *The girth of $D(k,q)$ is at least $k+4$ if k is even, and $k+5$ if k is odd.*

Further, it is known that for $k \geq 6$ the graphs $D(k,q)$ start to get disconnected into pairwise isomorphic components at regular intervals. These connected components are called $CD(k,q)$. The graphs $CD(2k-3,q)$ give the currently best known asymptotic lower bounds on $\mathrm{ex}(n, C_{2k})$ for $k \geq 3$. We omit the proof of Proposition 2.2 here.

In the following subsection, we extend $D(k,q)$ to the 3-graph case.

2.1 The 3-graphs $D_3(k,q)$

Definition 2.3 (The 3-partite 3-graphs $D_3(q)$)**.** For a prime power q, let A, B, and C be three copies of the countably infinite dimensional vector space V over $\mathbb{F}_q$. We use the following coordinate representations for $a \in A$, $b \in B$, $c \in C$:

$$\begin{aligned}
a &= (a_1, a_{11}, a_{12}, a_{21}, a_{22}, a'_{22}, a_{23}, \ldots, a_{ii}, a'_{ii}, a_{i,i+1}, a_{i+1,i}, \ldots),\\
b &= (b_1, b_{11}, b_{12}, b_{21}, b_{22}, b'_{22}, b_{23}, \ldots, b_{ii}, b'_{ii}, b_{i,i+1}, b_{i+1,i}, \ldots),\\
c &= (c_1, c_{11}, c_{12}, c_{21}, c_{22}, c'_{22}, c_{23}, \ldots, c_{ii}, c'_{ii}, c_{i,i+1}, c_{i+1,i}, \ldots).
\end{aligned}$$

Let $A \sqcup B \sqcup C$ be the vertex set of $D(q)$, and say that $\{a,b,c\}$ is a hyperedge if the following coordinate relations (call them I) hold ($i \geq 2$):

$$\begin{aligned}
a_{11} + b_{11} + c_{11} + a_1 b_1 + b_1 c_1 + c_1 a_1 &= 0\\
a_{12} + b_{12} + c_{12} + a_1 b_{11} + b_1 c_{11} + c_1 a_{11} &= 0\\
a_{21} + b_{21} + c_{21} + a_{11} b_1 + b_{11} c_1 + c_{11} a_1 &= 0\\
&\vdots\\
a_{ii} + b_{ii} + c_{ii} + a_{i-1,i} b_1 + b_{i-1,i} c_1 + c_{i-1,i} a_1 &= 0\\
a'_{ii} + b'_{ii} + c'_{ii} + a_1 b_{i,i-1} + b_1 c_{i,i-1} + c_1 a_{i,i-1} &= 0\\
a_{i,i+1} + b_{i,i+1} + c_{i,i+1} + a_1 b_{ii} + b_1 c_{ii} + c_1 a_{ii} &= 0\\
a_{i+1,i} + b_{i+1,i} + c_{i+1,i} + a'_{ii} b_1 + b'_{ii} c_1 + c'_{ii} a_1 &= 0.
\end{aligned} \tag{2.3}$$

Define $D_3(k,q)$ to be the 3-graph obtained by truncation of A, B, and C to the first k coordinates and I to the first $k-1$ relations.

The graphs $D_3(q)$ are designed in such a way that the link of the vertex $\vec{0}$ from any part is isomorphic to $D(q)$. In fact, note that $D_3(q)$ has the natural cyclic automorphism $a_* \mapsto b_*$, $b_* \mapsto c_*$, and $c_* \mapsto a_*$, under which all the defining equations of $D_3(q)$ remain invariant. Hence, for any vertex $v \in V$, the links of v in $D_3(q)$ taken from either of the three parts are all isomorphic, and it is enough to consider the link graphs from a fixed part, say, A.

One would hope that the link graphs of other vertices in $D_3(k,q)$ also have similar high girth properties as $D(k,q)$. This inspires us to analyze the links of every vertex in $D_3(k,q)$. To that end, we analyze $\mathrm{Aut}(D_3(q))$.

Proposition 2.4. *Suppose $\mathbb{F}_q$ has characteristic 3, and consider $D_3(q)$ with parts A, B, C. Let $a \in A$ be fixed, and suppose $s \geq 1$. Then there is an automorphism $\varphi \in Aut(D_3(q))$ such that $\varphi(a) = (a_1, 0, \dots, 0, *, *, \dots) \in A$, where the second through the $(s+1)$'th coordinates are mapped to 0.*

The proof of Proposition 2.4 is technical. Before looking at the proof, we note an important consequence: it is sufficient to analyze the girths of the link graphs of the vertices $(a_1, 0, \dots, 0) \in A$ for $a_1 \in \mathbb{F}_q$. In fact, it is seen that the truncated 3-graphs $D_3(k, q)$ have exactly two kinds of links.

Proposition 2.5. *If $3 \mid q$, then the 3-graph $D_3(k, q)$ admits exactly two classes of link graphs, one of which is $D(k, q)$.*

Now, we present the proofs of Propositions 2.4 and 2.5.

2.1.1 Proof of Proposition 2.4

Recall that $3 \mid q$, and we wish to construct an automorphism φ of $D(q)$ sending any vertex $a \in A$ to $(a_1, 0, \dots, 0, *, *, \dots) \in A$, where the coordinates have s zeros followed by a_1.

We construct φ via a product of automorphisms of $D_3(q)$. First, we may rewrite the system of equations (2.3) into the following form:

$$\left.\begin{aligned} a_{ii} + b_{ii} + c_{ii} + a_{i-1,i}b_1 + b_{i-1,i}c_1 + c_{i-1,i}a_1 &= 0 \\ a'_{ii} + b'_{ii} + c'_{ii} + a_1 b_{i,i-1} + b_1 c_{i,i-1} + c_1 a_{i,i-1} &= 0 \\ a_{i,i+1} + b_{i,i+1} + c_{i,i+1} + a_1 b_{ii} + b_1 c_{ii} + c_1 a_{ii} &= 0 \\ a_{i+1,i} + b_{i+1,i} + c_{i+1,i} + a'_{ii} b_1 + b'_{ii} c_1 + c'_{ii} a_1 &= 0 \end{aligned}\right\} , i \geq 1, \tag{2.4}$$

where we set the convention $a_{01} = a_1, b_{01} = b_1, c_{01} = c_1$; $a'_{11} = a_{11}, b'_{11} = b_{11}, c'_{11} = c_{11}$; and $a_{10} = a_1, b_{10} = b_1, c_{10} = c_1$, with the implication that the first and second equations coincide for $i = 1$. Further, for the sake of ease in defining the automorphisms, we give meaningful interpretations for the equations in (2.4) when $i = 0$. We set $a'_{00} = b'_{00} = c'_{00} = a_{00} = b_{00} = c_{00} = -1$; and $a_{0,-1} = b_{0,-1} = c_{0,-1} = a_{-1,0} = b_{-1,0} = c_{-1,0} = 0$. Notice that the first and the second equations reduce to $-3 = 0$ for $i = 0$, which is true in characteristic 3.

Now, we define five different automorphisms of $D(q)$ in Table 1 below, by noting where each coordinate is sent to. For example, for fixed $x \in \mathbb{F}_q$, we denote $t_{1,1}(x)$ to be the automorphism that sends $a_1 \mapsto a_1 + a_{-1,0}x = a_1$, $a_{11} \mapsto a_{11} + a_{00}x = a_{11} - x$, and so on. A "-" as a table entry denotes a coordinate fixed by that map, e.g $t_{m+1,m}(a_{ii}) = a_{ii}$.

Claim 2.6. The maps defined in Table 1 are Automorphisms of $D(q)$.

Proof of Claim 2.6. Observe that each of the maps defined have inverses given by x replaced with $-x$, respectively, once we check that they are homomorphisms.

- $t_{1,1}(x)$: We observe that the map $t_{1,1}(x)$ keeps a_1, b_1, c_1 fixed as $a_1 = a_{0,1} \mapsto a_{0,1} + a_{-1,0}x = a_{0,1}$, etc. And, for $i \geq 1$, we need to check that the equations (2.4) are preserved after the transformation given by $t_{1,1}$. Suppose the equations (2.4) hold, then note that we also have for $i \geq 1$,

$$a_{ii} + b_{ii} + c_{ii} + a_{i-1,i}b_1 + b_{i-1,i}c_1 + c_{i-1,i}a_1 = 0,$$
$$(a_{i-1,i-1} + b_{i-1,i-1} + c_{i-1,i-1} + a_{i-2,i-1}b_1 + b_{i-2,i-1}c_1 + c_{i-2,i-1}a_1)x = 0,$$

  and adding these up verifies that the first equation is preserved under the image of $t_{1,1}(x)$. Similarly, the other three equations can be verified for each $i \geq 1$.

Coordinates ($i \geq 0$)	$t_{1,1}(x)$	$t_{m,m+1}(x), m \geq 1$ $r = i - m$	$t_{m+1,m}(x), m \geq 1$ $r = i - m$	$t_{m,m}(x), m \geq 2$ $r = i - m$	$t'_{m,m}(x), m \geq 2$ $r = i - m$
a_{ii}	$+a_{i-1,i-1}x$	$+a_{r,r-1}x$, $r \geq 1$	-	$+a_{rr}x$, $r \geq 0$	-
$a_{i,i+1}$	$+a_{i-1,i}x$	$+a'_{rr}x$, $r \geq 0$	-	$+a_{r,r+1}x$, $r \geq 0$	-
$a_{i+1,i}$	$+a_{i,i-1}x$	-	$+a_{rr}x$, $r \geq 0$	-	$+a_{r+1,r}x$, $r \geq 0$
a'_{ii}	$+a'_{i-1,i-1}x$	-	$+a_{r-1,r}x$, $r \geq 1$	-	$+a'_{rr}x$, $r \geq 0$
b_{ii}	$+b_{i-1,i-1}x$	$+b_{r,r-1}x$, $r \geq 1$	-	$+b_{rr}x$, $r \geq 0$	-
$b_{i,i+1}$	$+b_{i,i-1}x$	$+b'_{rr}x$, $r \geq 0$	-	$+b_{r,r+1}x$, $r \geq 0$	-
$b_{i+1,i}$	$+b_{i,i-1}x$	-	$+b_{rr}x$, $r \geq 0$	-	$+b_{r+1,r}x$, $r \geq 0$
b'_{ii}	$+b'_{i-1,i-1}x$	-	$+b_{r-1,r}x$, $r \geq 1$	-	$+b'_{rr}x$, $r \geq 0$
c_{ii}	$+c_{i-1,i-1}x$	$+c_{r,r-1}x$, $r \geq 1$	-	$+c_{rr}x$, $r \geq 0$	-
$c_{i,i+1}$	$+c_{i-1,i}x$	$+c'_{rr}x$, $r \geq 0$	-	$+c_{r,r+1}x$, $r \geq 0$	-
$c_{i+1,i}$	$+c_{i,i-1}x$	-	$+c_{rr}x$, $r \geq 0$	-	$+c_{r+1,r}x$, $r \geq 0$
c'_{ii}	$+c'_{i-1,i-1}x$	-	$+c_{r-1,r}x$, $r \geq 1$	-	$+c'_{rr}x$, $r \geq 0$

Table 1: Automorphisms of $D(q)$
($a'_{00} = b'_{00} = c'_{00} = a_{00} = b_{00} = c_{00} = -1$, $a_{0,-1} = b_{0,-1} = c_{0,-1} = a_{-1,0} = b_{-1,0} = c_{-1,0} = 0$)

- $t_{m,m+1}(x), m \geq 1$: Again, note that this map fixes $a_1 = a_{0,1}$, $b_1 = b_{0,1}$ and $c_1 = c_{0,1}$ as for $i = 0$ and $m \geq 1$, $r = i - m < 0$. It also fixes all a_{ii}, $i \leq m$ and all $a_{i,i+1}$, $i < m$. Therefore, all of (2.4) are satisfied for $i < m$. When $i = m$, the first equation is still preserved as $a_{mm}, a'_{m-1,m}$ are fixed. For the third equation, we observe that $a_{m,m+1} \mapsto a_{m,m+1} + a'_{00}x = a_{m,m+1} - x$, $b_{m,m+1} \mapsto b_{m,m+1} - x$ and $c_{m,m+1} \mapsto c_{m,m+1} - x$. Thus, the third equation becomes

$$(a_{m,m+1} - x) + (b_{m,m+1} - x) + (c_{m,m+1} - x) + a_1 b_{mm} + b_1 c_{mm} + c_1 a_{mm} = 0,$$

which is still true as $3x = 0$ in $\mathbb{F}_q$. Finally, for $i > m$, we need to check the validity of the first and third equations from (2.4). However, note that for $i > m$ and $r = i - m \geq 1$,

$$\begin{gathered} a_{ii} + b_{ii} + c_{ii} + a_{i-1,i}b_1 + b_{i-1,i}c_1 + c_{i-1,i}a_1 = 0, \\ (a_{r,r-1} + b_{r,r-1} + c_{r,r-1} + a'_{r-1,r-1}b_1 + b'_{r-1,r-1}c_1 + c'_{r-1,r-1}a_1)x = 0, \end{gathered}$$

and adding these up verifies the first equation, since $t_{m,m+1}(x)(a_{i-1,i}) = a_{i-1,i} + a'_{r-1,r-1}x$. In a similar fashion, we verify the third equation by adding up:

$$\begin{gathered} a_{i,i+1} + b_{i,i+1} + c_{i,i+1} + a_1 b_{ii} + b_1 c_{ii} + c_1 a_{ii} = 0, \\ (a'_{rr} + b'_{rr} + c'_{rr} + a_1 b_{r,r-1} + b_1 c_{r,r-1} + c_1 a_{r,r-1})x = 0, \end{gathered}$$

for $i > m$ and $r = i - m \geq 1$. The second and fourth equations are unchanged by $t_{m,m+1}$.

- $t_{m+1,m}(x), m \geq 1$: Similar to $t_{m,m+1}$, this map fixes a_{ii} and $a_{i,i+1}$ for every i, and hence does not change the first and third set of equations of (2.4). It changes $a_{m+1,m} \mapsto a_{m+1,m} - x$, yet fixes a'_{mm}, hence satisfies

$$(a_{m+1,m} - x) + (b_{m+1,m} - x) + (c_{m+1,m} - x) + a'_{mm}b_1 + b'_{mm}c_1 + c'_{mm}a_1 = 0.$$

  Finally, when $i > m$, the following four equations vouch for the validity of the second and fourth equations of (2.4):

$$\left\| \begin{array}{c} \left\| \begin{array}{c} a_{i+1,i} + b_{i+1,i} + c_{i+1,i} + a'_{ii}b_1 + b'_{ii}c_1 + c'_{ii}a_1 = 0 \\ (a_{rr} + b_{rr} + c_{rr} + a_{r-1,r}b_1 + b_{r-1,r}c_1 + c_{r-1,r}a_1)x = 0 \end{array} \right\| \\ a'_{ii} + b'_{ii} + c'_{ii} + a_1b_{i,i-1} + b_1c_{i,i-1} + c_1a_{i,i-1} = 0 \\ (a_{r-1,r} + b_{r-1,r} + c_{r-1,r} + a_1b_{r-1,r-1} + b_1c_{r-1,r} + c_1a_{r-1,r})x = 0 \end{array} \right\|.$$

- $t_{m,m}(x), m \geq 2$: Same as before, we start by observing that $t_{m,m}(a_{mm}) = a_{mm} - x$, $t_{m,m}(a_{m-1,m}) = a_{m-1,m}$, preserving the first equation of (2.4) for $i = m$. On the other hand, as $a_{m,m+1} \mapsto a_{m,m+1} + a_{0,1}x = a_{m,m+1} + a_1x$, we can rewrite the third equation into:

$$(a_{m,m+1} + a_1x) + (b_{m,m+1} + b_1x) + (c_{m,m+1} + c_1x) + a_1(b_{mm} - x) + b_1(c_{mm} - x) + c_1(a_{mm} - x) = 0.$$

  For $i > m$ and $r = i - m \geq 1$, we only add the first and third equations to themselves for $i = i$ and $i = r$.

- $t'_{m,m}(x), m \geq 2$: For this map, $t'_{m,m}(a'_{mm}) = a'_{mm} - x$, $t'_{m,m}(a_{m,m-1}) = a_{m,m-1}$, verifying the second equation of (2.4) for $i = m$. And, as $t'_{m,m}(a_{m+1,m}) = a_{m+1,m} + a_{1,0}x = a_{m+1,m} + a_1x$, we again have

$$(a_{m+1,m} + a_1x) + (b_{m+1,m} + b_1x) + (c_{m+1,m} + c_1x) + (a'_{mm} - x)b_1 + (b'_{mm} - x)c_1 + (c'_{mm} - x)a_1 = 0.$$

  For $i > m$ and $r = i - m \geq 1$, adding the first and third equations to themselves for $i = i$ and $i = r$ completes the verification.

This calculation shows that the maps defined in Table 1 are all homomorphisms. Since replacing x by $-x$ doesn't change the verification of the equations, and since $f(x) \circ f(-x)$ is the identity map for $f = t_{1,1}$, $t_{m,m+1}$, $t_{m+1,m}$, $t_{m,m}$ and $t'_{m,m}$, this implies that all these maps are automorphisms. This finishes the proof of Claim 2.6. ■

We now return to the proof of Proposition 2.4. In the proof of Claim 2.6, we checked that $t_{1,1}(x)$ keeps a_1 fixed, and moves $a_{11} \mapsto a_{11} + a_{00}x = a_{11} - x$. Therefore, given an edge $\{a, b, c\}$ of $D(q)$, we can perform $t_{1,1}(a_{11})$ to map a_{11} to 0. After applying this map, an application of $t_{1,2}(a_{12})$ sends a_{12} to 0. Therefore, the map φ given by

$$\varphi = \cdots \circ t_{i+1,i}(a_{i+1,i}) \circ t_{i,i+1}(a_{i+1,i}) \circ t'_{ii}(a'_{ii}) \circ t_{ii}(a_{ii}) \circ \cdots \circ t_{1,2}(a_{12}) \circ t_{1,1}(a_{11}),$$

where φ is truncated to s compositions, sends the second through $(s+1)$'th coordinates of a to 0. It also preserves all edges through a, being an automorphism of $D(q)$. This completes the proof. □

2.1.2 Proof of Proposition 2.5

Our goal in this section is to prove that $D_3(k, q)$ admits two different link graphs. We shall consider the link graphs of $a = (a_1, 0, \ldots, 0) \in A$ for $a_1 \in \mathbb{F}_q$. Let L_a denote the link graph of a. We see that $bc \in E(L_a)$ if and only if the following equations hold ($i \geq 2$):

$$\begin{aligned}
b_{11} + c_{11} + a_1 b_1 + b_1 c_1 + c_1 a_1 &= 0 \\
b_{12} + c_{12} + a_1 b_{11} + b_1 c_{11} &= 0 \\
b_{21} + c_{21} + b_{11} c_1 + c_{11} a_1 &= 0 \\
&\vdots \\
b_{ii} + c_{ii} + b_{i-1,i} c_1 + c_{i-1,i} a_1 &= 0 \\
b'_{ii} + c'_{ii} + a_1 b_{i,i-1} + b_1 c_{i,i-1} &= 0 \\
b_{i,i+1} + c_{i,i+1} + a_1 b_{ii} + b_1 c_{ii} &= 0 \\
b_{i+1,i} + c_{i+1,i} + b'_{ii} c_1 + c'_{ii} a_1 &= 0.
\end{aligned} \tag{2.5}$$

Here we consider two different cases.

- **Case 1:** $a_1 = 0$. In this case, we note that the relations (2.5) reduce to the relations (2.2) defining $D(k, q)$, implying $L_a \cong D(k, q)$.
- **Case 2:** $a_1 \neq 0$. In this case, let us define another automorphism ψ on L_a as follows:

$$\left\{\begin{array}{rl}
\psi(b_1) &= \frac{b_1}{a_1} \\
\psi(b_{ii}) &= \frac{b_{ii}}{a_1^{2i}}, \\
\psi(b'_{ii}) &= \frac{b'_{ii}}{a_1^{2i}}, \\
\psi(b_{i,i+1}) &= \frac{b_{i,i+1}}{a_1^{2i+1}}, \\
\psi(b_{i+1,i}) &= \frac{b_{i+1,i}}{a_1^{2i+1}};
\end{array}\right\} \text{ and } \left\{\begin{array}{rl}
\psi(c_1) &= \frac{c_1}{a_1}, \\
\psi(c_{ii}) &= \frac{c_{ii}}{a_1^{2i}}, \\
\psi(c'_{ii}) &= \frac{c'_{ii}}{a_1^{2i}}, \\
\psi(c_{i,i+1}) &= \frac{c_{i,i+1}}{a_1^{2i+1}}, \\
\psi(c_{i+1,i}) &= \frac{c_{i+1,i}}{a_1^{2i+1}}.
\end{array}\right\}$$

  By dividing the equations in (2.5) by appropriate powers of a_1, it can be seen that ψ is an automorphism. Therefore, this implies $L_a \cong L_{(1,0,\ldots,0)}$, completing the proof.

$\square$

Proposition 2.5 naturally leads us to investigate the links of the vertex $(1, 0, \ldots 0)$ in $D_3(q)$. The defining equations for the link of $c = (1, 0, \ldots, 0) \in C$ is

$$\begin{aligned}
a_{11} + b_{11} + a_1 + a_1 b_1 + b_1 &= 0 \\
a_{12} + b_{12} + a_{11} + a_1 b_{11} &= 0 \\
a_{21} + b_{21} + a_{11} b_1 + b_{11} &= 0 \\
&\vdots \\
a_{ii} + b_{ii} + a_{i-1,i} b_1 + b_{i-1,i} &= 0 \\
a'_{ii} + b'_{ii} + a_{i,i-1} + a_1 b_{i,i-1} &= 0 \\
a_{i,i+1} + b_{i,i+1} + a_{ii} + a_1 b_{ii} &= 0 \\
a_{i+1,i} + b_{i+1,i} + a'_{ii} b_1 + b'_{ii} &= 0.
\end{aligned} \tag{2.6}$$

We can reduce this further by replacing a_1 with $a_1 + 1$ and b_1 with $b_1 + 1$. Noting that $(a_1 + 1) + (a_1 + 1)(b_1 + 1) + (b_1 + 1) = a_1 b_1 - a_1 - b_1$ in characteristic 3, we get a new set of equations, namely (2.8). We call this new series of graphs $D'(k, q)$, and take a closer look at them in the next subsection.

2.2 The bipartite graphs $D'(k, q)$

We now take a detour into the series of graphs $D'(k, q)$. It is worth clarifying that in this subsection, we look at $\mathbb{F}_q$ of general characteristic.

Definition 2.7 (The bipartite graphs $D'(q)$)**.** For a prime power q, let A and B be two copies of the countably infinite dimensional vector space V over $\mathbb{F}_q$. We use the following coordinate representations for $a \in A$, $b \in B$:

$$\begin{aligned} a &= (a_1, a_{11}, a_{12}, a_{21}, a_{22}, a'_{22}, a_{23}, \ldots, a_{ii}, a'_{ii}, a_{i,i+1}, a_{i+1,i}, \ldots), \\ b &= (b_1, b_{11}, b_{12}, b_{21}, b_{22}, b'_{22}, b_{23}, \ldots, b_{ii}, b'_{ii}, b_{i,i+1}, b_{i+1,i}, \ldots), \end{aligned} \tag{2.7}$$

Let $D'(q)$ consist of vertex set $A \sqcup B$, and let us join $a \in A$ to $b \in B$ iff the following equations hold ($i \geq 2$):

$$\begin{aligned} a_{11} - a_1 + b_{11} - b_1 + a_1 b_1 &= 0 \\ a_{12} + a_{11} + b_{12} + b_{11} + a_1 b_{11} &= 0 \\ a_{21} + a_{11} + b_{21} + b_{11} + a_{11} b_1 &= 0 \\ &\vdots \\ a_{ii} + a_{i-1,i} + b_{ii} + b_{i-1,i} + a_{i-1,i} b_1 &= 0 \\ a'_{ii} + a_{i,i-1} + b'_{ii} + b_{i,i-1} + a_1 b_{i,i-1} &= 0 \\ a_{i,i+1} + a_{ii} + b_{i,i+1} + b_{ii} + a_1 b_{ii} &= 0 \\ a_{i+1,i} + a'_{ii} + b_{i+1,i} + b'_{ii} + a'_{ii} b_1 &= 0 \end{aligned} \tag{2.8}$$

Define $D'(k,q)$ to be the graph obtained by truncation of A and B to the first k coordinates in (2.7) and, the first $k-1$ relations in (2.8).

It is natural to inquire whether $D'(k,q)$ and $D(k,q)$ are related in any way, in particular, whether they're the same graph. The answer turns out to be yes for small values of k, but no for larger k:

Theorem 2.8. *(a) For $2 \leq k \leq 6$, $D'(k,q) \cong D(k,q)$.*

(b) $D'(11,3) \ncong D(11,3)$.

Proof. First, we prove part (a).

The main idea of the proof is as follows. Observe that it is enough to show that $D'(6,q) \cong D(6,q)$, as an isomorphism $D'(6,q) \to D(6,q)$ can be restricted to fewer coordinates to give isomorphisms $D'(k,q) \to D(k,q)$ for $k \leq 10$. To demonstrate that $D'(6,q) \cong D(6,q)$, we shall define a map $x \mapsto \bar{x}$ sending $a, b \in V(D'(6,q))$ to vectors $\bar{a}, \bar{b} \in \mathbb{F}_q^6$, such that $ab \in E(D'(6,q))$ implies $\bar{a}\bar{b} \in E(D(6,q))$. By construction, this map will be linear and invertible, which would then complete the proof.

We define the map $x \mapsto \bar{x}$ as described in Table 2.

$a \in V(D(k,q))$	$\bar{a} \in \mathbb{F}_q^{10}$	$b \in V(D(k,q))$	$\bar{b} \in \mathbb{F}_q^{10}$
a_1	a_1	b_1	b_1
a_{11}	$a_{11} - a_1$	b_{11}	$b_{11} - b_1$
a_{12}	$a_{12} + a_1$	b_{12}	$b_{12} + b_1$
a_{21}	$a_{21} + a_1$	b_{21}	$b_{21} + b_1$
a_{22}	$a_{22} + a_{12} + a_{11} - a_1$	b_{22}	$b_{22} + b_{12} + b_{11} - b_1$
a'_{22}	$a'_{22} + a_{21} + a_{11} - a_1$	b'_{22}	$b'_{22} + b_{21} + b_{11} - b_1$

Table 2: The isomorphism $D'(6,q) \to D(6,q)$

Suppose $a, b \in V(D'(k,q))$ with $ab \in E(D'(k,q))$. This implies:

$$\begin{aligned}
a_{11} - a_1 + b_{11} - b_1 + a_1 b_1 &= 0\\
a_{12} + a_{11} + b_{12} + b_{11} + a_1 b_{11} &= 0\\
a_{21} + a_{11} + b_{21} + b_{11} + a_{11} b_1 &= 0\\
a_{22} + a_{12} + b_{22} + b_{12} + a_{12} b_1 &= 0\\
a'_{22} + a_{21} + b'_{22} + b_{21} + a_1 b_{21} &= 0.
\end{aligned}$$

Now observe that, $\bar{a}_1 = a_1$ and $\bar{b}_1 = b_1$. Further,

$$\begin{aligned}
&\bullet \begin{cases} \bar{a}_{11} + \bar{b}_{11} + a_1 b_1 = a_{11} - a_1 + b_{11} - b_1 + a_1 b_1 \\ \qquad\qquad\qquad\quad = 0, \end{cases}\\
&\bullet \begin{cases} \bar{a}_{12} + \bar{b}_{12} + a_1 \bar{b}_{11} = a_{12} + a_1 + b_{12} + b_1 + a_1(b_{11} - b_1) \\ \qquad\qquad\qquad\quad = a_{12} + a_1 + b_{12} + b_1 + a_1 b_{11} + (a_{11} - a_1 + b_{11} - b_1) \\ \qquad\qquad\qquad\quad = a_{12} + a_{11} + b_{12} + b_{11} + a_1 b_{11} \\ \qquad\qquad\qquad\quad = 0, \end{cases}\\
&\bullet \begin{cases} \bar{a}_{21} + \bar{b}_{21} + \bar{a}_{11} b_1 = a_{21} + a_1 + b_{21} + b_1 + (a_{11} - a_1) b_1 \\ \qquad\qquad\qquad\quad = a_{21} + a_1 + b_{21} + b_1 + a_{11} b_1 + (a_{11} - a_1 + b_{11} - b_1) \\ \qquad\qquad\qquad\quad = a_{21} + a_{11} + b_{21} + b_{11} + a_{11} b_1 \\ \qquad\qquad\qquad\quad = 0, \end{cases}\\
&\bullet \begin{cases} \bar{a}_{22} + \bar{b}_{22} + \bar{a}_{12} b_1 = a_{22} + a_{12} + a_{11} - a_1 + b_{22} + b_{12} + b_{11} - b_1 + (a_{12} + a_1) b_1 \\ \qquad\qquad\qquad\quad = a_{22} + a_{12} + b_{22} + b_{12} + a_{12} b_1 \\ \qquad\qquad\qquad\quad = 0, \end{cases}\\
&\bullet \begin{cases} \bar{a}'_{22} + \bar{b}'_{22} + \bar{a}_1 \bar{b}_{21} = a'_{22} + a_{21} + a_{11} - a_1 + b'_{22} + b_{21} + b_{11} - b_1 + a_1(a_{21} + b_1) \\ \qquad\qquad\qquad\quad = a'_{22} + a_{21} + b'_{22} + b_{21} + a_1 b_{21} \\ \qquad\qquad\qquad\quad = 0. \end{cases}
\end{aligned} \tag{2.9}$$

Therefore the map $x \mapsto \bar{x}$ is an isomorphism from $D'(6,q)$ to $D(6,q)$, as desired. ■

Our proof of part (b) is purely computational. In summary, it has been computed that the diameter of the component of $D(11,3)$ containing $\vec{0}$ is 22 whereas the same number for $D'(11,3)$ is 20, implying they're not isomorphic (as it is known that $D(11,3)$ is edge-transitive). Further, $D(11,3)$ has 112 cycles through the edge $\{\vec{0}, \vec{0}\}$ whereas $D'(11,3)$ has only 4. This also implies $D(11,3) \ncong D'(11,3)$.

The github repository `https://github.com/Potla1995/hypergraphSuspension/` contains further details on how to reproduce these results. □

Remark. Computer calculations for small values of q suggest that $D'(k,q)$ and $D(k,q)$ are isomorphic for $7 \le k \le 10$. However, the proof method used for $k \le 6$ does not extend to this range.

Note that proving that $D'(k,q)$ has high girth is synonymous to proving lower bounds on $\mathrm{ex}(n, \widetilde{C}_{2k})$ by the machinery we've built so far in this section, and we believe there is enough evidence, computational, and otherwise, to make the following conjecture, analogous to $D(k,q)$.

Conjecture 2.9. $D'(k,q)$ has girth at least $k+4$ if k is even, and $k+5$ if k is odd.

2.3 Proof of Theorem 1.2

We have now built all the machinery required to complete our proof of Theorem 1.2, and will delve into the proof.

Proof of Theorem 1.2. Recall that we have to check three properties of $D_3(k,q)$, and that q is a power of 3.

1. First, we check that $D_3(k,q)$ has $3q^k$ vertices and q^{2k+1} edges. It is clear that every part of $D_3(k,q)$ has q^k vertices. Since there is exactly one free variable when we fix a and b for a hyperedge $\{a,b,c\}$, this gives us a total of $q^k \cdot q^k \cdot q = q^{2k+1}$ edges.
2. Next, we shall prove that the link graphs of every vertex of $D_3(k,q)$ is isomorphic, in fact, to $D(k,q)$ for $k \leq 6$. By Proposition 2.5, the link of every vertex of $D_3(k,q)$ is isomorphic to $D(k,q)$ or $D'(k,q)$ as $3 \mid q$. However, $D(k,q) \cong D'(k,q)$ for $k \leq 6$, implying the required assertion.
3. Finally, it remains to show that $D_3(3,q)$ is $\widetilde{C}_6$-free and $D_3(5,q)$ is $\widetilde{C}_8$-free. From the previous point, and since $D(3,q)$ and $D(5,q)$ are known to have girths 8 and 10 respectively (Proposition 2.2 pt. 2), this completes the proof.

□

3 The arc construction and Wenger's construction

In this section, we relate the arc construction and Wenger's construction via Proposition 1.4, and provide a new set of C_6-free graphs with n vertices and $\Theta(n^{4/3})$ edges via proving Theorem 1.5.

3.1 Proof of Proposition 1.4

Our main goal is to algebraically parametrize the constructions $G_{\texttt{arc}}(k,q,\alpha_0)$ for $k \geq 2$, prime powers q and the normal rational curve α_0, which would lead us to Wenger's construction $H(k,q)$. To this end, we would require the use of the Plücker embedding [7], an algebraic geometric tool that allows us to parametrize the set L.

Lemma 3.1 (Plücker Embedding). *Every line ℓ passing through points $[a_1 : \cdots : a_{t+1}]$ and $[b_1 : \cdots : b_{t+1}]$ in $PG(t,q)$ can be parametrized using $\binom{t+1}{2}$ coordinates $\{w_{ij} : 1 \leq i < j \leq t+1\}$, where w_{ij} is given by the i,j'th minor of the matrix*

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{t+1} \\ b_1 & b_2 & \cdots & b_{t+1} \end{bmatrix}.$$

For further details on the Plücker embedding, the reader is referred to [7], p.211.

We are now well-equipped to prove Proposition 1.4, which asserts that $G_{\texttt{arc}}(k,q,\alpha_0^-) \cong H(k,q)$.

Proof of Proposition 1.4. Recall that in the $G_{\texttt{arc}}(k,q,\alpha_0^-)$ construction, $P = \Sigma \setminus \Sigma_0$ and

$$L = \{\text{projective lines } \ell : \ell \cap \Sigma_0 \in \alpha_0^-\}.$$

Therefore, $|P| = q^k$ and $|Q| = q^{k-1}|\alpha_0^-| = q^k$.

Observe that the lines in L pass through the points $[1 : a_1 : \cdots : a_k] \in P$ and $[0 : 1 : x : \cdots : x^{k-1}] \in \alpha_0$. Let $\{w_{ij} : 1 \leq i < j \leq k+1\}$ parametrize lines in L. Then, for $2 \leq j \leq k+1$,

$$w_{1j} = \det \begin{bmatrix} 1 & a_{j-1} \\ 0 & x^{j-2} \end{bmatrix} = x^{j-2}, \tag{3.1}$$

and for $2 \leq i < j$,

$$w_{ij} = \det \begin{bmatrix} a_{i-1} & a_{j-1} \\ x^{i-2} & x^{j-2} \end{bmatrix} = x^{i-2}(a_{i-1}x^{j-i} - a_{j-1}). \tag{3.2}$$

This set of relations imply

$$x = w_{13}; w_{1j} = w_{13}^{j-2}, 2 \le j; \text{ and } w_{ij} = w_{13}^{i-2}(w_{2j} - w_{2,j-1}w_{13}), 2 \le i < j \le k+1. \tag{3.3}$$

As w_{1j} are all dependent on w_{13} and $\{w_{ij} : i \ge 3\}$ are all dependent on $\{w_{2j} : j \ge 3\}$ by (3.3), we may reduce our variables to only the set $\{w_{13}\} \cup \{w_{2j} : 3 \le j \le k+1\}$. Let $b_1 := x = w_{13}$ and $b_{j-1} = w_{2j}, 3 \le j \le k+1$. Then, the equation (3.2) for $i = 2$ reduces to

$$b_{j-1} = a_1 b_1^{j-2} - a_{j-1},\ 3 \le j \le k+1,$$

Which is exactly the defining set of equations for the graph $H(k,q)$. As P consists of q^k points parametrized by $\{w_{13}\} \cup \{w_{2j} : 3 \le j \le k+1\}$, this implies $G_{\mathtt{arc}}(k,q,\alpha_0^-) \cong H(k,q)$. □

3.2 Proof of Theorem 1.5

We remark that Theorem 1.5 can be proved completely analogously to the proof of Proposition 1.4 via using the arc α of $PG(2,2^r)$ given by $\alpha = \{[1 : t : t^{2^s}] : t \in \mathbb{F}_q\}$. However, for the sake of simplicity, we provide an alternative and more direct proof following Wenger's proof in [21]. Recall that $q = 2^r$, $(s,r) = 1$, and $G(2^r, s)$ is the bipartite graph with parts $A = B = \mathbb{F}_q^3$ such that $(a_1,a_2,a_3) \in A$ and $(b_1,b_2,b_3) \in B$ are adjacent iff

$$b_2 + a_2 = a_1 b_1 \text{ and } b_3 + a_3 = a_1 b_1^{2^s}.$$

Proof of Theorem 1.5. Let $a = (a_1,a_2,a_3), b = (b_1,b_2,b_3), \ldots, f = (f_1,f_2,f_3)$ form a C_6 in $G(2^r,s)$ where $a,c,e \in A$ are distinct, and $b,d,f \in B$ are distinct.

Then, as ab and bc are edges, we have $a_2 + b_2 = a_1 b_1, c_2 + b_2 = c_1 b_1$ implying $a_2 + c_2 = b_1(a_1 + c_1)$ (due to characteristic 2). Similarly, $a_3 + c_3 = b_1^{2^s}(a_1 + c_1)$. We can write these equations as,

$$\begin{bmatrix} a_1 + c_1 \\ a_2 + c_2 \\ a_3 + c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ b_1 \\ b_1^{2^s} \end{bmatrix} \cdot (a_1 + c_1).$$

As similarly

$$\begin{bmatrix} c_1 + e_1 \\ c_2 + e_2 \\ c_3 + e_3 \end{bmatrix} = \begin{bmatrix} 1 \\ d_1 \\ d_1^{2^s} \end{bmatrix} \cdot (c_1 + e_1) \text{ and } \begin{bmatrix} e_1 + a_1 \\ e_2 + a_2 \\ e_3 + a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ f_1 \\ f_1^{2^s} \end{bmatrix} \cdot (e_1 + a_1),$$

Adding these up and using characteristic 2, we have

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ b_1 \\ b_1^{2^s} \end{bmatrix} \cdot (a_1 + c_1) + \begin{bmatrix} 1 \\ d_1 \\ d_1^{2^s} \end{bmatrix} \cdot (c_1 + e_1) + \begin{bmatrix} 1 \\ f_1 \\ f_1^{2^s} \end{bmatrix} \cdot (e_1 + a_1) \\ &= \begin{bmatrix} 1 & 1 & 1 \\ b_1 & d_1 & f_1 \\ b_1^{2^s} & d_1^{2^s} & f_1^{2^s} \end{bmatrix} \begin{bmatrix} a_1 + c_1 \\ c_1 + e_1 \\ e_1 + a_1 \end{bmatrix}. \end{aligned} \tag{3.4}$$

Let $M(x,y,z) = \begin{bmatrix} 1 & 1 & 1 \\ x & y & z \\ x^{2^s} & y^{2^s} & z^{2^s} \end{bmatrix}$. We shall now show that if $x,y,z \in \mathbb{F}_q$ are all distinct, then $M(x,y,z)$ is invertible, i.e. $\frac{x^{2^s}+y^{2^s}}{x+y} \ne \frac{y^{2^s}+z^{2^s}}{y+z}$. To prove this, it is enough to check that for a fixed $t \in \mathbb{F}_q$,

$$\left| \left\{ \frac{(x+t)^{2^s} + t^{2^s}}{x} : x \in \mathbb{F}_q \setminus \{t\} \right\} \right| = q - 1.$$

Observe that, by the binomial theorem and using the fact that $\binom{2^s}{i}$ is even for every $0 < i < 2^s$, $\frac{(x+t)^{2^s}+t^{2^s}}{x} = x^{2^s-1}$. Hence, it suffices to show that the map $x \mapsto x^{2^s-1}$ is a permutation of $\mathbb{F}_q$. However, as the multiplicative group $\mathbb{F}_q^*$ has order $q-1$, this happens only when $(2^s-1, q-1) = 1$, which is true since

$$(2^s-1, 2^r-1) = 2^{(s,r)} - 1 = 1$$

by assumption.

Further, note that if $b_1 = d_1$, then, as

$$b_2 + c_2 = b_1c_1 = c_1d_1 = c_2 + d_2$$

and

$$b_3 + c_3 = b_1^{2^s}c_1 = c_1d_1^{2^s} = c_3 + d_3,$$

we would obtain $b = d$, a contradiction. Thus, b_1, d_1, f_1 are pairwise distinct, and therefore $M(b_1, d_1, f_1)$ is invertible. Hence, (3.4) implies

$$a_1 + c_1 = c_1 + e_1 = e_1 + a_1 = 0,$$

i.e., $a_1 = c_1 = e_1$. However, as

$$a_2 + b_2 = a_1b_1 = c_1b_1 = b_2 + c_2$$

and

$$a_3 + b_3 = a_1b_1^{2^s} = c_1b_1^{2^s} = b_3 + c_3,$$

this would imply $a = c$, a contradiction. $\square$

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