MATH 215 **Hour Exam II Solution** **Radford 07/29/2009**

Name (print) ____________________________

(1) There are *four questions* on this exam. (2) *Return* this exam copy with your test booklet. (3) *You are expected to abide by the University's rules concerning academic honesty.*

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1. (30 points) For sets $A_1, \ldots, A_n$ the union $A_1 \cup \cdots \cup A_n$ and intersection $A_1 \cap \cdots \cap A_n$ are defined inductively by

$$A_1 \cup \cdots \cup A_n = \begin{cases} A_1 & : \ n = 1; \\ (A_1 \cup \cdots \cup A_{n-1}) \cup A_n & : \ n > 1 \end{cases}$$

and

$$A_1 \cap \cdots \cap A_n = \begin{cases} A_1 & : \ n = 1; \\ (A_1 \cap \cdots \cap A_{n-1}) \cap A_n & : \ n > 1 \end{cases}$$

respectively.

(a) Working from definitions, for sets A, B and C show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

Solution: Let $x \in A \cup (B \cap C)$ (**2**). Then $x \in A$ or $x \in B \cap C$. (**2**) If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ so $x \in (A \cup B) \cap (A \cup C)$ (**3**). If $x \in B \cap C$ then $x \in B$ and $x \in C$; therefore $x \in A \cup B$ and $x \in A \cup C$ which means $x \in (A \cup B) \cap (A \cup C)$ (**3**).

(b) Given that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ for all sets A, B, and C, show by induction that $A \cup (A_1 \cap \cdots \cap A_n) = (A \cup A_1) \cap \cdots \cap (A \cup A_n)$ for all sets $A, A_1, \ldots, A_n$ where $n \geq 1$.

Solution: If $n = 1$ then the left hand side and the right hand side of the equation are $A \cup A_1$; therefore the equation is true for $n = 1$. (**3**) Suppose $n \geq 1$ and the equation holds for all sets $A, A_1, \ldots, A_n$ and let $A, A_1, \ldots, A_{n+1}$ be sets. (**3**) Then

$$\begin{aligned} & A \cup (A_1 \cap \cdots \cap A_{n+1}) \\ = \ & A \cup ((A_1 \cap \cdots \cap A_n) \cap A_{n+1}) \\ = \ & (A \cup (A_1 \cap \cdots \cap A_n)) \cap (A \cup A_{n+1}) \\ = \ & ((A \cup A_1) \cap \cdots \cap (A \cup A_n)) \cap (A \cup A_{n+1}) \\ = \ & (A \cup A_1) \cap \cdots \cap (A \cup A_{n+1}). \end{aligned}$$

(**3** for each equation) Therefore the equation holds for all $n \geq 1$ and sets $A, A_1, \ldots, A_n$ (**2**).

2. (25 points) Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be functions.

(a) Using quantifiers define what it means for f to be a surjection.

Solution: $\forall y \in Y$, (**2**) $\exists x \in X$, (**2**) $y = f(x)$ (**2**).

(b) Using quantifiers define what it means for f to be an injection.

Solution: $\forall x_1, x_2 \in X$, (**2**) $(f(x_1) = f(x_2)) \Longrightarrow (x_1 = x_2)$ (**4**); OR equivalently $\forall x_1, x_2 \in X$, $(x_1 \neq x_2) \Longrightarrow (f(x_1) \neq f(x_2))$.

(c) Suppose f, g are injections. Show that $g \circ f : X \longrightarrow Z$ is an injection.

Solution: A proof using the first definition of injectivion. Suppose $x_1, x_2 \in X$ and $(g \circ f)(x_1) = (g \circ f)(x_2)$. (**2**) Then $g(f(x_1)) = g(f(x_2))$. (**3**) Since g is an injection $f(x_1) = f(x_2)$. (**3**) Since f is an injection $x_1 = x_2$. (**3**) Therefore $g \circ f$ is an injection. (**2**)

3. (20 points) Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a function and $a, b \in \mathbf{R}$. A compact definition of $\lim_{x \longrightarrow a} f(x) = b$ is: $\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in \mathbf{R},\ (0 < |x - a| < \delta) \Longrightarrow (|f(x) - b| < \epsilon)$.

(a) Use quantifiers to express "not ($\lim_{x \longrightarrow a} f(x) = b$)" without using "not".

Solution: $\exists \epsilon > 0$, (**2**) $\forall \delta > 0$, (**2**) $\exists x \in \mathbf{R}$, (**2**) $(0 < |x - a| < \delta)$ (**2**) and (**2**) $(|f(x) - b| \geq \epsilon)$ (**2**).

(b) Let $f(x) = \begin{cases} 13x - 2 & : \ x \neq 3 \\ 429 & : \ x = 3 \end{cases}$ Prove that $\lim_{x \longrightarrow 3} f(x) = 37$ from the definition of limit above.

Solution: Let $\epsilon > 0$. Then $|f(x) - 37| = |(13x - 2) - 37| = |13x - 29| = 13|x - 3|$ (**3**) $< \epsilon$ (**2**) when $|x - 3| < \epsilon/13$. Now $\epsilon/13 > 0$ since $\epsilon > 0$. Set $\delta = \epsilon/13$ (**3**).

4. (25 points) *In this problem binomial symbols must be computed.* A committee of 6 persons is to be formed a group of 9 people.

(a) Find the number of such committees.

Solution: $\binom{9}{6} = \dfrac{9{\cdot}8{\cdot}7}{3{\cdot}2{\cdot}1} = 3{\cdot}4{\cdot}7 = 84.$ (**5**)

(b) Find the number of such committees, given that a particular individual is to be *included*.

Solution: $\binom{9-1}{6-1} = \binom{8}{5}_{=} \dfrac{8{\cdot}7{\cdot}6}{3{\cdot}2{\cdot}1} = 4{\cdot}7{\cdot}2 = 56.$ (**5**)

(c) Find the number of such committees, given that a particular individual is to be *excluded*.

Solution: $\binom{9-1}{6} = \binom{8}{6}_{=} \dfrac{8{\cdot}7}{2{\cdot}1} = 4{\cdot}7 = 28.$ (**5**)

(d) Use the Principle of Inclusion-Exclusion to find the number of such committees, given that at least one of two particular individuals is to be *excluded*.

Specific instructions for part (d): Let A and B be these individuals, let X be the set of committees which exclude A, and let Y be the set of committees which exclude B. Express the set of committees of part (d) in terms of X and Y and count them.

Solution: $|X \cup Y| = |X| + |Y| - |X \cap Y|$ (**5**) $= \binom{9-1}{6} + \binom{9-1}{6} - \binom{9-2}{6} = 28 + 28 - 7 = 49$ (**5**).