1. (**30 points total**) We first construct a table for small values of n and for them the values of the terms a_n of the Fibonacci sequence and the values of 2^{n-3}.

n	a_n	2^{n-3}
1	1	1/4
2	1	1/2
3	2	1
4	3	2
5	5	4
6	8	8
7	13	16.

a) From the table we that $a_n > 2^{n-3}$ for $1 \leq n \leq 5$ and $a_n \leq 2^{n-3}$ for $n = 6, 7$. (**10**)

b) From the table it follows that $a_6 = 8 \leq 8 = 2^3 = 2^{6-3}$; thus the inequality is true for $n = 6$. (**5**) Suppose $n \geq 6$ and the inequality is true. We need to show that it is true for $n+1$; that is $a_{n+1} \leq 2^{(n+1)-3} = 2^{n-2}$. If $n = 7$ then from the table $a_7 = 13 < 16 = 2^4 = 2^{7-3}$. (**5**)

Suppose $n > 6$. Then $n - 1 \geq 6$ and therefore

$$a_{n+1} = a_n + a_{n-1} \leq 2^{n-3} + 2^{n-4} \leq 2^{n-3} + 2^{n-3} = 2{\cdot}2^{n-3} = 2^{n-2} = 2^{(n+1)-3}. \quad (\mathbf{5})$$

We have shown that $a_{n+1} \leq 2^{(n+1)-3}$. Thus $a_n \leq 2^{n-3}$ for all $n \geq 6$ by the strong induction principle (with base case $n = 6$.) (**5**)

2. (**40 points total**) When $m = 1$ the left hand side of the equation is $1^2 = 1$ and the right hand side is $\dfrac{1(2{\cdot}1 - 1)(2{\cdot}1 + 1)}{3} = \dfrac{1{\cdot}1{\cdot}3}{3} = 1$. Thus the equation holds in the base case $n = 1$. (**10**)

Suppose $n \geq 1$ the equation holds. We must show it holds for $n + 1$; that is

$$1^2 + 3^2 + 5^2 + \cdots (2(m+1) - 1)^2 = \frac{(m+1)(2(m+1) - 1)(2(m+1) + 1)}{3},$$

or equivalently

$$1^2 + 3^2 + 5^2 + \cdots (2m+1)^2 = \frac{(m+1)(2m+1)(2m+3)}{3}.$$

Now

$$\begin{aligned} & 1^2 + 3^2 + 5^2 + \cdots + (2m+1)^2 \\ = \; & \underbrace{1^2 + 3^2 + 5^2 + \cdots + (2m-1)^2} + (2m+1)^2 \quad (\mathbf{5}) \end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\frac{m(2m-1)(2m+1)}{3}} + (2m+1)^2 \quad (\mathbf{5}) \\
&= \left(\frac{2m+1}{3}\right)(m(2m-1)+3(2m+1)) \\
&= \left(\frac{2m+1}{3}\right)(2m^2+5m+3) \quad (\mathbf{5}) \\
&= \left(\frac{2m+1}{3}\right)(2m+3)(m+1) \quad (\mathbf{5}) \\
&= \frac{(m+1)(2m+1)(2m+3)}{3} \\
&= \frac{(m+1)(2(m+1)-1)(2(m+1)+1)}{3} \quad (\mathbf{5})
\end{aligned}
$$

as required. We have shown that if the formula holds for n it holds for $n+1$. Therefore the the formula holds for all $n \geq 1$ by induction on n. (**5**)

3. (**30 points total**) Suppose $x \in A$. Then $x \notin B$, in which case $x \in A - B$, or $x \in B$, in which case $x \in A \cap B$. Therefore $x \in (A - B) \cup (A \cap B)$. We have shown $x \in A$ implies $x \in (A - B) \cup (A \cap B)$. (**10**)

Conversely, suppose $x \in (A-B) \cup (A \cap B)$. Then $x \in A - B$ or $x \in A \cap B$. By definition $x \in A$ and $x \notin B$, or $x \in A$ and $x \in B$. In either case $x \in A$. Therefore $x \in A$. We have shown that $x \in (A - B) \cup (A \cap B)$ implies $x \in A$. Therefore $A = (A - B) \cup (A \cap B)$. (**10**)

To show that $(A - B) \cap (A \cap B) = \emptyset$, assume $(A - B) \cap (A \cap B) \neq \emptyset$. Then there is an $x \in (A - B) \cap (A \cap B)$. Now $x \in A - B$ and $x \in A \cap B$. Therefore $x \in A$, $x \notin B$ and $x \in A$, $x \in B$. But $x \notin B$ and $x \in B$ is impossible. This contradiction shows that $(A - B) \cap (A \cap B) = \emptyset$ after all. (**10**)