1. (20 points total) We are assuming (i) \( A \cap B \subseteq A \cap C \) and (ii) \( A \cup B \subseteq A \cup C \).
   (a) We wish to show \( B \subseteq C \). Let \( b \in B \). Suppose that \( b \in A \). Then \( b \in A \cap B \) which means \( b \in A \cap C \) by (i). Therefore \( b \in C \). (4) Now suppose \( b \not\in A \). Since \( b \in B \) it follows \( b \in A \cup B \). Therefore \( b \in A \cup C \) by (ii). Since \( b \not\in A \) necessarily \( b \in C \). (4) In any event \( b \in C \). We have shown \( B \subseteq C \). (4)

(b) We are given \( A \cap B = A \cap C \) and \( A \cup B = A \cup C \). In particular \( A \cap B \subseteq A \cap C \) and \( A \cup B \subseteq A \cup C \). Thus \( B \subseteq C \) by part (a). (3) The equations also imply \( A \cap C \subseteq A \cap B \) and \( A \cup C \subseteq A \cup B \). Therefore \( C \subseteq B \) by part (a) as well. (3) We have shown \( B = C \). (2)

2. (20 points total) The functions \( f : X \rightarrow Y \) and \( g : Y \rightarrow X \) satisfy \( g \circ f = I_X \). The latter is equivalent to \( g(f(x)) = x \) for all \( x \in X \). (4) We use this equivalence in our proofs.
   First of all we show \( f \) is injective. Suppose \( x, x' \in X \) and \( f(x) = f(x') \). (3) Then \( x = x' \) since \( x = g(f(x)) = g(f(x')) = x' \). (3) Therefore \( f \) is injective. (2)
   Next we show that \( g \) is surjective. Suppose \( x \in X \) and set \( y = f(x) \). (3) Then \( g(y) = g(f(x)) = x \). (3) Therefore \( f \) is surjective. (2)

3. (20 points total) \( f : [1/2, \infty) \rightarrow [-1/4, \infty) \) is defined by \( f(x) = x^2 - x = (x - 1/2)^2 - 1/4 \). We base arguments on facts derived about increasing functions and quadratics in class; that increasing functions are injective and the function \( g : [0, \infty) \rightarrow [0, \infty) \) given by \( g(x) = x^2 \) is bijective.
   (a) For \( x \geq 1/2 \) observe that \( x - 1/2 \geq 0 \) and therefore \( f(x) \) is increasing (as \( g \) is increasing). Therefore \( f \) is increasing and hence injective. (6)
   (b) Let \( y \in [-1/4, \infty) \) or equivalently \( y \geq -1/4 \). Then \( y + 1/4 \geq 0 \) so \( \sqrt{y + 1/4} \) exists. Since the latter is non-negative \( x = \sqrt{y + 1/4} + 1/2 \geq 1/2 \) which means \( x \in [1/2, \infty) \). (4) The calculation
   \[
   f(x) = f(\sqrt{y + 1/4} + 1/2) = (\sqrt{y + 1/4} + 1/2)^2 - 1/4 = (\sqrt{y + 1/4})^2 - 1/4 = (y + 1/4) - 1/4 = y
   \]
   shows that \( y = f(x) \). (4) Therefore \( f \) is surjective.

Comment: Given \( y \) one discovers the solution \( x \) to \( y = f(x) \) by working backwards. Here we omit those details and show that our \( x \) is indeed a solution.
   (c) From part (b) the inverse \( f^{-1} : [-1/4, \infty) \rightarrow [1/2, \infty) \) is given by
   \[
   f^{-1}(y) = \sqrt{y + 4} + 1/2 \quad (4)
   \]
for all \( y \in [-1/4, \infty) \), or in more standard notation, \( f^{-1}(x) = \sqrt{x + 1/4} + 1/2 \) for all \( x \in [-1/4, \infty) \).

4. (20 points total) Recall \( G^\text{op}_f = \{(y, x) \mid (x, y) \in G_f\} \) is the graph of a function if and only if (a) \( \forall y \in Y, \exists x \in X, (y, x) \in G^\text{op}_f \) and (b) \( (y, x), (y, x') \in G^\text{op}_f \) implies \( x = x' \). This is given.

Therefore \( G^\text{op}_f \) is the graph of a function if and only if (a’) \( \forall y \in Y, \exists x \in X, (x, y) \in G_f \) and (b’) \( (x, y), (x', y) \in G_f \) implies \( x = x' \). (6)

Note \((x, y) \in G_f\) if and only if \( x \in X, y \in Y, \) and \( y = f(x) \). Thus (a’) holds if and only if \( f \) is surjective (7) and (b’) holds if and only if \( f \) is injective (7).

5. (20 points total) Here we show two functions are the same by using a modified truth table.

(a) From the table

\[
\begin{array}{cccccc}
 x \in A & x \in B & \chi_A(x) & \chi_B(x) & \chi_{A\cap B}(x) & \chi_A(x)\chi_B(x) \\
 T & T & 1 & 1 & 1 & 1 \\
 T & F & 1 & 0 & 0 & 0 \\
 F & T & 0 & 1 & 0 & 0 \\
 F & F & 0 & 0 & 0 & 0 \\
\end{array}
\]

(6 for the table) we see that \( \chi_{A\cup B} = \chi_A\chi_B \) since these functions agree in all cases as columns 5 and 6 of the table are identical (4).

(b) From the table

\[
\begin{array}{cccccc}
 x \in A & x \in B & \chi_A(x) & \chi_B(x) & \chi_{A\cup B}(x) & \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x) \\
 T & T & 1 & 1 & 1 & 1 \\
 T & F & 1 & 0 & 1 & 1 \\
 F & T & 0 & 1 & 1 & 1 \\
 F & F & 0 & 0 & 0 & 0 \\
\end{array}
\]

(6 for the table) we see that \( \chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B} \) since these functions agree in all cases as columns 5 and 6 of the table are identical (4).