1. (**20 points total**) The number of $m$-element subset of an $n$-element set, where $0 \le m \le n$, is $\binom{n}{m} = \dfrac{n!}{m!(n-m)!}$.

(a) $\binom{11}{7} = \dfrac{11!}{7!4!} = \dfrac{11{\cdot}10{\cdot}9{\cdot}8}{4{\cdot}3{\cdot}2{\cdot}1} = 11{\cdot}10{\cdot}3 = 330$. (**3 points**)

(b) Since two particular individuals are to be included on the committee, these committees are formed by choosing $7-2=5$ from the remaining $11-2=9$. Thus the number is $\binom{11-2}{7-2} = \binom{9}{5} = \dfrac{9!}{5!4!} = \dfrac{9{\cdot}8{\cdot}7{\cdot}6}{4{\cdot}3{\cdot}2{\cdot}1} = 9{\cdot}7{\cdot}2 = 126$. (**3 points**)

(c) Since two particular individuals are to be excluded from the committee, these committees are formed by choosing 7 from the remaining $11-2=9$. Thus the number is $\binom{11-2}{7} = \binom{9}{7} = \dfrac{9!}{7!2!} = \dfrac{9{\cdot}8}{2{\cdot}1} = 9{\cdot}4 = 36$. (**3 points**)

(d) See part (c). Thus the number is $\binom{11-1}{7} = \binom{10}{7} = \dfrac{10!}{7!3!} = \dfrac{10{\cdot}9{\cdot}8}{3{\cdot}2{\cdot}1} = 10{\cdot}3{\cdot}4 = 120$. (**3 points**)

(e) Let $X$ be the set of committees of 7 with the first individual excluded and $Y$ be the set of committees of 7 with the second excluded. Then $X \cup Y$ is the set of committees with one or the other excluded and $X \cap Y$ is the set of committees with both excluded. Thus

$$|X \cup Y| = |X| + |Y| - |X \cap Y| \ \ (\textbf{4 points}) = 120 + 120 - 36 = 204. \ \ (\textbf{4 points})$$

2. (**20 points total**) This exercise is best done by systematic listings.

(a) Isomorphisms $f : \{a,b,c,d\} \longrightarrow \{a,b,c,d\}$ such that $f(a) = c$:

| $x$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $f_1(x)$ | $c$ | $a$ | $b$ | $d$ |

| $x$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $f_2(x)$ | $c$ | $a$ | $d$ | $b$ |

| $x$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $f_3(x)$ | $c$ | $b$ | $a$ | $d$ |

| $x$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $f_4(x)$ | $c$ | $b$ | $d$ | $a$ |

| $x$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $f_5(x)$ | $c$ | $d$ | $a$ | $b$ |

| $x$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $f_6(x)$ | $c$ | $d$ | $b$ | $a$ |

Inverses are obtained by exchanging the value parts of the rows.

| $x$ | $c$ | $a$ | $b$ | $d$ |
|---|---|---|---|---|
| $f(x)_1^{-1}$ | $a$ | $b$ | $c$ | $d$ |

| $x$ | $c$ | $a$ | $d$ | $b$ |
|---|---|---|---|---|
| $f_2^{-1}(x)$ | $a$ | $b$ | $c$ | $d$ |

| $x$ | $c$ | $b$ | $a$ | $d$ |
|---|---|---|---|---|
| $f_3^{-1}(x)$ | $a$ | $b$ | $c$ | $d$ |

| $x$ | $c$ | $b$ | $d$ | $a$ |
|---|---|---|---|---|
| $f_4^{-1}(x)$ | $a$ | $b$ | $c$ | $d$ |

| $x$ | $c$ | $d$ | $a$ | $b$ |
|---|---|---|---|---|
| $f_5^{-1}(x)$ | $a$ | $b$ | $c$ | $d$ |

| $x$ | $c$ | $d$ | $b$ | $a$ |
|---|---|---|---|---|
| $f_6^{-1}(x)$ | $a$ | $b$ | $c$ | $d$ |

(**6** for each of the two tables)

(b) Surjections $f : \{\pi, e, 19\} \longrightarrow \{c, x\}$:

| $x$ | $\pi$ | $e$ | 19 |
|---|---|---|---|
| $f_1(x)$ | $c$ | $c$ | $x$ |

| $x$ | $\pi$ | $e$ | 19 |
|---|---|---|---|
| $f_2(x)$ | $c$ | $x$ | $c$ |

| $x$ | $\pi$ | $e$ | 19 |
|---|---|---|---|
| $f_3(x)$ | $x$ | $c$ | $c$ |

| $x$ | $\pi$ | $e$ | 19 |
|---|---|---|---|
| $f_4(x)$ | $x$ | $x$ | $c$ |

| $x$ | $\pi$ | $e$ | 19 |
|---|---|---|---|
| $f_5(x)$ | $x$ | $c$ | $x$ |

| $x$ | $\pi$ | $e$ | 19 |
|---|---|---|---|
| $f_6(x)$ | $c$ | $x$ | $x$ |

(**4 points**)

*Comment*: Note that $x$ plays two roles in the tables above. This is known as "abuse of notation". We should use "$z$" for the input, or some other letter not $c$ or $x$.

(c) Injections $f : \{c, x\} \longrightarrow \{\pi, e, 19\}$:

| $z$ | $c$ | $x$ |
|---|---|---|
| $f_1(z)$ | $\pi$ | $e$ |

| $z$ | $c$ | $x$ |
|---|---|---|
| $f_2(z)$ | $\pi$ | 19 |

| $z$ | $c$ | $x$ |
|---|---|---|
| $f_3(z)$ | $e$ | $\pi$ |

| $z$ | $c$ | $x$ |
|---|---|---|
| $f_4(z)$ | $e$ | 19 |

| $z$ | $c$ | $x$ |
|---|---|---|
| $f_5(z)$ | 19 | $\pi$ |

| $z$ | $c$ | $x$ |
|---|---|---|
| $f_6(z)$ | 19 | $e$ |

(**4 points**)

3. (**20 points total**) Let $X$ be the set of residents of this small town.

(a) For $x \in X$ let $f(x)$ be the number of denominations resident $x$ is carrying. Then $f(x) \in \{0, 1, \ldots, 7\}$ as there are 7 denominations. The question can be rephrased as how large does $|X|$ have to be to guarantee that $f(x_1) = f(x_2)$ for some $x_1 \neq x_2$; that is for $f$ not to be injective. Answer: $|X| > 8$. (**10 points**)

(b) Let $f(x)$ be the set of types of denominations resident $x$ is carrying. Then $f(x) \in P(\{\$1, \$5, \$10, \$20, \$50, \$100, \$500\})$ which has $2^7 = 128$ elements. In light of the solution to part (a), $|X| > 128$. (**10 points**)

4. (**20 points total**) Observe that a $2m-1$, where $m \geq 1$, describes all positive odd integers as does $2m+1$, where $m \geq 0$.

$f$ is surjective. Suppose $\ell \in \mathbf{O}^+$. Then $\ell = 2m-1$ for some $m \geq 1$. Suppose $m$ is even. Then $m = 2n$ for some $n \geq 1$. Therefore $\ell = 2m-1 = 4n-1 = f(n)$. Suppose $m$ is odd. Then $m = 2n'+1$ for some $n' \geq 0$. In this case $\ell = 2m-1 = 2(2n'+1)-1 = 4n'+1 = -4n+1$, where $n = -n' \leq 0$. In this case $\ell = f(n)$. We have shown that $f$ is surjective. (**8 points**)

$f$ is injective. Let $n, n' \in \mathbf{Z}$ and suppose that $f(n) = f(n')$.

*Case* 1: $n, n'$ both positive or $n, n'$ both non-negative. Then $4n-1 = f(n) = f(n') = 4n'-1$ or $-4n+1 = f(n) = f(n') = -4n+1$. Thus $4n = 4n'$ or $-4n = -4n'$ either one of which $n = n'$. (**6 points**)

*Case* 2: $n$ positive and $n'$ non-negative, or vice versa. We may assume the former. In this case $4n-1 = f(n) = f(n') = -4n'+1$ which implies $4(n+n') = 2$, or equivalently $2(n+n') = 1$, a contradiction. This case does not exist. (**6 points**)

We have shown that $f(n) = f(n')$ implies $n = n'$. Therefore $f$ is injective.

5. (**20 points total**) The Principle of Inclusion-Exclusion: If $X, Y$ are finite sets then $|X \cup Y| = |X| + |Y| - |X \cap Y|$.

(a) Using DeMorgan's Law $|A^c \cap B^c| = |(A \cup B)^c| = |U| - |A \cup B|$. (**4 points**) Since $|A \cup B| = |A| + |B| - |A \cap B| = 9 + 5 - 2 = 12$ (**4 points**) and $|U| = 23$, $|A^c \cap B^c| = 23 - 12 = 11$. (**2 points**)

(b) Let $S$ and $C$ be the sets of square tiles and circular tiles respectively, and let Let $R$ and $G$ be the sets of red tiles and green tiles respectively. Let $U$ be the set of tiles. Then $S \cup C = U = G \cup R$ and these are disjoint unions. Thus by the Addition Principle (a special case of the Inclusion-Exclusion Principle)

$$|S| + |C| = |U| = |G| + |R|.$$

Since we are given that $|U| = 22$, $|S| = 9$, and $|R| = 14$, we conclude that $|C| = 13$ and $|G| = 8$.

(i) $|S \cup G| = |S| + |G| - |S \cap G| = 9 + 8 - 6 = 11$ (**3 points**) as $|S \cap G| = 6$ (given).

(ii) $S = (S \cap G) \cup (S \cap R)$ and is a disjoint union. Therefore

$$|S| = |S \cap G| + |S \cap R|,$$

or $9 = 6 + |S \cap R|$ which means $|S \cap R| = 3$.

Now $R = (R \cap S) \cup (R \cap C)$ and is a disjoint union. Therefore

$$|R| = |S \cap R| + |C \cap R|,$$

or $14 = 3 + |C \cap R|$ which means $|C \cap R| = 11$. (**3 points**)

(iii) $|C \cup R| = |C| + |R| - |C \cap R| = 13 + 14 - 8 = 16$. (**4 points**)