MATH 215 Final Examination **Radford 12/05/05**

Name (print) ______________________________

(1) With the exception of part a) of Problem 1, write yours answers in the exam booklet provided. (2) The last problem, Problem 8, has two versions. **Do one or the other but not both**. (3) *Return* this exam copy with your test booklet. (4) *You are expected to abide by the University's rules concerning academic honesty.*

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1. (25 pts. total) Let P and Q be statements.

a) (12 pts.) Complete the following three truth tables (on this sheet if you wish):

P	Q	P implies Q
T	T	
T	F	
F	T	
F	F	

P	Q	P and Q
T	T	
T	F	
F	T	
F	F	

P	Q	not P	(not P) or Q
T	T		
T	F		
F	T		
F	F		

b) (8 pts.) What are the logical relationships (implication, equivalence, negation) between the statements "P implies Q", "P and Q", and "(not P) or Q"? Justify your answer in terms of the truth tables of parts a).

c) (5 pts.) Given the logical equivalence of "P" and "not (not P)", of "not (P or Q)" and "(not P) and (not Q)", deduce that "P and (not Q)" is the negation of "(not P) or Q".

2. (30 pts. total) Consider the statement "$(a-3)(a-5) > 0$ implies $a < 3$ or $5 < a$".

a) (5 pts.) What is the converse of the statement?

b) (5 pts.) What is the contrapositive of the statement? Write it without "not".

c) (10 pts.) Prove the statement by contradiction.

d) (10 pts.) Prove the *converse* of the statement directly.

3. (15 pts. total) Let $a_1, a_2, a_3, \ldots$ be a sequence of real numbers and $\ell \in \mathbf{R}$. The statement $\lim_{n\longrightarrow\infty} a_n = \ell$ expressed in terms of quantifiers is $\forall\epsilon > 0, \exists N > 0, \forall n \geq N, |a_n - \ell| < \epsilon$. (It is understood that $\epsilon \in \mathbf{R}^+$, $N \in \mathbf{Z}^+$, and $n \in \{N, N+1, \ldots\}$.)

a) (5 pts.) Express the statement $\lim_{n\longrightarrow\infty} a_n = \ell$ in English with the quantifiers translated.

b) (10 pts.) Express the statement $\lim_{n \longrightarrow \infty} a_n \neq \ell$ (the negation of $\lim_{n \longrightarrow \infty} a_n = \ell$) in terms of quantifiers, without the use of "not".

4. (25 pts. total) Let U be a universal set.

a) (7 pts.) State De Morgan's Laws for subsets A, B of U.

b) (18 pts.) For sets $A_1, \ldots, A_n$ we define the union $A_1 \cup \cdots \cup A_n$ and intersection $A_1 \cap \cdots \cap A_n$ inductively by

$$A_1 \cup \cdots \cup A_n = \begin{cases} A_1 & : \; n = 1; \\ (A_1 \cup \cdots \cup A_{n-1}) \cup A_n & : \; n > 1 \end{cases}$$

and

$$A_1 \cap \cdots \cap A_n = \begin{cases} A_1 & : \; n = 1; \\ (A_1 \cap \cdots \cap A_{n-1}) \cap A_n & : \; n > 1 \end{cases}.$$

Now suppose that $A_1, \ldots, A_n \subseteq U$. Use De Morgan's Laws and the definitions above to construct a proof by induction that

$$(A_1 \cap \cdots \cap A_n)^c = A_1^c \cup \cdots \cup A_n^c$$

for $n \geq 1$. [Comment: You may assume $A_1 \cup \cdots \cup A_m, A_1 \cap \cdots \cap A_m \subseteq U$ for all $A_1, \ldots, A_m \subseteq U$. *The steps of your proof must at least be implicitly justified.*]

Suppose that $n \geq 1$ and the assertion is true. Let $A_1, \ldots, A_{n+1} \subseteq U$. Then $A_1, \ldots, A_n \subseteq U$ and

$$\begin{aligned} (A_1 \cap \cdots \cap A_{n+1})^c &= ((A_1 \cap \cdots \cap A_n) \cap A_{n+1})^c \quad (\textbf{3 points}) \\ &= (A_1 \cap \cdots \cap A_n)^c \cup A_{n+1}^c \quad (\textbf{4 points}) \\ &= (A_1^c \cup \cdots \cup A_n^c) \cup A_{n+1}^c \quad (\textbf{4 points}) \\ &= A_1^c \cup \cdots \cup A_{n+1}^c. \quad (\textbf{3 points}) \end{aligned}$$

Thus the assertion holds for $n+1$, By induction the assertion holds for all $n \geq 1$. (**14 points**)

5. (30 pts. total) Let X and Y be sets.

a) (8 pts.) Give the conditional definitions of $X \cup Y$, $X \cap Y$, and $X \times Y$.

b) (4 pts.) Give the definition of $X \subseteq Y$.

c) (2 pts.) What does it mean for $x \notin X \cap Y$?

d) (12 pts.) Construct a proof of the equivalence of: (1) $X = X \cap Y$, (2) $X \subseteq X \cap Y$, and (3) $X \subseteq Y$ based on the definitions in parts a) and b).

e) (4 pts.) Now suppose that $X = \{2, \pi, -6\}$. Find $P(X)$.

6. (25 pts. total) Suppose that $f : A \longrightarrow B$ is a function.

   a) (10 pts.) Suppose further $g : B \longrightarrow A$ is a function and $g \circ f = I_A$. Show that f is an injection and g is a surjection.

For the remainder of this problem $A = B = \mathbf{Z}^+$ and $f(n) = 2n$ for all $n > 0$.

   b) (10 pts.) Find $\overrightarrow{f}(\{1, 2, 3, 4\})$ and $\overleftarrow{f}(\{1, 2, 3, 4\})$.

   c) (5 pts.) *Define* a function $g : B \longrightarrow A$ such that $g \circ f = I_A$. [You may assume these basic facts about the integers: any integer n can be written $n = 2m$ for some integer m or $n = 2m + 1$ for some integer m, but not both, and in either case the integer m is unique.]

7. (25 pts. total) A committee of 5 is to be formed from a group of 11 people.

   a) (8 pts.) Suppose a certain 3 individuals from this group are to be *excluded*. How many such committees are there?

   b) (5 pts.) Suppose that a certain 2 individuals from this group are to be *included*. How many such committees are there?

   c) (12 pts.) Suppose *at least one* of 2 certain individuals from this group is to be included. Let X be the set of committees which include the first and Y the set of committees which include the second. Use the principle of inclusion-exclusion to calculate the number of elements of $X \cup Y$, the set of committees which include at least one of the two.

Do ONE version of Problem 8 but NOT both.

8. (version 1) (25 pts. total) Let A, B be sets.

   a) (4 pts.) Give the definition for "A and B are equipotent".

   b) (4 pts.) Give the definition for "A is a finite set".

   c) (4 pts.) Give the definition for "A is a denumerable set".

   d) (3 pts.) Give the definition for "A is a countable set".

   e) (10 pts.) Let $A = \{7, 9, 11, 13, 15, \ldots\}$. Show that A is denumerable by showing that the definition of denumerable set is satisfied.

8. (version 2) (25 pts. total) Determine:

   a) (12 pts.) the rational number $32.7\overline{38}$;

   b) (13 pts.) the greatest common divisor of 344 and 60 by the Euclidean Algorithm.