Math 180 Fall 2004 Final Examination Solution; reformatted by Radford 12/12/04

1. (15 points total) Find the derivative of each of the following functions. Do NOT simplify after taking derivatives.

(a) \( \ln(e^x + 1) \)

Solution: \( \frac{e^x}{e^x + 1} \) (5 points)

(b) \( e^{2x} \sinh(3x) \)

Solution: \( 2e^{2x} \sinh(3x) + 3e^{2x} \cosh(3x) \) (5 points)

(c) \( \frac{x^2}{x + 2} \)

Solution: \( \frac{2(x + 2) - x^2}{(x + 2)^2} \) (5 points)

2. (25 points total) A curve is given by the equation \( y^2 + 2y = x^3 - 4x \).

(a) Use implicit differentiation to find a formula for \( \frac{dy}{dx} \).

Solution: \( 2yy' + 2y' = 3x^2 - 4 \), so \( y'(2y + 2) = 3x^2 - 4 \). Thus \( y' = (3x^2 - 4)/(2y + 2) \). (10 points)

(b) Find the equation of the tangent line to this curve at the point \((2, -2)\).

Solution: At \((2, -2)\) the slope is \((3*4-4)/(-4+2) = -4\). Tangent line is: \((y+2)/(x-2) = -4\) or \(y = -4x + 6\). (10 points)

(c) Use part (b) to find the approximate value of \( y \) when \( x = 1.98 \).

Solution: If \( x = 1.98 \), then \( y \) is approximately \(-4(1.98) + 6 = -1.92\). Or using the linear approximation: \(-2 + -4(1.98 - 2) = -1.92\). (5 points)

3. (25 points total) Let \( f(x) = x^2e^{-2x} \). You must use calculus and show all your work in this problem.

(a) Find and classify all critical points of \( f \) (as to local maximum, local minimum, or neither)

Solution: \( f'(x) = 2xe^{-2x} - 2xe^{-2x} = 2xe^{-2x}(1 - x) \). Critical points are \( f'(x) = 0 \), so \( x = 0 \) and \( x = 1 \) are the only critical points. At \( x = 0 \) the value of \( f' \) changes from - to + so \( x = 0 \) (or \( (0,0) \)) is a local minimum. At \( x = 1 \) the value of \( f' \) changes from + to - so \( x = 1 \) or \( (1, e^{-2}) \) is a local maximum. Can also be done using the second derivative test. (15 points)

(b) Find the global maximum and global minimum of \( f \) for \(-1 \leq x \leq 2\).

Solution: \( f(0) = 0, f(1) = e^{-2} \approx .135, f(-1) = e^2 \) and \( f(2) \approx .073 \). So \( e^2 \approx 7.39 \) is the global max and 0 is the global minimum. (10 points)

4. (20 points total)

(a) If \( a \) is a constant, use algebra to find \( \lim_{h \to 0} \frac{(a+h)^2+2(a+h)-a^2-2a}{h} \).

Solution: The fraction simplifies to \((2ah + h^2 + 2h)/h = 2a + h + 2\). As \( h \to 0 \) this becomes \(2a + 2\). (10 points)

(b) Use L’Hôpital’s Rule to find \( \lim_{x \to 0} \frac{1 - \cos(2x)}{x^2} \).

Solution: At \( x = 0 \) we have 0/0. L’Hopital gives \( 2 \sin(2x)/2x = \sin(2x)/x \). Again we have 0/0 at \( x = 0 \) so use L’Hopital again to get \( 2 \cos(2x)/1 \). The limit as \( x \to 0 \) of this is 2. (10 points)
5. (20 points total) Let \( f(x) = \begin{cases} 3 - x & \text{if } x < 3; \\ x^2 + x + b & \text{if } x \geq 3. \end{cases} \)

(i) Find a value for \( b \) so that \( f(x) \) is a continuous function.

Solution: We need \( 3 - 3 = 3^2 + 3 + b \). So \( b = -12 \). (10 points)

(ii) If \( b \) is as in part (i), is \( f(x) \) differentiable for all \( x \)? Justify your answer.

Solution: Must have \((3 - x)' = (x^2 + x + 12)'\) at \( x = 3 \). Then \(-1 = 2 * 3 + 1\), which is impossible. So answer is NO. Other justifications: graph has sharp corner at \( x = \), slopes of the tangent lines to \( 3 - x \) and \( x^2 + x + 12 \) are different. (10 points)

6. (25 points total) Consider the following table of values for the function \( y = f(x) \).

(a) Estimate \( \int_0^3 f(x) \, dx \) using the Left-hand Sum for \( a = 0, b = 3, \) and \( n = 3 \) rectangles;

Solution: \( \Delta x = (b - a)/n = (3 - 0)/3 = 1 \). Thus \( LS = (2 + 3 + 4) * 1 = 9 \). (12 points)

(b) Estimate \( \int_5^3 f(x) \, dx \) using the Right-hand Sum for \( a = 0.5, b = 3, \) and \( n = 5 \) rectangles.

Solution: \( \Delta x = (b - a)/n = (3 - .5)/5 = .5 \). Thus \( RS = (3+7+4-11+10)(.5) = 113/2 = 56.5 \). (13 points)

<table>
<thead>
<tr>
<th>t</th>
<th>f(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0.25</td>
<td>3</td>
</tr>
<tr>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>0.75</td>
<td>-4</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1.25</td>
<td>10</td>
</tr>
<tr>
<td>1.5</td>
<td>7</td>
</tr>
<tr>
<td>1.75</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>2.25</td>
<td>3</td>
</tr>
<tr>
<td>2.5</td>
<td>-11</td>
</tr>
<tr>
<td>2.75</td>
<td>98</td>
</tr>
<tr>
<td>3</td>
<td>110</td>
</tr>
</tbody>
</table>

7. (25 points total) The graph below is a graph of \( f' \) (NOT the graph of \( f \)). Use the graph of \( f' \) to answer the following questions about the function \( f \). To receive full credit you must justify your answers.

(a) Write down all \( x \)-values of critical points of \( f \).

Solution: Critical points are where \( f'(x) = 0 \). Answer: -3, -1, 2. (5 points)

(b) Write down all \( x \)-values for which \( f''(x) = 0 \).

Solution: \( f''(x) = 0 \) at critical point for \( f'(x) \), where tangent line to \( f' \) is has slope 0. Answer: -2, 0, 2. (5 points)

(c) For each critical point, indicate whether \( f \) has a local maximum, a local minimum, or neither at that point.

Solution: \( f' \) changes from + to – at \( x = -3 \), so \( x = 3 \) is a local max. \( f' \) changes from – to +
at } x = -1, \text{ so } x = -1 \text{ is local min. } f' \text{ does not change sign at } x = 2, \text{ so neither local max nor local min. (5 points)} \\
(d) \text{ For each value of } x \text{ for which } f''(x) = 0, \text{ indicate whether it is an inflection point of } f. \\
\textbf{Solution}: \text{ All three of } -2, \text{ 0, 2 are inflection points since } f' \text{ changes from an increasing function to a decreasing function or from a decreasing function to an increasing function at each point. (5 points)} \\
(e) \text{ For which value of } x \text{ is } f(x) \text{ decreasing most rapidly.} \\
\textbf{Solution}: f' \text{ is the most negative at } x = -2. \text{ (5 points)} \\
8. \text{ (25 points total) A farmer has 1,200 feet of fencing and she wants use this fencing to form a rectangular pen consisting of two sections sharing a common side as in the diagram. Let } x \text{ and } y \text{ be the lengths as shown in the diagram.} \\
(a) \text{ Express the area of the pen as a function of } x. \\
\textbf{Solution}: A = 2xy \text{ and } y = (1200 - 4x)/3. \text{ So } A(x) = 2x(1200 - 4x)/3 = 800x - 8x^2/3. \text{ (5 points)} \\
(c) \text{ If } x \text{ and } y \text{ are as in the diagram, what values of } x \text{ and } y \text{ give the maximum total area of the pen?} \\
\textbf{Solution}: A'(x) = 800 - 16x/3. \text{ So } A'(x) = 0 \text{ at } x = 3 * 800/16 = 150 \text{ ft. This is a global maximum since } A = 0 \text{ when } x = 0 \text{ or } x = 1200/4 \text{ (since } y = 0 \text{ in that case). If } x = 150 \text{ then } y = 600/3 = 200 \text{ ft. (15 points)} \\
(b) \text{ What is the maximum area that the pen can have?} \\
\textbf{Solution}: A = 2 * 150 * 200 = 60,000 \text{ square feet. (5 points)} \\
9. \text{ (20 points total) Let } f(x) = 3x^2 + 2x. \\
(a) \text{ Write a definite integral whose value is the area of the region that is bounded on the left by the vertical line } x = 1, \text{ bounded on the right by the vertical line } x = \pi, \text{ bounded above by the curve } y = f(x) \text{ and bounded below by the } x\text{-axis.} \\
\textbf{Solution}: \int_1^\pi (3x^2 + 2x) \, dx. \text{ (5 points)} \\
(b) \text{ Find the } exact \text{ value of the area of the region described in part (a). (Your answer should involve } \pi\text{.)} \\
\textbf{Solution}: \int_1^\pi (3x^2 + 2x) = \pi^3 + \pi^2 - (1^3 + 1^2) = \pi^3 + \pi^2 - 2. \text{ (10 points)} \\
(c) \text{ Find the average value of } y = f(x) \text{ on the interval } [1, \pi]. \\
\textbf{Solution}: \frac{1}{\pi - 1} \int_1^\pi f(x) \, dx = (\pi^3 + \pi^2 - 2)/(\pi - 1), \text{ (5 points) or } \pi^2 + 2\pi + 2 \approx 18.153.