The parameterized family of polynomials $f(x) = x^4 + ax^2$

Let a be a fixed real number. We will analyze the graph of the fourth degree polynomial $y = f(x) = x^4 + ax^2$ by considering two cases. First we note that

$$f(x) = x^4 + ax^2 = x^2(x^2 + a), \tag{1}$$

$$f'(x) = 4x^3 + 2ax = 2x(2x^2 + a), \tag{2}$$

$$f''(x) = 12x^2 + 2a = 2(6x^2 + a). \tag{3}$$

There are two natural cases to consider.

Case 1: $a \geq 0$. Using (1) we note that $f(x) > 0$ when $x \neq 0$ and $f(x) = 0$ when $x = 0$.

f	positive		positive
f	$+$	0	$+$

.

Since $2x^2 + a \geq x^2 \geq 0$ for all x we conclude from (2) that $f'(x) = 0$ if and only if $x = 0$ and

f	decreasing		increasing
f'	$-$	0	$+$

.

Since $6x^2 + a \geq 0$ for all x we conclude from (3) that $f''(x) \geq 0$ for all x and $f''(x) = 0$ if and only if $x = 0$ and $a = 0$. Thus if $a > 0$ we have

(graph of) f	concave up
f''	$+$

and if $a = 0$ we have

(graph of) f	concave up		concave up
f''	$+$	0	$+$

.

In any event f' is increasing on $(-\infty, 0]$ and $[0, \infty)$ since its derivative f'' is positive on $(-\infty, 0)$ and $(0, \infty)$. Therefore f' is increasing on $(-\infty, \infty)$ which is to say that f' is always increasing. Our conclusion: the graph of f is *always* concave up.

Use your graphing calculator to plot the graphs of $y = f(x)$ where $a = 0, 0.5, 1, 3$ on the same screen so that they can be compared.

Case 2: $a < 0$. By (1) we note that $f(x) = 0$ if and only if $x^2 = 0$ or $x^2 + a = 0$; that is $x = 0$, $-\sqrt{-a}$, or $\sqrt{-a}$. Since the graph of the quadratic $y = x^2 + a$ opens upward and crosses the x-axis at $-\sqrt{-a}, \sqrt{-a}$, we see from (1) that

f	positive		negative		negative		positive
f	$+$	$-\sqrt{-a}$	$-$	0	$-$	$\sqrt{-a}$	$+$

By (2) we see that $f'(x) = 0$ if and only if $2x = 0$ or $2x^2 + a = 0$; that is $x = 0$, $-\sqrt{\frac{-a}{2}}$, or $\sqrt{\frac{-a}{2}}$. Since the graph of the quadratic $y = 2x^2 + a$ opens upward and crosses the x-axis at $x = -\sqrt{\frac{-a}{2}}, \sqrt{\frac{-a}{2}}$, by virtue of (2)

f	decreasing		increasing		decreasing		increasing
f'	$-$	$-\sqrt{\frac{-a}{2}}$	$+$	0	$-$	$\sqrt{\frac{-a}{2}}$	$+$

In particular $f(x)$ has local minima which occur at $x = -\sqrt{\frac{-a}{2}}, \sqrt{\frac{-a}{2}}$ and $f(x)$ has one local maximum which occurs at $x = 0$. Since

$$f(-\sqrt{\frac{-a}{2}}) = f(\sqrt{\frac{-a}{2}}) = (\sqrt{\frac{-a}{2}})^2((\sqrt{\frac{-a}{2}})^2 + a) = (\frac{-a}{2})(\frac{-a}{2} + a) = -\frac{a^2}{4}$$

and $f(0) = 0$,

$$\boxed{\text{local mimima: } (-\sqrt{\frac{-a}{2}}, -\frac{a^2}{4}), (\sqrt{\frac{-a}{2}}, -\frac{a^2}{4})}$$

$$\boxed{\text{local maxima: } (0, 0).}$$

Observe that $f(x)$ has no maximum, but

$$\boxed{f(x) \text{ has a minimum which is } -\frac{a^2}{4}.}$$

By (3) we see that $f''(x) = 0$ if and only if $6x^2 + a = 0$ or equivalently $x = -\sqrt{\frac{-a}{6}}, \sqrt{\frac{-a}{6}}$. Since the graph of the quadratic $y = 6x^2 + a$ opens upward and crosses the x-axis at $x = -\sqrt{\frac{-a}{6}}, \sqrt{\frac{-a}{6}}$, as a consequence of (3)

(graph of) f	concave up		concave down		concave up
f''	$+$	$-\sqrt{\frac{-a}{6}}$	$-$	$\sqrt{\frac{-a}{6}}$	$+$

Therefore the graph of $y = f(x)$ has two inflection points which are at $x = -\sqrt{\frac{-a}{6}}, \sqrt{\frac{-a}{6}}$. Since

$$f(-\sqrt{\frac{-a}{6}}) = f(\sqrt{\frac{-a}{6}}) = (\sqrt{\frac{-a}{6}})^2((\sqrt{\frac{-a}{6}})^2 + a) = (\frac{-a}{6})(\frac{-a}{6} + a) = -\frac{5a}{36},$$

$$\boxed{\text{inflection points: } (-\sqrt{\frac{-a}{6}}, -\frac{5a}{36}), (\sqrt{\frac{-a}{6}}, -\frac{5a}{36})}$$

Use your graphing calculator to plot the graphs of $y = f(x)$ where $a = -0.5, -1, -8$ on the same screen so that they can be compared.

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