Math 431 **Spring 2009** **Radford**

Final Examination 05/07/2009

Name (PRINT) ______________________________

(1) *Return* this exam copy. (2) *Write* your solutions in your exam booklet. (3) *Show* your work; justification is required for credit. (4) There are *eight questions* on this exam. (5) Each question counts 25 points. (6) Problems 4–7 constitute a version of Hour Test II. (7) *You are expected to abide by the University's rules concerning academic honesty.*

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1. Determine whether or not the following polynomials are irreducible over $\mathbf{Q}$:

(a) $9x^{100} + 25x^4 - 15$;

(b) $11x^4 - 21x + 27$.

For part (b) you may assume $x^2 + x + 1$ is the only irreducible quadratic in $\mathbf{Z}_2[x]$.]

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2. Let R be an integral domain.

(a) Suppose $a, b \in R$ are irreducible. Show that $a|b$ implies that a and b are associates.

(b) Suppose that $a, b, c, d \in R$ are distinct irreducibles and no two are associates. If $ab = cd$, show that a is not prime.

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3. Let $R = \mathbf{Z}[\sqrt{5}] = \{m + n\sqrt{5} \,|\, m, n \in \mathbf{Z}\}$. Recall that $N : R \longrightarrow \{0, 1, 2, 3, \ldots\}$ defined by $N(m+n\sqrt{5}) = |m^2 - 5n^2| = |(m+n\sqrt{5})(m-n\sqrt{5})|$ satisfies $N(rr') = N(r)N(r')$ for all $r, r' \in R$ and $N(r) = 1$ if and only if r is a unit of R. You may assume these properties of the function N.

(a) Suppose $r \in R$ and $N(r) = p$ is a prime integer. Show that r is irreducible.

(b) Suppose $r \in R$ and $N(r) = p$ is a prime integer. Show that p is not an irreducible element, and also not a prime element, of R.

(c) Use part (b) to show that 11 is not a prime element of R.

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4. Suppose that G is a finite group and $|G| = 825 = 3{\cdot}5^2{\cdot}11$.

(a) Show that G has a subgroup of order 55.

(b) Show that G has an element of order 33.

You may assume the following from group theory. Let $H, K \leq G$. Then $|HK| = |H||K|/|H \cap K|$ and $H \trianglelefteq G$ implies $HK \leq G$.

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5. Let $E = \mathbf{Q}(3^{1/4}, 19^{1/7})$.

(a) Given that $[E : \mathbf{Q}] \leq 28$ find $[E : \mathbf{Q}]$.

(b) Show that $f(x) = x^5 + 27x^2 - 21$ has no root in E.

(c) Show that $3^{1/8} \notin E$.

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6. Let E be a splitting field of $x^4 - 19$ over $\mathbf{Q}$.

(a) Show that $[E : \mathbf{Q}] = 8$.

(b) Find a basis for E as a vector space over $\mathbf{Q}$.

(c) The Galois group $\mathrm{Gal}(E/\mathbf{Q}) \simeq D_4$. Describe generators and relations for $\mathrm{Gal}(E/\mathbf{Q})$. (Justification not needed.)

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7. Let $E = \mathbf{Q}(\sqrt{3}, \imath\sqrt{7}) = \mathbf{Q}(\sqrt{3})(\imath\sqrt{7})$.

(a) Use the fact that $x^2 + 7$ is irreducible over $\mathbf{Q}(\sqrt{3})$ to find $[E : \mathbf{Q}]$.

(b) Show that $E = \mathbf{Q}(2\sqrt{3} - \imath\sqrt{7})$ and find the minimal polynomial of $\alpha = 2\sqrt{3} - \imath\sqrt{7}$ over $\mathbf{Q}$.

(c) Find the minimal polynomial of $\alpha = 2\sqrt{3} - \imath\sqrt{7}$ over $\mathbf{Q}(\sqrt{3})$.

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8. Let F be a field of characteristic 0.

(a) Suppose E is a splitting field of an irreducible $p(x) \in F[x]$ of degree 3 and $[E : F] > 3$. Show $[E : F] = 6$ and $\mathrm{Gal}(E/F) \simeq S_3$.

(b) For the field E of part (a) and each positive divisor d of 6 find the number of subfields K of E which satisfy $F \subseteq K \subseteq E$ and $[K : F] = d$.

(c) Suppose that L is a field extension of F and $[L : F] = 2$. Show that L is a Galois extension of F; that is a splitting field of some $f(x) \in F[x]$ over F.

For part (a) you may use the fact that $\sigma \in \mathrm{Gal}(E/F)$ permutes the set S of roots of $p(x)$ in E and the restriction map $\pi : \mathrm{Gal}(E/F) \longrightarrow \mathrm{Sym}(S)$ given by $\pi(\sigma) = \sigma|_S$ is an injective group homomorphism, where $\mathrm{Sym}(S)$ is the group of permutations of S under composition.