MATH 425 # Hour Exam I **Radford 02/16/2009**

Name (print) ________________________________

(1) *Return* this exam copy with your exam booklet. (2) *Write* your solutions in your exam booklet. (3) *Show* your work. (4) There are *four questions* on this exam. (5) Each question counts 25 points. (6) *You are expected to abide by the University's rules concerning academic honesty.*

For a commutative ring R with unity recall that $R^\times$ denotes the multiplicative group of units of R. $\mathbf{Z}$ denotes the ring of integers, $\mathbf{Q}$ and $\mathbf{R}$ denote the field of rational numbers and real numbers respectively.

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1. Let R be a commutative ring with unity. Define a relation on R by $a \sim b$ if and only if $a = ub$ for some $u \in R^\times$.
(a) Show that "$\sim$" is an equivalence relation on R.
(b) Suppose $a, b \in R$ and $a \sim b$. Show that $Ra = Rb$.

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2. Determine whether or not the following polynomials are irreducible over $\mathbf{Q}$:
(a) $7x^4 + 15x^3 + 12$;
(b) $7x^4 + 15x^3 + 9$. [**Hint**: You may assume $x^2 + x + 1$ is the only irreducible quadratic in $\mathbf{Z}_2[x]$.]

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3. Let $d \in \mathbf{Z}$ and $R = \{\begin{pmatrix} m & n \\ dn & m \end{pmatrix} \,|\, m, n \in \mathbf{Z}\}$.
(a) Show that R is a subring of the ring $\mathrm{M}_2(\mathbf{R})$ of 2×2 matrices with real coefficients.
(b) Suppose that $x^2 - d \in \mathbf{Q}[x]$ is irreducible. Show that $f : \mathbf{Z}[\sqrt{d}] \longrightarrow R$ defined by

$$f(m + n\sqrt{d}) = \begin{pmatrix} m & n \\ dn & m \end{pmatrix}$$

for all $m, n \in \mathbf{Z}$ is a ring isomorphism.
[**Comment**: $x^2 - d \in \mathbf{Q}[x]$ irreducible means f is well-defined; that is $m + n\sqrt{d} = m' + n'\sqrt{d}$ implies $m = m'$ and $n = n'$ for all $m, m', n, n' \in \mathbf{Z}$. You may assume this.]
(c) Recall that $N : \mathbf{Z}[\sqrt{d}] \longrightarrow \{0, 1, 2, \ldots\}$ is defined by $N(m + n\sqrt{d}) = |m^2 - dn^2|$. Write N in terms of f, the determinant function $\mathrm{Det} : \mathrm{M}_2(\mathbf{R}) \longrightarrow \mathbf{R}$, and the absolute value function.

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4. We continue Problem 3 with $R = \mathbf{Z}[\sqrt{5}]$. Recall that $N(1) = 1$, $N(xy) = N(x)N(y)$ for all $x, y \in R$, and $N(x) = 1$ if and only if $x \in R^\times$. You may assume this.
(a) Let $x = m + n\sqrt{5} \in R$ and suppose that $N(x)$ is a prime integer. Show that x is irreducible.
(b) Show that $\sqrt{5}$, $1 - \sqrt{5}$, and $1 + \sqrt{5}$ are irreducible. [**Comment**: You may assume that there are no $x \in R$ with $N(x) = 2$.]
(c) Show that $5, 19$ are not irreducible. [**Hint**: $19 = 20 - 1$.]
(d) Are $5, 19$ prime elements of $\mathbf{Z}[\sqrt{5}]$? Justify your answer.