MATH 431 Written Homework 1 Solution Radford 02/01/09

1. Page 268, number 4: (**20 points**) Let $S = \{(n, n) \,|\, n \in \mathbf{Z}\}$ (**7**). Then $S \neq \emptyset$ as $(0, 0) \in S$. Suppose $(m, m), (n, n) \in S$. The calculations

$$(m, m) - (n, n) = (m - n, m - n) \in S \ \ (\mathbf{4})$$

and

$$(m, m)(n, n) = (mn, mn) \in S \ \ (\mathbf{4})$$

show that S is a subring of $\mathbf{Z}\oplus\mathbf{Z}$. Since $(1, 1) \in S$ and $(1, 0)(1, 1) = (1, 0) \notin S$ it follows that S is not an ideal of $\mathbf{Z}\oplus\mathbf{Z}$ (**5**).

2. Page 269, number 16: (**20 points**) Let $a \in A$ and $b \in B$. Then $ab \in A$, since $a \in A$ and $ab \in B$ since $b \in B$. Therefore $ab \in A\cap B$. Since AB *consists of finite sums of the form* ab, and $A\cap B$ is an additive subgroup of R, it follows that $AB \subseteq A\cap B$ (**8**).

Conversely, let $c \in A\cap B$. Since $A + B = R$, and R is a ring with unity, there are $a \in A$ and $b \in B$ such that $1 = a + b$ (**4**). Thus $c = c1 = ca + cb = ac + cb$. Now $c \in B$ implies $ac \in AB$ and $c \in A$ which implies $cb \in AB$. Therefore $c \in AB$. We have shown $A\cap B \subseteq AB$. Thus $A\cap B = AB$ (**8**).

3. Page 286, number 12: (**20 points**) The authors seem to be assuming that $\mathbf{Z}[\sqrt{2}]$ and H are rings which really should be established. We do not require this for the exercise.

Define $f : \mathbf{Z}[\sqrt{2}] \longrightarrow H$ by

$$f(a + b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$$

for all $a + b\sqrt{2} \in \mathbf{Z}[\sqrt{2}]$ (**3**). The function f is *well defined* since $\sqrt{2}$ is not rational; that is $a + b\sqrt{2} = a' + b'\sqrt{2}$ implies $a = a'$ and $b = b'$ for all $a, a', b, b' \in \mathbf{Z}$. This point needs to be mentioned (**2**).

Let $a, a', b, b' \in \mathbf{R}$. The calculations

$$\begin{aligned} f((a + b\sqrt{2}) + (a' + b'\sqrt{2})) &= f((a + a') + (b + b')\sqrt{2}) \\ &= \begin{pmatrix} a + a' & 2(b + b') \\ b + b' & a + a' \end{pmatrix} \\ &= \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} + \begin{pmatrix} a' & 2b' \\ b' & a' \end{pmatrix} \\ &= f(a + b\sqrt{2}) + f(a' + b'\sqrt{2}) \ \ (\mathbf{5}) \end{aligned}$$

and

$$\begin{aligned} f((a + b\sqrt{2})(a' + b'\sqrt{2})) &= f((aa' + 2bb') + (ab' + a'b)\sqrt{2}) \\ &= \begin{pmatrix} aa' + 2bb' & 2(ab' + a'b) \\ ab' + a'b & aa' + 2bb' \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \begin{pmatrix} a' & 2b' \\ b' & a' \end{pmatrix} \\
&= f(a+b\sqrt{2})f(a'+b'\sqrt{2}) \ \ (\mathbf{5})
\end{aligned}
$$

show that f is a ring homomorphism.

By definition f is surjective. Suppose that $f(a + b\sqrt{2}) = f(a' + b'\sqrt{2})$. Then $\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} = \begin{pmatrix} a' & 2b' \\ b' & a' \end{pmatrix}$ which implies $a = a'$ and $b = b'$. Therefore $a+b\sqrt{2} = a'+b'\sqrt{2}$. We have shown that f is injective. Therefore f is bijective (**5**).

4. Page 299, number 24: (**20 points**) Let $d(x) = f(x) - g(x)$ (**7**). If $a \in F$ and $f(a) = g(a)$ then $d(a) = f(a) - g(a) = 0$. Thus by assumption $d(x)$ has an infinite number of roots (**7**). By §16 Corollary 3 $d(x) = 0$ which means $f(x) = g(x)$ (**6**).

5. Page 299, number 24: (**20 points**) Since $f(a) = 0$, by §16 Corollary 2 $f(x) = (x - a)g(x)$ for some $g(x) \in \mathbf{R}[x]$ (**5**). Write $f(x) = (x - a)^n h(x)$, where $n \geq 1$ and $h(x) \in \mathbf{R}[x]$ (**5**). Then $f'(x) = n(x - a)^{n-1}h(x) + (x - a)^n h'(x)$ (**5**) means that $0 \neq f'(a) = n(a - a)^{n-1}h(a) + (a - a)^n h'(a) = n(a - a)^{n-1}h(a)$ and thus $n = 1$ ($n > 1$ implies $(a - a)^{n-1} = 0$) (**5**).