1. Page 315, number 4: (20 points) Write \( r = p/q \), where \( p, q \in \mathbb{Z} \) and have no common prime factor. Since \( r \) is a root of \( f(x) \) we may write \( f(x) = (x - r)g(x) \) for some \( g(x) \in \mathbb{Q}[x] \). Clearing denominators \( ag(x) = bh(x) \) for some non-zero \( a, b \in \mathbb{Z} \), where \( h(x) \in \mathbb{Z}[x] \) and is primitive. Thus \( agf(x) = (qx - p)ag(x) = b(qx - p)h(x) \) (10). Now \((qx - p)h(x) \) is the product of primitive polynomials and is thus primitive. Therefore \( ag = \pm b \) which means \( f(x) = \pm (qx - p)h(x) \) (5).

By assumption \( f(x) = a_0 + \cdots + a_nx^n \in \mathbb{Z}[x] \), where \( n \geq 0 \) and \( a_n = 1 \). Write \( h(x) = b_0 + \cdots + b_mx^m \in \mathbb{Z}[x] \). Then \( 1 = a_n = \pm qb_m \) which means \( q, b_m \in \{-1, 1\} \). Therefore \( r = p/q \in \mathbb{Z} \) (5).

2. Page 316, number 10: (20 points) The polynomials of (a) and (c) are irreducible over \( \mathbb{Q} \) by the Eisenstein Criterion with \( p = 3 \) (7, 7). Let \( f(x) \) be the polynomial of (c) and \( u \in \mathbb{Q} \) be non-zero, that is a unit \( u \) of \( \mathbb{Q} \). Then \( f(x) \) is irreducible over \( \mathbb{Q} \) if and only if \( uf(x) \) is irreducible over \( \mathbb{Q} \). Now \( 14f(x) \) is irreducible over \( \mathbb{Q} \) by the Eisenstein criterion with \( p = 3 \) again. Therefore \( f(x) \) is irreducible over \( \mathbb{Q} \) (6).

Parts (b) and (d) were not graded; however here are solutions. We apply the mod 2 test in both cases.

Part (b). \( x^4 + x + 1 \in \mathbb{Z}_2[x] \) is primitive. Thus \( x^4 + x + 1 \in \mathbb{Q}[x] \) is irreducible if and only if \( x^4 + x + 1 \in \mathbb{Z}[x] \) is irreducible.

The mod 2 reduction of \( x^4 + x + 1 \in \mathbb{Z}[x] \) is \( f(x) = x^4 + x + 1 \in \mathbb{Z}_2[x] \). Since \( f(a) = 1 \neq 0 \) for all \( a \in \mathbb{Z}_2 \) it follows that \( f(x) \) has no linear factors.

Suppose that \( f(x) \) is reducible. Then it must be the product of quadratic factors.

There are 3 quadratic reducible polynomials in \( \mathbb{Z}_2[x] \); see the solution of Exercise 3 below. Thus there is 1 irreducible quadratic in \( \mathbb{Z}_2[x] \) which is \( x^2 + x + 1 \) since this polynomial has no roots in \( \mathbb{Z}_2[x] \). Therefore \( f(x) = (x^2 + x + 1)^2 = x^4 + x^2 + 1 \) which is not the case.

We have shown \( f(x) \in \mathbb{Z}_2[x] \) is irreducible. Thus \( x^4 + x + 1 \in \mathbb{Z}[x] \) is irreducible which means \( x^4 + x + 1 \in \mathbb{Q}[x] \) is also.

Part (d). Note \( x^5 + 5x^2 + 1 \in \mathbb{Q}[x] \) as a polynomial in \( \mathbb{Z}[x] \) is primitive and the mod 2 reduction is \( f(x) = x^5 + x^2 + 1 \in \mathbb{Z}_2[x] \). This polynomial has no roots in \( \mathbb{Z} \) which means \( f(x) \) has no linear factors.

Suppose \( f(x) \) is reducible. Then it is divisible by an irreducible quadratic which must be \( x^2 + x + 1 \) from part (b). Since \( x^5 + x^2 + 1 = (x^3 + x^2)(x^2 + x + 1) + 1 \), this is not possible by the Division Algorithm. Thus \( f(x) \in \mathbb{Z}_2[x] \) is irreducible and therefore \( x^5 + 5x^2 + 1 \in \mathbb{Q}[x] \) is as well by the argument of part (b).

3. Page 316, number 16: (20 points) (a) We count the number of reducible polynomials in \( \mathbb{Z}_p[x] \) of the form \( x^2 + ax + b \). These are of type \((x-d)^2 \) or \((x-d)(x-e) = (x-e)(x-d)\), where \( d \neq e \). There are \( p \) of the first type, \( \left( \frac{p}{2} \right) \) of the second, and \( p^2 \) of the form
The number of irreducible polynomials of the form \( x^2 + ax + b \) is therefore
\[
p^2 - (p + \left( \frac{p}{2} \right)) = p(p - 1) - \frac{p(p - 1)}{2} = \frac{p(p - 1)}{2} \quad (10).
\]

(b) Let \( f(x) \in \mathbb{Z}_p[x] \) and \( 0 \neq u \in \mathbb{Z}_p \). Then \( f(x) \) is irreducible if and only if \( uf(x) \) is irreducible. Thus the number of irreducible polynomials in \( \mathbb{Z}_p[x] \) of the form \( ax^2 + bx + c \), where \( a \neq 0 \), is \( p - 1 \) times the answer in part (a). There are \( \frac{(p - 1)^2}{2} \) of them \((10)\).

4. Page 316, number 22: \( 20 \) points \ Suppose \( \pi^2 = a\pi + b \), or equivalently \( \pi^2 - a\pi - b = 0 \) \((10)\), for some \( a, b \in \mathbb{Q} \). Then \( \pi \) is a zero, or root, of the polynomial \( f(x) = x^2 - ax - b \in \mathbb{Q}[x] \) which contradicts the given of the problem. Therefore \( \pi^2 \neq a\pi + b \) for all \( a, b \in \mathbb{Q} \) \((10)\).

5. Page 316, number 24: \( 20 \) points \( f(x) = 3x^2 + x + 4 \in \mathbb{Z}_7[x] \). By the quadratic formula the roots of \( f(x) \) in \( \mathbb{Z}_7 \) are given by
\[
((-1) \pm \sqrt{(-1)^2 - 4\cdot3\cdot4})(2\cdot3)^{-1} = ((-1) \pm \sqrt{2})(-1)^{-1} = (-1 \pm 3)(-1) = 4, -2 \quad (5)
\]
or \( 4, 5 \) \((5)\). Substitution yields \( f(4) = 0 = f(5) \) \((5)\). The quadratic formula holds if and only if \( b^2 - 4ac \) has a square root in \( \mathbb{Z}_p \) \((5)\).