Let $R$ be a commutative ring with unity. Recall that $R^\times$ denotes the multiplicative group of units of $R$. Let $a \in R$. We have shown that

$$<a> = R,$$
that is $Ra = R$, if and only if $a \in R^\times$. \hfill (1)

Throughout $R = D$ is an integral domain.

1. Page 333, number 2: (20 points) Suppose that $a, b \in D$ are associates. We show that $<a> = <b>$.

   By definition $a = ub$ for some $u \in D^\times$. The calculation $ra = r(ub) = (ra)b$ for all $r \in R$ shows that $<a> = Ra \subseteq Rb = \subseteq <b>$ \hfill (4).

   Now $u^{-1} \in D^\times$ and $a = ub$ implies $b = u^{-1}a$. We have shown $<b> \subseteq <a>$ \hfill (4). Therefore $<a> = <b>$ \hfill (2).

   Conversely, suppose that $<a> = <b>$. We show that $a$ and $b$ are associates.

   Since $a = 1a \in Ra = <a> = <b> = Rb$ it follows that $a = rb$ for some $r \in R$ \hfill (4). $<a> = <b>$ implies $<b> = <a>$. Therefore there is an $s \in D$ such that $b = sa$. Thus

$$1a = a = rb = r(sa) = (rs)a.$$

If $a \neq 0$ then $1 = rs$ by cancellation which means $r, s \in D^\times$. Therefore $a$ and $b$ are associates \hfill (4).

   Suppose $a = 0$. Then $b = 0$ in which case $a, b$ are associates ($0 = 1\cdot 0$) \hfill (2). We have shown that $a$ and $b$ are associates in any case.

2. Page 333, number 4: (20 points) Suppose $a \in D$ is irreducible and $u \in D^\times$. We show that $ua$ is irreducible.

   First of all $ua \neq 0$ since $u, a \neq 0$ and $D$ is an integral domain. Now $ua \notin D^\times$; else $ua \in D^\times$ and therefore $a = u^{-1}(ua) \in D^\times$. We have shown that $ua$ is a non-zero non-unit \hfill (3).

   Suppose that $ua = bc$, where $b, c \in D$ \hfill (7). Then $a = (u^{-1}b)c$. Since $a$ is irreducible either $u^{-1}b \in D^\times$, in which case $b = u(u^{-1}b) \in D^\times$, or $c \in D^\times$. We have shown that $ua$ is irreducible \hfill (10).

3. Page 333, number 6: (20 points) Let $a \in D$. Then $a \sim b$ since $a = 1a$ \hfill (6). Suppose $a, b \in D$ and $a \sim b$. Then $a = ub$ for some $u \in D^\times$. Since $b = u^{-1}a$ and $u^{-1} \in D^\times$, by definition $b \sim a$ \hfill (7).

   Suppose that $a, b, c \in D$ and $a \sim b$, $b \sim c$. Then $a = ub$ and $b = vc$ for some $u, v \in D^\times$. Since $uv \in D^\times$ and $a = ub = u(vc) = (uv)c$ by definition $a \sim c$ \hfill (7).

   We have shown that “$\sim$” is an equivalence relation on $D$.

4. Page 333, number 10: (20 points) We must assume $p \neq 0$ for the conclusion of the problem to be correct. Here $D$ is a PID.

   Suppose that $<p>$ is a maximal ideal. We show that $p$ is irreducible.
If $p \in D^\times$ then $<p> = D$. Since maximal ideals are proper by definition, $p \not\in D^\times$. Thus $p$ is a non-zero non-unit (2).

Let $a, b \in D$ and suppose $p = ab$. We must show that $a$ or $b$ is a unit, that is $a \in D^\times$ or $b \in D^\times$ (2).

Now $<p> \subseteq <a>$. Since $<p>$ is maximal, either $<a> = D$, in which case $a \in D^\times$ by (1) (2), or $<a> = <p>$, in which case $p, a$ are associates by Exercise 2 (2). In the latter case $p = ua$ for some $u \in D^\times$. But then $ua = p = ab = ba$. Now $a \neq 0$ since $p \neq 0$; thus $b = u$ by cancellation (2). We have shown $a \in D^\times$ or $b \in D^\times$; thus $p$ is irreducible.

Conversely, suppose that $p$ is irreducible. We will show that $<p>$ is a maximal ideal of $D$.

Since $p \not\in D^\times$ the ideal $<p>$ is proper by (1) (2). Suppose that $I$ is an ideal of $D$ and $<p> \subseteq I$. Since $D$ is a PID, $I = <a>$ for some $a \in D$. Now $p \in <p> \subseteq I = <a>$ implies $p = ra = ar$ for some $r \in D$ (2). Since $p$ is irreducible $a \in D^\times$, in which case $I = <a> = D$, or $r \in D^\times$ (2), in which case $p$ and $a$ are associates and thus $<p> = <a> = I$ by Exercise 1 (2). We have shown that $<p>$ is a maximal ideal of $D$ (2).

5. Page 333, number 12: (20 points) Suppose that $I$ is a non-zero proper ideal of $D$. Then $I = <a>$ for some $a \in D$ since $D$ is a PID. Now $a \not\in D^\times$ by (1). $a \neq 0$ since $I \neq (0)$. Therefore $a$ is a non-zero non-unit (4).

Now $D$ is a UFD since it is a PID. Therefore $a$ has a factorization into irreducibles (4) which means $a = pc$ for some irreducible $p \in D$ and $c \in D$ (4). Consequently $I = <a> = Ra \subseteq Rp = <p>$ (4) and the latter is a maximal ideal of $D$ by Exercise 4.

Suppose $I = (0)$. We have shown that if $D$ has a proper non-zero ideal then it has a maximal ideal $J$ and necessarily $I = (0) \subseteq J$. If $D$ has no non-zero proper ideals then $I = (0)$ is maximal (4). (In this case $D$ is a field by (1)).