MATH 431 Written Homework 3 Solution Radford 02/07/09

Let R be a commutative ring with unity. Recall that $R^{\times}$ denotes the multiplicative group of units of R. Let $a \in R$. We have shown that

$$<a> = R, \text{ that is } Ra = R, \text{ if and only if } a \in R^{\times}. \tag{1}$$

Throughout $R = D$ is an integral domain.

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1. Page 333, number 2: (**20 points**) Suppose that $a, b \in D$ are associates. We show that $<a> = <b>$.

By definition $a = ub$ for some $u \in D^{\times}$. The calculation $ra = r(ub) = (ru)b$ for all $r \in R$ shows that $<a> = Ra \subseteq Rb =\subseteq <b>$ (**4**). Now $u^{-1} \in D^{\times}$ and $a = ub$ implies $b = u^{-1}a$. We have shown $<b> \subseteq <a>$ (**4**). Therefore $<a> = <b>$ (**2**).

Conversely, suppose that $<a> = <b>$. We show that a and b are associates.

Since $a = 1a \in Ra = <a> = <b> = Rb$ it follows that $a = rb$ for some $r \in R$ (**4**). $<a> = <b>$ implies $<b> = <a>$. Therefore there is an $s \in D$ such that $b = sa$. Thus

$$1a = a = rb = r(sa) = (rs)a.$$

If $a \neq 0$ then $1 = rs$ by cancellation which means $r, s \in D^{\times}$. Therefore a and b are associates (**4**).

Suppose $a = 0$. Then $b = 0$ in which case a, b are associates $(0 = 1{\cdot}0)$ (**2**). We have shown that a and b are associates in any case.

2. Page 333, number 4: (**20 points**) Suppose $a \in D$ is irreducible and $u \in D^{\times}$. We show that ua is irreducible.

First of all $ua \neq 0$ since $u, a \neq 0$ and D is an integral domain. Now $ua \notin D^{\times}$; else $ua \in D^{\times}$ and therefore $a = u^{-1}(ua) \in D^{\times}$. We have shown that ua is a non-zero non-unit (**3**).

Suppose that $ua = bc$, where $b, c \in D$ (**7**). Then $a = (u^{-1}b)c$. Since a is irreducible either $u^{-1}b \in D^{\times}$, in which case $b = u(u^{-1}b) \in D^{\times}$, or $c \in D^{\times}$. We have shown that ua is irreducible (**10**).

3. Page 333, number 6: (**20 points**) Let $a \in D$. Then $a \sim b$ since $a = 1a$ (**6**). Suppose $a, b \in D$ and $a \sim b$. Then $a = ub$ for some $u \in D^{\times}$. Since $b = u^{-1}a$ and $u^{-1} \in D^{\times}$, by definition $b \sim a$ (**7**).

Suppose that $a, b, c \in D$ and $a \sim b$, $b \sim c$. Then $a = ub$ and $b = vc$ for some $u, v \in D^{\times}$. Since $uv \in D^{\times}$ and $a = ub = u(vc) = (uv)c$ by definition $a \sim c$ (**7**).

We have shown that "$\sim$" is an equivalence relation on D.

4. Page 333, number 10: (**20 points**) *We must assume $p \neq 0$ for the conclusion of the problem to be correct.* Here D is a PID.

Suppose that $<p>$ is a maximal ideal. We show that p is irreducible.

If $p \in D^\times$ then $<p> = D$. Since maximal ideals are proper by definition, $p \notin D^\times$. Thus p is a non-zero non-unit (**2**).

Let $a, b \in D$ and suppose $p = ab$. We must show that a or b is a unit, that is $a \in D^\times$ or $b \in D^\times$ (**2**).

Now $<p> \subseteq <a>$. Since $<p>$ is maximal, either $<a> = D$, in which case $a \in D^\times$ by (1) (**2**), or $<a> = <p>$, in which case p, a are associates by Exercise 2 (**2**). In the latter case $p = ua$ for some $u \in D^\times$. But then $ua = p = ab = ba$. Now $a \neq 0$ since $p \neq 0$; thus $b = u$ by cancellation (**2**). We have shown $a \in D^\times$ or $b \in D^\times$; thus p is irreducible.

Conversely, suppose that p is irreducible. We will show that $<p>$ is a maximal ideal of D.

Since $p \notin D^\times$ the ideal $<p>$ is proper by (1) (**2**). Suppose that I is an ideal of D and $<p> \subseteq I$. Since D is a PID, $I = <a>$ for some $a \in D$. Now $p \in <p> \subseteq I = <a>$ implies $p = ra = ar$ for some $r \in D$ (**2**). Since p is irreducible $a \in D^\times$, in which case $I = <a> = D$, or $r \in D^\times$ (**2**), in which case p and a are associates and thus $<p> = <a> = I$ by Exercise 1 (**2**). We have shown that $<p>$ is a maximal ideal of D (**2**).

5. Page 333, number 12: (**20 points**) Suppose that I is a non-zero proper ideal of D. Then $I = <a>$ for some $a \in D$ since D is a PID. Now $a \notin D^\times$ by (1). $a \neq 0$ since $I \neq (0)$. Therefore a is a non-zero non-unit (**4**).

Now D is a UFD since it is a PID. Therefore a has a factorization into irreducibles (**4**) which means $a = pc$ for some irreducible $p \in D$ and $c \in D$ (**4**). Consequently $I = <a> = Ra \subseteq Rp = <p>$ (**4**) and the latter is a maximal ideal of D by Exercise 4.

Suppose $I = (0)$. We have shown that if D has a proper non-zero ideal then it has a maximal ideal J and necessarily $I = (0) \subseteq J$. If D has no non-zero proper ideals then $I = (0)$ is maximal (**4**). (In this case D is a field by (1)).