1. Page 377, number 6: \(20\) points\) Regard \(g(x) \in F[x]\) as a polynomial with coefficients in \(F[a]\) and let \(b\) be a zero of \(g(x)\) belonging to some field extension of \(F[a]\). We need only show that \(\text{Deg } g(x) = [F[a][b] : F[a]] (4)\).

Now \(m = [F[a][b] : F[a]] \leq \text{Deg } g(x)\) since \(g(b) = 0\) (3). By Theorem 21.5 note \([F[a][b] : F] = [F[a][b] : F[a]] [F[a] : F] = m \text{Deg } f(x)\) (3). On the other hand, from the relations \(F \subseteq F[a], F[b] \subseteq F[a][b] = F[a][b]\) we deduce that \(\text{Deg } f(x) = [F[a] : F]\) and \(\text{Deg } g(x) = [F[b] : F]\). Divide \([F[a][b] : F] = m \text{Deg } f(x)\) by the same theorem (3). Since \(\text{Deg } f(x), \text{Deg } g(x)\) are relatively prime \(\text{Deg } g(x) \text{Deg } f(x)\) divides \(m \text{Deg } f(x)\) (3). Therefore \(\text{Deg } g(x) = m = [F[a][b] : F[a]]\) as \(m \leq \text{Deg } g(x)\) (4).

2. Page 378, number 8: \(20\) points\) First of all \(Q[\sqrt{15}] \subseteq Q[\sqrt{3} + \sqrt{5}]\) since \((\sqrt{3} + \sqrt{5})^2 = 3 + 2 \sqrt{15} + 5 = 8 + 2 \sqrt{15}\) implies \(\sqrt{15} \in Q[\sqrt{3} + \sqrt{5}]\) (3).

We show that \(Q[\sqrt{3} + \sqrt{5}] = Q[\sqrt{3}, \sqrt{5}]\). The left hand side of the equation is contained in the right. The right hand side is contained in the left as from the calculation \((\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) = 5 - 2 = 2\) it follows that \(\sqrt{5} + \sqrt{3}, \sqrt{5} - \sqrt{3} \notin 0\), \((\sqrt{5} + \sqrt{3})^{-1} = (1/2)(\sqrt{5} - \sqrt{3})\), and thus \(\sqrt{5}, \sqrt{3} \in Q[\sqrt{3} + \sqrt{5}]\) as these elements are rational linear combinations of \(\sqrt{5} + \sqrt{3}, \sqrt{5} - \sqrt{3}\) (3).

Let \(a = \sqrt{3} + \sqrt{5}\). We have shown that \(a^2 - (8 + 2 \sqrt{15}) = 0\). Thus \(a\) is a root of \(p(x) = x^2 - (8 + 2 \sqrt{15}) \in Q[\sqrt{15}]\). Therefore \(Q[\sqrt{3} + \sqrt{5}] : Q[\sqrt{15}] \leq 2\). We will show that this dimension is 2 which means \(a\) has degree 2 over \(Q[\sqrt{15}]\) and thus \(1, \sqrt{3} + \sqrt{5}\) is a basis for \(Q[\sqrt{3} + \sqrt{5}]\) over \(Q\) (3).

Suppose \(Q[\sqrt{3} + \sqrt{5}] : Q[\sqrt{15}] = 1\). Then \(Q[\sqrt{3} + \sqrt{5}] = Q[\sqrt{15}] = Q[\sqrt{3}, \sqrt{5}]\). Now \(x^2 - 15, x^2 - 3, x^2 - 5 \in Q[x]\) are irreducible by the Eisenstein Condition with \(p = 3, 5, 15\) respectively. In particular, 3, 5, 15 have no rational square roots and \(\{1, \sqrt{15}\}\) is a basis for \(Q[\sqrt{15}]\) over \(Q\). Thus \(\sqrt{5} = r + s \sqrt{15}\) for some \(r, s \in Q\). Squaring both sides of this equation we have \(5 = (r^2 + 15s^2) + 2rs \sqrt{15}\). Therefore \(5 = r^2 + 15s^2\) and \(2rs = 0\). Exactly one of \(r\) and \(s\) is zero. If \(r = 0\) then \(s = s^{-2}\) and if \(s = 0\) then \(5 = r^2\). In both cases we have a contradiction. We have shown \(Q[\sqrt{3} + \sqrt{5}] : Q[\sqrt{15}] = 2\) (3).

For the second part observe that \(Q[\sqrt{21}/2, 2^{1/3}, 2^{1/4}] = Q[21/3, 2^{1/4}]\) since \((2^{1/4})^2 = 2^{1/2}\). Now \(x^3 - 2, x^4 - 2 \in Q[x]\) are irreducible by the Eisenstein condition with \(p = 2\). Thus \([Q[21/3, 2^{1/4}] : Q] = 12\) by our solution to Problem 1 above; \([Q[21/3, 2^{1/4}] : Q[21/3]] = 4\) and \(Q[21/3, 2^{1/4}]\) has basis \(\{1, 2^{1/4}, 2^{3/4}, 2^{1/2}\}\) over \(Q[21/3]\) (2); \([Q[21/3] : Q] = 3\) and \(Q[21/3]\) has basis \(\{1, 21/3, 2^{3/3}\}\) over \(Q\) (2). A basis for \(Q[21/3, 2^{1/4}]\) over \(Q\) is obtained by multiplying these two bases (2).

3. Page 378, number 14: \(20\) points\) Let \(a = \sqrt{-3} + \sqrt{2} = \sqrt{2} + \sqrt{3}i\) and \(F = Q[a]\). By the calculation \((\sqrt{2} + \sqrt{3}i)(\sqrt{2} - \sqrt{3}i) = (\sqrt{2})^2 - (\sqrt{3}i)^2 = 5\) we deduce neither factor is zero and \(a^{-1} = (1/5)(\sqrt{2} - \sqrt{3}i)\). Therefore \(\sqrt{2} + \sqrt{3}i, \sqrt{2} - \sqrt{3}i \in F\) which means \(\sqrt{2}, \sqrt{3}i \in F\) and \(F = Q[\sqrt{2}, \sqrt{3}i]\) (4).
Now $\mathbb{Q}[\sqrt{2} : \mathbb{Q}] = 2$ as $\sqrt{2}$ is a root of $x^2 - 2 \in \mathbb{Q}[x]$ and is irreducible by the Eisenstein Criterion with $p = 2$ (4). Since $\sqrt{3} i \not\in \mathbb{Q}[\sqrt{2}]$ and $(\sqrt{3} i)^2 = -3 \in \mathbb{Q}[\sqrt{2}]$ it follows that $[\mathbb{Q}[\sqrt{2}][\sqrt{3} i] : \mathbb{Q}[\sqrt{2}]] = 2$ (4). Therefore $[F : \mathbb{Q}] = 4$ by Theorem 21.5 (4).

Therefore the minimal polynomial of $a$ over $\mathbb{Q}$ has degree 4; it is the only monic polynomial of degree 4 which has $a$ as a root.

4. Page 378, number 18: (20 points) We need to assume $E \subseteq \mathbb{C}$ for this problem. Since $[E : \mathbb{Q}] = 2$ there is an $\alpha \in E \setminus \mathbb{Q}$. Choose any such $\alpha$. Then $E = \mathbb{Q}[\alpha]$ and $\alpha$ is a root of a quadratic $x^2 + bx + c \in \mathbb{Q}[x]$ (4). The completing the square calculation

$$0 = \alpha^2 + b\alpha + c = (\alpha + b/2)^2 + (c - b^2/4)$$

shows that $\beta = \alpha + b/2$ is a root of $x^2 - r \in \mathbb{Q}[x]$, where $r = b^2/4 - c$ (4). Note $E = \mathbb{Q}[\beta]$.

Write $\beta^2 = m/n$ where $m, n \in \mathbb{Z}$ and $n > 0$ (4). Since $\sqrt{m/n} = (1/n)\sqrt{mn}$,

$E = \mathbb{Q}[\beta] = \mathbb{Q}[\sqrt{m/n}] = \mathbb{Q}[\sqrt{d}]$ (4). Write $mn = \ell^2 d$, where $\ell, d \in \mathbb{Z}$ and $d$ is square free. Since $\sqrt{mn} = \pm \sqrt{\ell \sqrt{d}}$ we have $E = \mathbb{Q}[(\sqrt{d})] = \mathbb{Q}[\sqrt{d}]$ as required (4).

5. Page 379, number 28: (20 points) Since $a \in \mathbb{C}$ is algebraic over $\mathbb{Q}$, $F = \mathbb{Q}[a]$ is an algebraic extension of $\mathbb{Q}$ (7). Write $r = m/n$ where $m, n \in \mathbb{Z}$ and $n > 0$. Since $a^{1/n}$ is a root of $x^n - a \in F[x]$, $E = F[a^{1/n}]$ is an algebraic extension of $F$ (7). Therefore $E$ is an algebraic extension of $\mathbb{Q}$ by Theorem 21.7. Since $a^r = (a^{1/n})^m \in E$ it follows that $a^r$ is algebraic over $\mathbb{Q}$ (6).