MATH 431 Written Homework 8 Solution Radford 04/09/2009

n_p denotes the number of Sylow p-subgroups in a finite group G.

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1. Page 413, number 8: (**20 points**) Since $|A_4| = 4!/2 = 12 = 2^2{\cdot}3$, each Sylow 3-subgroup of A_4 has 3 elements, and is thus generated by s3-cycle, (**2**) and the number of these subgroups is 1 or 4 (**2**). There are 4 of them:

$$\begin{aligned} <(1\,2\,3)> &= \{\mathrm{I}, (1\,2\,3), (1\,3\,2)\} \ \ (\mathbf{4}) \\ <(1\,2\,4)> &= \{\mathrm{I}, (1\,2\,4), (1\,4\,2)\} \ \ (\mathbf{4}) \\ <(1\,3\,4)> &= \{\mathrm{I}, (1\,3\,4), (1\,4\,3)\} \ \ (\mathbf{4}) \\ <(2\,3\,4)> &= \{\mathrm{I}, (2\,3\,4), (2\,4\,3)\} \ \ (\mathbf{4}) \end{aligned}$$

2. Page 413, number 10: (**20 points**) H is a normal subgroup of $\mathrm{N}(H)$ (**6**). Let K be a subgroup of G and suppose that $H \subseteq K$. Then H is a Sylow p-subgroup of K as $|K|$ divides $|G|$ (**7**).

Let L be a Sylow p-subgroup of G and suppose $L \subseteq \mathrm{N}(H)$. Then L is a Sylow p-subgroup of $\mathrm{N}(H)$. By Sylow's Third Theorem $L = gHg^{-1}$ for some $g \in \mathrm{N}(H)$. Since $aHa^{-1} = H$ for all $a \in \mathrm{N}(H)$ it follows that $L = H$ (**7**).

3. Page 413, number 12: (**20 points**) $|G| = 56 = 2^3{\cdot}7$. Let n_7 be the number of Sylow 7-subgroups of G. Then $n_7|2^3$ and $n_7 = 1 + 7\ell$ for some $\ell \geq 0$. Therefore $n_7 = 1$ or $n_7 = 8$.

Suppose that $n_7 = 1$. Then the Sylow 7-subgroup is a 7-element normal subgroup of G by the corollary to Theorem 24.5 (**4**).

Suppose that $n_7 = 8$ and let H, K be two different Sylow 7-subgroups of G. Then every element of H, except e, has order 7 and generates H. Thus $H \cap K = (e)$. Note every element of G of order 7 generates a Sylow 7-subgroup of G. Thus G has $8{\cdot}6 = 48$ elements of order 7 (**4**).

Let S be the subset of G consisting of the elements of G whose order is not 7. Then $|S| = |G| - 48 = 56 - 48 = 8$ (**4**). Let L be a Sylow 2-subgroup of G. Then elements of L have order a power of 2; thus $L \subseteq S$. Since $|L| = 8 = |S|$ necessarily $L = S$ (**4**). We have shown that G has a unique Sylow 2-subgroup which is thus a normal subgroup of G by the corollary to Sylow's Third Theorem again (**4**).

4. Page 414, number 18: (**20 points**) $|G| = 175 = 5^2{\cdot}7$. Since $n_5|7$ and $n_5 = 1 + 5\ell$ for some $\ell \geq 0$ it follows that $n_5 = 1$. Likewise $n_7|5^2$ and $n_7 = 1 + 7\ell$ for some $\ell \geq 0$; therefore $n_7 = 1$. By the corollary to Theorem 24.5 there is a unique Sylow 5-subgroup H of G and a unique Sylow 7-subgroup K of G and both are normal subgroups of G (**4**). Note H is abelian and K is cyclic. That $G = HK$ and is abelian is the argument of Problem 3 of Written Homework #9 (**16**).

5. Page 414, number 22: (**20 points**) $|G| = 375 = 3{\cdot}5^3$. Here we need to glean an observation from the proof Sylow's Third Theorem; $n_p = [G : \mathrm{N}(H)]$, where H is a Sylow p-subgroup of G (**5**). (This I will grant without proof.)

Consider $p = 3$. Then $n_3 | 5^3$, thus $n_3 = 1$, 5, 25, or 125, and $n_3 = 1 + 3\ell$ for some $\ell \geq 0$. Therefore $n_3 = 1$ or $n_3 = 25$ (**5**).

Suppose $n_3 = 1$. Then $G = \mathrm{N}(H)$, that is H is a normal subgroup of G. By Cauchy's Theorem there is an element $a \in G$ of order 5. Thus $K = <a>$ has order 5. Now HK is a subgroup of G since H is a normal subgroup of G. Since $3, 5$ divide $|HK| \leq |H||K| = 15$, it follows that $|HK| = 15$ (**5**).

Suppose that $n_3 = 25$. Then $25 = [G : \mathrm{N}(H)] = |G|/|\mathrm{N}(H)| = 125/|\mathrm{N}(H)|$ implies that $|\mathrm{N}(H)| = 15$ (**5**).